Suppose there is no such pair of integers. For positive integer a, consider the integers

$$a, \qquad a+9, \qquad a+16, \qquad a+25.$$

Then a, $a+16$, and $a+25$ must all be of different colors and a, $a+9$, and $a+25$ are of different colors. Thus $a+9$ and $a+16$ are the same color. Repeating this argument with the integers

$$a+16, \qquad a+25, \qquad a+32, \qquad a+41,$$

we see that $a+41$ is the same color as a.

Continueing this argument, next with the numbers $a+41$, $a+50$, $a+57$, $a+66$ and $a+57$, $a+66$, $a+73$, $a+82 = a+2\cdot 41$, we find that

$$a,\ a+41,\ a+2\cdot 41, \ldots, a+41\cdot 41 \ldots$$

all have the same color. However, a and $a+41^2$ differ by a square, contradicting our initial assumption. This there must always be two integers of the same color that differ by a square.