Problems about Fractions

A warm-up problem. Find the fraction that is strictly between $\frac{48}{97}$ and $\frac{49}{99}$ and has the smallest possible denominator. Can you give reasons in support of your answers?

Farey Sequences.

Let n be a positive integer and consider the set of all fractions between 0 and 1 which, in reduced form, have a denominator less than or equal to n. List these fractions in increasing order, starting with $\frac{0}{1}$ and ending with $\frac{1}{1}$. This list of fractions is called the *n-th Farey sequence* and is denoted F_n. For example, F_1 is

$$\frac{0}{1} \quad \frac{1}{1}$$

and F_5 is

$$\frac{0}{1} \quad \frac{1}{5} \quad \frac{1}{4} \quad \frac{1}{3} \quad \frac{2}{5} \quad \frac{1}{2} \quad \frac{3}{5} \quad \frac{2}{3} \quad \frac{3}{4} \quad \frac{4}{5} \quad \frac{1}{1}.$$

1. Write out the Farey sequences F_1, $F_2, \ldots, F_6$.
2. What are the relations between one Farey sequence and another? Can these relations help you write F_7, F_8, etc?
3. What things do you observe about the fractions in any Farey sequence?
4. How many more fractions will be in F_{36} than in F_{35}?

More with Fractions.

1. Use the four integers 94, 95, 96, 97 to make two fractions (so two of these numbers are in the numerator and two in the denominator.) Which arrangement gives the largest possible sum of these two fractions? The smallest? For example, one such sum is $\frac{95}{97} + \frac{96}{94}$.
2. Let a, b, c, d be distinct positive digits (e.g. 1, 2, 3, 4, 5, 6, 7, 8, 9.) Which choice of digits makes
$$\frac{a}{b} + \frac{c}{d} < 1,$$
but as large as possible? What if we want
$$\frac{a}{b} + \frac{c}{d} > 1,$$
but as small as possible?
3. Are there six different positive digits a, b, c, d, e, f with $\frac{a}{b} + \frac{c}{d} = \frac{e}{f}$?
4. Find six different positive digits a, b, c, d, e, f with
$$\frac{a}{b} + \frac{c}{d} + \frac{e}{f} < 1,$$
but as large as possible. What if we want
$$\frac{a}{b} + \frac{c}{d} + \frac{e}{f} > 1,$$
but as small as possible?
5. Use eight distinct positive digits to form four two-digit numbers ab, cd, ef, gh so that
$$\frac{ab}{cd} + \frac{ef}{gh}$$
is as large as possible. Repeat the problem to make the sum as small as possible. (Here $ab = 10a + b$ is the two digit number with tens digit a and units digit b.)