

Problems about parity

Central Iowa Middle School Math Teachers' Circle, November 18, 2008

A warm-up problem. A *lattice point* is a point in the plane for which both coordinates are integers.

- a. Consider the following five lattice points in the plane:

$$A = (13, 25), B = (-24, 14), C = (21, -48), D = (31, 17), E = (26, 43).$$

These points determine many different line segments with two of these points as endpoints (e.g. AB, AC, CE , etc.) Do any of these line segments have a midpoint that is also a lattice point?

- b. Repeat the problem for the lattice points

$$A = (1234567, -7654321), B = (2345789, 9875432), C = (4236638, -34236802),$$

$$D = (-73648265, 10382776), E = (39647554, 3758490013).$$

- c. Suppose you have 4 lattice points in the plane. Must there always be a segment with two of these points as endpoints and with a lattice point as midpoint? What if you have 5 lattice points?

1. There are three frogs in the x, y -plane. Initially one is at each of the points $(1, 0)$, $(0, 0)$, and $(1, 0)$. Every minute one frog, picked at random, jumps over one of the other frogs, also picked at random. When a frog jumps, it jumps in a straight line over the frog being jumped and lands as far from the frog as it was before it jumped, but on the other side.

Is it possible that at some time a frog lands on the point $(1, 1)$?

2. Can you put the numbers $1, 2, 3, \dots, 2009$ in a line in some order so that with the exception of the numbers on the ends, each number is either the sum of the numbers on either side of it or is the absolute value of the difference of the numbers to either side? For example, if a, b, c are three consecutive numbers in the list, then either $b = a + c$ or $b = |a - c|$.
3. A chocolate bar is scored with vertical and horizontal grooves that make a 4×8 grid of small rectangles. Alice and Bob play the following game: Alice goes first. On her first turn she picks up the chocolate bar and breaks it into two pieces along one of the grooves, then sets both pieces down. Next, Bob picks up one of the pieces and breaks it along one of the straight grooves and puts the pieces down. Play continues in this way, with each player picking up a piece, breaking it two, and returning the pieces to the table. The winner is the last person who can break a piece. Is there a winning strategy for either player? (Note: the small rectangular pieces cannot be broken; all breaks must be straight line breaks along the groove lines.)