Two Player Games

Warm-up problem. Two players play the following game: there are 25 pennies on a table. Players alternate turns, and for a turn a player may pick up 1, 2, 3, or 4 of the pennies. The player who takes the last coin wins.

a. Is there a winning strategy for either player? If so what is it?
b. What if the game starts with 24 pennies instead of 25?
c. What is the situation if there are N pennies?

In the traditional two person game setting, there are two players, Alice and Bob. Alice always goes first, Bob second, and they then alternate turns. We will learn to play several two player games. Our goal is for each game to decide if there is (or is not) a winning strategy for one of players and, of course, to completely describe the winning strategy.
To explain a winning strategy we must explain what the winning player should do regardless of the opponent's moves.

1. There are 26 pennies on a table. For a turn, a player may pick up 1, 2, or 4 of the pennies.
   a. The player who takes the last coin wins.
   b. The player who takes the last coin loses.
2. Alice and Bob produce a 20-digit number, writing one digit at a time, from left to right.
   a. Alice wins if the number produced is not divisible by 7, while Bob wins if the number is divisible by 7.
   b. What if the number 7 is replaced by the number 13? (I needed a calculator for this one!)
3. A rook is on the top left corner of an 8×8 chess board. On a turn a player can move the rook to the right or down as many squares as desired. The last person to move wins.
4. There are two piles of matches; one pile contains 10 matches while the other contains 7. A player can take one match from the first pile, one match from the second pile, or one match from each of the two piles. The player who takes the last match wins.
5. The game starts with the number 60 on the board. For a turn, a player reduces the number that is currently on the board by any of its positive divisors. The player who writes the number 0 is the loser.
6. Players take turns placing pennies on a table. The pennies cannot overlap and they can overhand the edge of the table but cannot fall off. The player who places the last penny wins.
7. There are several minuses written in a line. For a turn, a player can replace one minus by a plus or replace two adjacent minuses by two pluses. The player who replaces the last minus sign wins. How does the strategy change if the minuses are written around a circle rather than in a line?

8. There are nine cards, numbered $1, 2, \ldots, 9$ on a table. Alice and Bob take turns choosing one card. A player is declared the winner when he or she has collected three distinct cards whose sum is 15.

9. Given a convex n-gon, players take turns drawing diagonals that do not intersect any of the diagonals already drawn. The player unable to draw a diagonal loses.

10. The game starts with one pile of pebbles on a table. For a move, a player must split one pile into two nonempty piles in such a way that when the turn is completed, no two piles on the table have the same number of pebbles. The player that is unable to move loses.

    a. After the first move the piles contain 5 and 11 pebbles. Find a winning strategy for Bob.
    b. After the first move the piles contain 5 and 11 pebbles. Give an example of a bad move after which Bob can lose.
    c. Which player has a winning strategy if the games starts with a pile of 11 pebbles?
    d. Which player has a winning strategy if the games starts with a pile of 22 pebbles?
    e. What if the game starts with one pile of N pebbles?

Two Player Games—Notes

Warm-up problem. The key here is to be the player who can always leave his/her opponent with a multiple of 5 pennies. If you can do this then you win because on some move you will leave your opponent with 5 pennies. How to arrive at this? One way is to look at smaller games and in particular, seek out simple winning positions. For example, you can discover that if your opponent has to choose from a pile of 5 pennies, then you win. So how to you leave the opponent with 5 pennies? Well, if on the previous turn you left 10 pennies then you know you could leave 5 the next time.
This problem illustrates a valuable general idea in solving problems of this nature: *Find a state of the game that you can repeatedly achieve and will lead to a win.* Aiming for this state for each turn can help you find the correct moves. A recurring state of this kind is sometimes called an *invariant* for the problem or game.

1. Find a small pile such that if you leave your opponent with that pile you win (you can find such a position for each of a. and b.) Then back up. What positions that you leave can lead into this position or to a win?
2. What is it about 7 and 13 that allows Bob to always win the first game but allows Alice to always win the second? What should Bob and Alice be thinking about on their last moves? Again, playing a shorter game can help. 2-digits, 4-digits, etc.
3. Who will win the 2×2 game? 3×3? Who can always make a move to achieve a smaller game that they can win?
4. If it is my turn and the piles are $0, 1$ or $1, 1$ I win! What positions lead into one of these? Can you model this game on a 7×10 checkerboard?
5. What number can you leave to guarantee a win? Even and odd crop up in a surprising way here.
6. Player 1 will win by placing a penny in the exact center of the table. After that no matter how 2 plays, 1 can be assured he can play. On what other shapes of tables will this work?
7. This is very similar to game 6. Why?
8. Why is this game like tic-tc-toe?
9. Can diagonals intersect at end points? If yes, this is a pretty easy game, If no, it is much more interesting (e.g. difficult)

10. One way to analyze the game (at least for small numbers of pebbles) is to write the possible ending positions of the game. For the 11 pebble problem the game is over if it reaches one of the two states:

$$1, 2, 3, 5 \qquad \text{or} \qquad 1, 4, 6.$$

Note that in case no other moves are possible. It will take an odd number of moves to reach the first state and an even number to reach the second. Thus Player 1 can win if she can force the game to the first state. Is this possible? Can you analyze the 22 pebble game in this way? At first this seem daunting, however with some thought we can say a few things about a final state. For example if there are no piles of 1 or 2 pebbles, then the game is not over. Why?