Applied Mathematics Qualifying Exam
Fall 2011

Tuesday, August 16th, 9:00 am – 1:00 pm; Room: Carver 305

Instructions

• Write your student ID number on every page that you turn in. Do NOT write your name on any sheet you turn in.
• Turn in solutions to 6 problems. No credit will be given for additional problems.
• Start each problem on a separate sheet of paper, with the problem number clearly stated at the top. SHOW ALL WORK.
• In the event that you believe a problem has a misprint or is improperly stated, explain your concerns to the proctor. Problems should not be interpreted trivially.

Problems

(1) Let \( K : L^2([0, 1]) \to L^2([0, 1]) \) be the (Volterra) integral operator
\[
Kf(x) = \int_0^x f(y) \, dy.
\]
(a) Find the adjoint operator \( K^* \).
(b) Determine the spectral radius of \( K \).

(2) Use the method of characteristics to solve:
\[
\frac{\partial u}{\partial t} + u^2 \frac{\partial u}{\partial x} = 0, \quad u(x, 0) = \sqrt{x}, \quad x > 0.
\]
Where is the solution defined?

(3) Let \( A \) be a bounded self-adjoint linear operator on a Hilbert space \( H \); that is \( A : H \to H \). The Rayleigh quotient, \( R(x) \), of \( A \) is defined by
\[
R(x) = \frac{\langle Ax, x \rangle}{\|x\|^2}, \quad x \in H, \; x \neq 0.
\]
Show that \( R(x) \) is Frechet differentiable for all \( x \neq 0 \), and find \( R'(x) \).
(4) Let \( \Omega = \{ (x, y) \in \mathbb{R}^2 : 0 < x, y < 1 \} \) denote the unit square in the first quadrant of the \( xy \)-plane, and let \( V = \{ u \in H^1(\Omega) : u = 0 \text{ if } x = 0, 1 \} \). Define the functional \( J : V \to V \) by
\[
J(u) = \iint_{\Omega} (|\nabla u|^2 + xu) \, dA.
\]
Find the Euler-Lagrange equations for a minimizer of \( J \). Determine the minimizer.

(5) Let \( f \in C[-1, 1] \) and consider the boundary value problem (BVP)
\[
-u'' = f, \quad -1 < x < 1, \\
u'(-1) + \alpha u(-1) = 0, \quad u'(1) - u(1) = 0,
\]
(a) What are the adjoint boundary conditions for this problem?
(b) Prove that for \( \alpha \neq 1 \) there is a unique solution of the BVP.
(c) Suppose \( \alpha = 1 \). Find solvability conditions on \( f \) so that the problem has solutions, and find a modified Green’s function for the BVP.

(6) Let \( \phi_n(x) = c_n (1 + \cos x)^n \), \( n \) a positive integer) where
\[
c_n = \frac{2^{n-1}}{\pi \binom{2n}{n}} \quad \text{so that} \quad \int_{-\pi}^{\pi} \phi_n(x) \, dx = 1.
\]
It can be shown that for any \( \delta > 0 \)
\[
\lim_{n \to \infty} \int_{\delta \leq |x| \leq \pi} \phi_n(x) \, dx = 0.
\]
Use this fact to show that every continuous \( 2\pi \)-periodic function is the uniform limit of a sequence of trigonometric polynomials.

(7) Define the operator \( K \) on \( \mathcal{H} = L^2(0, \infty) \) by
\[
(Ku)(x) = \int_{0}^{\infty} (e^{-x-2y} + 2e^{-2x-y})u(y) \, dy.
\]
For which \( \lambda \in \mathbb{R} \) is there a unique solution in \( \mathcal{H} \) to \( Ku - \lambda u = f \) for all \( f \) in \( \mathcal{H} \)? For each \( \lambda \) that does not admit a unique solution, under what conditions on \( f \) does there exist (non-unique) solutions?
(8) (a) Let \( f \in L^1(\mathbb{R}^n) \), \( A \) be a nonsingular \( n \times n \) matrix and \( g(x) = f(Ax) \). Express the Fourier transform of \( g \) in terms of the Fourier transform of \( f \).

(b) Use the result in part a) to show that if \( f \) is radially symmetric (i.e. \( f(Ox) = f(x) \) for all \( x \) and any orthogonal matrix \( O \)) then so is its Fourier transform.

(9) (a) Prove that if \( M \) is a subspace of a Hilbert space \( \mathcal{H} \), then 
\[
(M^\perp)^\perp = \overline{M}.
\]

(b) Prove that for any two subspaces \( M_1, M_2 \) of a Hilbert space \( \mathcal{H} \) we have 
\[
(M_1 + M_2)^\perp = M_1^\perp \cap M_2^\perp.
\]

(c) Prove that for any two closed subspaces \( M_1, M_2 \) of a Hilbert space \( \mathcal{H} \) we have 
\[
(M_1 \cap M_2)^\perp = M_1^\perp + M_2^\perp.
\]