Part I
1. Let $G$ be a group and let $M$ be a proper subgroup of $G$. Prove that $M$ is a maximal normal subgroup of $G$ if and only if $G/M$ is simple.

2. Let $G$ be a finite $p$-group acting on a finite set $X$. Define 
$$X^G = \{x \in X : gx = x \text{ for all } g \in G\}.$$ 
(a) Show that $|X^G| \equiv |X| \text{ mod } p$.
(b) Show that the center of $G$ has at least $p$ elements.

3. Let $G$ be a group of order $p^2q$ where $p$ and $q$ are two distinct primes. Show that $G$ has either a normal $p$-Sylow subgroup or a normal $q$-Sylow subgroup.

4. Let $f(x) = x^6 + x^3 + 1$.
(a) Factorize (Decompose) $f(x)$ into irreducible factors in $\mathbb{Z}_3[x]$.
(b) Show that $f(x)$ is irreducible in $\mathbb{Q}[x]$.

5. Find a generator for the ideal $(1 + 13i, 10 + 11i)$ in $\mathbb{Z}[i]$.

[For Part II, see overleaf.]
Part II
Let $\mathbb{R}$ and $\mathbb{C}$ denote the field of real and complex, respectively, and let $\mathbb{F}^{n \times n}$ and $\mathbb{F}^n$ denote the set of all $n \times n$ matrices and that of all $n$-dimensional vectors over the field $\mathbb{F}$, respectively. $A^*$ and $u^*$ denote the (complex) conjugate transposes of $A \in \mathbb{C}^{n \times n}$ and the column vector $u \in \mathbb{C}^n$, respectively. Also $I$ denotes the identity matrix of appropriate size.

6. Give a list of complex matrices such that every $3 \times 3$ complex matrix $A$ satisfying $A^3 = 125I$ must be similar to exactly one of the matrices on your list.

7. a) Let $A \in \mathbb{C}^{n \times n}$ and let $W$ be a nonzero $A$-invariant subspace of $\mathbb{C}^n$ (that is, if $w \in W$ then $Aw \in W$). Prove that $W$ contains an eigenvector of $A$.
   b) Give an example of a matrix $A \in \mathbb{R}^{n \times n}$ and a nonzero $A$-invariant subspace $W$ of $\mathbb{R}^n$ such that $W$ does not contain an eigenvector of $A$.

8. Let $H \in \mathbb{C}^{n \times n}$ be a nonsingular Hermitian matrix. Define $m := \max_{\|u\|=1} u^*H^{-1}u$.
   Express $m$ in terms of the eigenvalues of $H$.

9. Let $\theta \in [0, 2\pi]$ and $A = \frac{1}{2} \begin{pmatrix} 2 - \cos^2 \theta & \cos \theta \sin \theta & 0 \\ \cos \theta \sin \theta & 2 - \sin^2 \theta & 0 \\ 0 & 0 & -1 \end{pmatrix}$.
   (a) Calculate the eigenvalues and eigenvectors of $A$.
   (b) Calculate $\lim_{n \to \infty} A^n$.

10. Let $A$ be a square complex matrix. Prove that $A$ is a normal matrix if and only if $A^*$ commutes with every matrix that commutes with $A$. 