Stretching Sinusoidal Data into More than Sine and Cosine

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• Hands-on
• Emphasis on standards from the Functions and Modeling Conceptual Categories
  – Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline
• Rich pieces which deal with sinusoidal data, rates of change, transformation of functions, decaying exponential, and includes a “twist” at the end.
Equipment

- Graphing Calculator
- Motion Detector connected to a Calculator Based Laboratory (CBL)
- Spring
- Weight with a reflective surface
- “cage” to protect the motion detector

Using the Motion Detector

- How does the detector work?
  - Limits: not closer than 2 feet
  - “cone” of detection
- “Play” with the motion detector by first making predictions and striving to get a plot of a given shape
- How often should we collect data?
- For how many seconds should we collect data? So this translates to how many readings?
Spring (sinusoidal) data

- Reading every .05 second
- 120 readings ➔ 6 seconds
- Height (distance) is recorded in meters.
- Saved the data in lists
- Used TI Connect to transfer the lists from the graphing calculator to the computer.

Data fitting

- Graphing calculator
- Excel
- Geogebra
What do we need to do to \( \sin(x) \) to get it to fit (or model) our data?
$f(t) = a \cdot \sin(b \cdot (t - c)) + d$

$a$ is the amplitude

$b$ is the frequency $= 2\pi/\text{period}$

$c$ is the phase shift

$d$ is the vertical shift
Finding a function to fit the data

- **Amplitude**
  \[ \text{Amplitude} = \frac{y_{\text{peak}} - y_{\text{valley}}}{2} = \frac{1.03942 - .734294}{2} = .152563 \]

- **Period**
  \[ \text{Period} = x_{\text{valley}2} - x_{\text{valley}1} = 1.63075 - .530755 = 1.099995 \]

- **Phase shift**
  \[ \text{Phase shift} = \frac{x_{\text{peak}} + x_{\text{valley}}}{2} = \frac{.530755 + 1.08076}{2} = .8057575 \]

- **Vertical shift**
  \[ \text{Vertical shift} = \frac{y_{\text{peak}} + y_{\text{valley}}}{2} = \frac{.734294 + 1.03942}{2} = .886857 \]

\[ h(t) = .152563 \cdot \sin \left( \frac{2\pi}{1.099995} \cdot (t - .8057575) \right) + .886857 \]
Decaying oscillations

- Consider the peaks (or the valleys or both), and find an exponential function which models their decay.

<table>
<thead>
<tr>
<th>Time</th>
<th>Peaks</th>
<th>Peak-vs</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0307584</td>
<td>1.0515</td>
<td>0.164643</td>
</tr>
<tr>
<td>1.08076</td>
<td>1.03942</td>
<td>0.152563</td>
</tr>
<tr>
<td>2.18072</td>
<td>1.03393</td>
<td>0.147073</td>
</tr>
<tr>
<td>3.28067</td>
<td>1.01747</td>
<td>0.130613</td>
</tr>
<tr>
<td>4.35565</td>
<td>1.01308</td>
<td>0.126223</td>
</tr>
<tr>
<td>5.43052</td>
<td>1.01637</td>
<td>0.129513</td>
</tr>
</tbody>
</table>

Using Exponential Regression

Peak - vertical shift

\[ \text{Peak}(t) = 0.161607 \cdot 0.951421^t \]
Now that we have an exponential function to fit the peaks, how do we use that with our sine function to better fit our data?

\[ h(t) = \left( .161607 \cdot .951421^t \right) \cdot \sin \left( \frac{2\pi}{1.099995} \cdot (t - .8057575) \right) + .886857 \]
Consider the velocity of the weight on the spring

- Average velocity = \( \frac{\Delta \text{height}}{\Delta \text{time}} = \frac{\text{height}_{\text{next}} - \text{height}_{\text{now}}}{\text{time}_{\text{next}} - \text{time}_{\text{now}}} \)

- With TI-89:
  \[ c3=(\text{shift}(c2,1)-c2)/(\text{shift}(c1,1)-c1) \]

- With TI-84:
  \[ \frac{\Delta L_2}{\Delta L_1} \]
  Note: TI-84 will give a “dimension-mismatch” error, so you need to delete the last entry in the time list.

- Relate the average velocity to the slope of the line connecting two consecutive data points
- Make a conjecture about the shape of (time, velocity).

\[ v(t) = 0.8671 \cdot \cos \left( \frac{2 \pi}{1.09996} \cdot (t - 0.780753) \right) - 0.01097 \]
• Considering that instantaneous velocity = \( h'(t) \), how does the derivative compare with our velocity equation and with the plot of \((\text{time}, \text{velocity})\)?

\[
v(t) = 0.8671 \cdot \cos\left(\frac{2\pi}{1.09996} \cdot (t - 0.780753)\right) - 0.01097
\]

\[
h'(t) = 0.871442 \cdot \cos\left(\frac{2\pi}{1.099995} \cdot (t - 0.8057575)\right)
\]
Acceleration

- Acceleration = \( \frac{\Delta velocity}{\Delta time} \)

\[
accel(t) = 5.92768 \cdot \sin \left( \frac{2\pi}{1.199994} \cdot (t - 1.330755) \right) + .220168
\]
\[ \text{accel}(t) = 5.92768 \cdot \sin \left( \frac{2\pi}{1.199994} \cdot (t - 1.330755) \right) + 0.220168 \]

\[ h''(t) = -4.97769 \cdot \sin \left( \frac{2\pi}{1.099995} \cdot (t - 0.8057575) \right) \]

**Summarizing the equations:**

\[ h(t) = 0.152563 \cdot \sin \left( \frac{2\pi}{1.099995} \cdot (t - 0.8057575) \right) + 0.886857 \]

\[ v(t) = 0.8671 \cdot \cos \left( \frac{2\pi}{1.09996} \cdot (t - 0.780753) \right) - 0.01097 \]

\[ h'(t) = 0.871442 \cdot \cos \left( \frac{2\pi}{1.099995} \cdot (t - 0.8057575) \right) \]

\[ \text{accel}(t) = 5.92768 \cdot \sin \left( \frac{2\pi}{1.199994} \cdot (t - 1.330755) \right) + 0.220168 \]

\[ h''(t) = -4.97769 \cdot \sin \left( \frac{2\pi}{1.099995} \cdot (t - 0.8057575) \right) \]
Here’s the “twist”

• Make a conjecture:
• What should the shape of \((\text{height, velocity})\) be? Why?

\[
h(t) = 0.152563 \cdot \sin \left( \frac{2\pi}{1.099995} \cdot (t - 0.8057575) \right) + 0.886857
\]

\[
v(t) = h'(t) = 0.871442 \cdot \cos \left( \frac{2\pi}{1.099995} \cdot (t - 0.8057575) \right)
\]
\[ h(t) = 0.152563 \sin \left( \frac{2\pi}{1.099995} (t - 0.8057575) \right) + 0.886857 \]
\[ \sin \left( \frac{2\pi}{1.099995} (t - 0.8057575) \right) = \frac{h - 0.886857}{0.152563} \]
\[ v(t) = h'(t) = 0.871442 \cos \left( \frac{2\pi}{1.099995} (t - 0.8057575) \right) \]
\[ \cos \left( \frac{2\pi}{1.099995} (t - 0.8057575) \right) = \frac{v}{0.871442} \]

\[ \sin^2 \theta + \cos^2 \theta = 1 \]
\[ \left( \frac{h - 0.886857}{0.152563} \right)^2 + \left( \frac{v}{0.871442} \right)^2 = 1 \]

\[ \text{velocity} = \pm 0.871442 \sqrt{1 - \left( \frac{h - 0.886875}{0.152563} \right)^2} \]
velocity = ±0.871442 \sqrt{1 - \left( \frac{h - 0.886875}{0.152563} \right)^2}

Make a conjecture: What should the shape of (height, acceleration) be?
\[ h(t) = 0.152563 \sin \left( \frac{2\pi}{1.099995} (t - 0.8057575) \right) + 0.886857 \]

\[ \sin \left( \frac{2\pi}{1.099995} (t - 0.8057575) \right) = \frac{h - 0.886857}{0.152563} \]

\[ \text{accel}(t) = h''(t) = -4.97769 \sin \left( \frac{2\pi}{1.099995} (t - 0.8057575) \right) \]

\[ \sin \left( \frac{2\pi}{1.099995} (t - 0.8057575) \right) = \frac{-\text{accel}}{4.97769} \]
\[
\frac{h - .886857}{.152563} = -\frac{accel}{4.97769}
\]

\[-accel = -4.97769 \left( \frac{h - .886857}{.152563} \right) = -32.6271 \cdot h + 28.9356\]
Conclusion

- With the collection of a single set of sinusoidal data, some very rich mathematics results.
  - Fitting sinusoidal data
  - Rates of change
  - Decaying oscillations (decaying exponential)
  - An elliptical relationship & a linear relationship

Thank you!

- For an electronic copy of the power point presentation and/or the data, please email me at: hbolles@iastate.edu