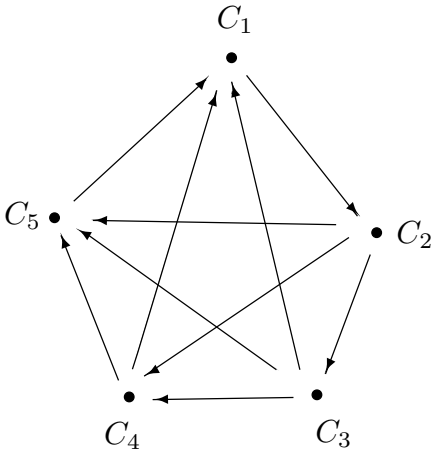
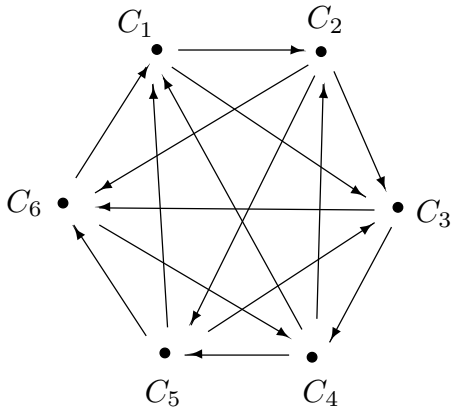
The Chicken Kings

Introduction: The term "pecking order" comes from the behavior of chickens (at least that's what Google reports.) One chicken demonstrates its dominance over another by pecking at it. In fact, chickens do not have "friends;" given any two chickens, one will dominate the other and display that dominance by pecking the other. It is natural to illustrate/describe the pecking order in a flock by means of a complete directed graph. In such a graph we draw a point to represent each chicken, then an arrow from C_1 to C_2 if chicken C_1 dominates (or pecks) chicken C_2. The graph is called "complete" because there is an arrow between every pair of chickens, that is, there are no chicken neutrals.

Directed graphs provide a useful way to represent and study any round robin tournament and are useful in analyzing transportation networks. Although the formal study of graph theory began only in the first half of the twentieth century, there has already been developed a tremendous body of knowledge related to this important subject.

To begin: Suppose we have a flock of one or more chickens. We label the chickens C_1, C_2, $C_3, \ldots$. Between every pair of chickens there is drawn exactly one arrow, depending on the pecking order. For example we write $C_i \to C_j$ if chicken C_i pecks chicken C_j. (In this we assume $i \neq j$, that is, no chicken pecks itself!) We assume that pecking is *not* transitive. That is, it is possible that $C_1 \to C_2$, $C_2 \to C_3$ and $C_3 \to C_1$. Don't ask for an explanation; these are chickens after all!! One possible pecking graph for a flock of 5 chickens and one for a flock of 6 chickens are illustrated below:





There are many ways that we might define a Chicken King. In what follows, we say that K is a King if, given any other chicken C, either K pecks C ($K \to C$) or there exists a Field Marshal chicken F such that $K \to F$ and $F \to C$. This is written compactly as $K \to F \to C$. For example in the 5 chicken flock illustration above, C_1 is a Chicken King with only one (very powerful) Field Marshal. Are there any other Kings? How many Kings are there in the 6 chicken flock example above? For each King, which Chickens are Field Marshalls?

Problems and Questions

1. Suppose we have a flock of chickens. Can we arrange for the flock to have exactly one King?
2. If a chicken is all alone, then it is the King. How many Kings can a two chicken flock have?
3. How many essentially different pecking orders are there for a flock of 3 chickens? What do you mean by different here? How many Chicken Kings are there in each case?
4. How many essentially different pecking orders are there for a flock of 4 chickens? How many Chicken Kings are there in each case?
5. Can a flock of 5 chickens have five Kings? four? three? two? one? none?
6. Find a flock of six chickens in which every chicken is a King. Is there a flock with four Kings? Three Kings? Two? One? No Kings?
7. Suppose we have a flock of n chickens with exactly k Kings. From this could you add a chicken to get a flock of $n+1$ chickens with exactly k Kings?
8. Suppose we have a flock of n chickens in which every chicken is a King. Describe how we can then construct a flock of $n+2$ chickens so that again every chicken is a King.
9. For each chicken in a flock, count the number of other chickens which that chicken pecks. Let K be the chicken with the highest peck count. (If there is a tie, let K be any one of the winners.) Prove that K is a Chicken King. This shows that every flock has at least one King, e.g., there can be no Kingless flocks.
10. Construct a pecking order for a flock with an odd number of birds so that every chicken is a King.
11. Prove the following assertion: Given a particular chicken C, if C is pecked by other chickens, then one of the chickens that pecks C must be a King.
12. Use this lemma to prove that no flock can have exactly two Kings.
13. Prove that if a flock has exactly one King K, then K must peck every other chicken.
14. "Let K be a King which is pecked by some other chicken. Then there must be a King among the field marshals of K." Is this statement true for every pecking order? Either prove it is, or find a counterexample.

Based on an activity from "Circle in a Box" by Sam Vanderwelde