ANALYSIS QUALIFYING EXAMINATION
Fall 2011
305 Carver Hall, August 2011

- Write your university identification number on every page you turn in. Do NOT write your name on any page you turn in.
- Your score will be based on a total of six problems. If you turn in more than six problems, all the problems will be graded and your scores will be based on the six lowest scoring problems. If you work fewer than six problems, you will receive a score of 0 for the missing problems.
- Work each problem on a separate sheet of paper, and clearly indicate the problem on each page.
- To pass, you must work on at least two problems from Part I and three problems from Part II, and receive substantial credits from both parts. In the grading, one completely correct solution will be counted as more than two “half correct” solutions.
- Every effort is made to proofread the exam, but misprints may occur. If you believe that a problem has been stated incorrectly, check with the proctor and indicate your interpretation in the solution. Do not interpret the problem in a way that it becomes trivial.
Part I. Complex Analysis

Problem 1. Let $f(z) = \frac{1}{1-z}$ and let $a$ and $b$ be real numbers with $1 < a < b$. For any $z_0 \in \mathbb{C} \setminus \{1\}$, there is an $r > 0$ and a power series $\sum_{k=0}^{\infty} c_k (z - z_0)^k$ such that the power series has radius of convergence $r$ and

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} c_k (z - z_0)^k$$

for $z$ in the disk $D_{z_0} = \{|z - z_0| < r\}$. Find all $z_0$ for which the segment $[a, b] \subset D_{z_0}$.

Problem 2. Let $f(z)$ be meromorphic on an open connected set containing $\{z : |z| \leq 1\}$ and with $|f(z)| = 1$ for all $z$ with $|z| = 1$. Prove that $f$ is a rational function.

Problem 3. Consider the integral

$$\int_{|z|=R} \frac{z^6}{z^8 + z + 1} \, dz.$$ 

Prove that there are real numbers $0 < R_0 < R_1 < \infty$ such that the integral is 0 for $R < R_0$ and for $R > R_1$.

Problem 4. Let $P(z) = z^9 + z^3 + 10z + 1$. Find the number of zeros for $p(z)$, counted with respect to multiplicity, in

a) $\{z : 1 < |z| < 2\}$

b) $\{z : \text{Im}(z) > 0\}$
Part II. Real Analysis

Problem 1. Let the function $f$ be defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} 
  x \sin \frac{1}{x}, & x \neq 0 \\
  0, & x = 0 
\end{cases}$$

Is $f$ uniformly continuous on $\mathbb{R}$? Is $f$ absolutely continuous?

Problem 2. Suppose $f$ is a measurable function on $[0, 1]$ such that for every $1 \leq p < \infty$, $f \in L^p[0, 1]$, and suppose there exists a $B$ such that $\|f\|_p \leq B$. Prove that $f \in L^\infty[0, 1]$.

Problem 3. Let $f : [0, 1] \to \mathbb{R}$ be continuous with $f(x) > 0$ for $x \in [0, 1]$. Show that

$$\exp(\int_0^1 \log f) \leq \int_0^1 f.$$ 

Problem 4. Find a function $f \in L^p[0, 1]$ for every $p$, $1 \leq p < \infty$ but $f \notin L^\infty[0, 1]$.

Problem 5. Let $f_n(x) = \sum_{k=1}^n \frac{1}{k^2}$, for $x \geq 1$. Let $f(x) = \lim_{n \to \infty} f_n(x)$. Show that $f$ is measurable, and that

$$\int_1^2 f = \lim_{n \to \infty} \int_1^2 f_n < \infty.$$ 

Problem 6. For $f \in L^1(\mathbb{R})$, prove that

$$\lim_{t \to 0} \int_{\mathbb{R}} |f(x - t) - f(x)| \, dx = 0.$$