Numerical Analysis Qualify Exam

9am – 1pm, August 20, 2013

Instructions

• Write your ISU ID number on every page that you turn in. DO NOT write your name on any sheet that you turn in.

• Work all 6 problems. Start each problem on a separate sheet of paper and clearly indicate the problem number.

1. Consider approximating the function \( f(x, y) \) with a Lagrange type interpolation polynomial on a triangular element \( K \). We use \( V_1(x_1, y_1), V_2(x_2, y_2), V_3(x_3, y_3) \) to denote the three vertices of the triangle \( K \). The three middle points on the edges are denoted by \( V_{12}, V_{23}, V_{13} \).

(a) Suppose we choose \( V_1, V_2, V_3 \) and \( V_{12}, V_{23}, V_{13} \) as the six interpolation points to construct the second order Lagrange interpolation polynomial \( P_2(x, y) \) to approximate \( f(x, y) \). Verify that \( P_2(x, y) \) can be written out as

\[
P_2(x, y) = \sum_{i=1}^{3} f(V_i) \lambda_i(x, y)(2\lambda_i(x, y) - 1) + 4 \sum_{i<j} f(V_{ij}) \lambda_i(x, y) \lambda_j(x, y),
\]

where \( \lambda_i(x_j, y_j) = \delta_{ij} \) is the linear Lagrange interpolation polynomial for vertex \( V_i \) with coordinates \( (x_i, y_i) \).

(b) Now consider a right triangle \( K^* = \{(x, y) : 0 \leq x \leq h, 0 \leq y \leq h, x + y \leq h\} \). Write out the above Lagrange interpolation polynomial \( P_2(x, y) \) on element \( K^* \).

(c) For any \( (x, y) \in K^* \), what is the order of the error \( f(x, y) - P_2(x, y) = O(h^k) \)? Fully justify your answer.

2. Consider approximating \( \int_a^b f(x) \, dx \) with numerical quadrature rules.

(a) Derive Simpson’s rule (three points closed Newton-Cotes formula).

(b) Prove the error term of Simpson’s rule is \( -\frac{h^5}{90} f^{(4)}(\xi) \), where \( \xi \in (a, b) \) and \( h = (b - a)/2 \). What is the degree of precision of Simpson’s rule?

(c) Write out the composite Simpson’s rule if the domain \( [a, b] \) is divided into total \( N = 2m \) subintervals. Write a pseudo or Matlab code to implement the composite Simpson’s rule.

3. Consider the following numerical method for the initial value problem \( y'(t) = f(y(t)) \):

\[
y^{n+1} = y^n + \frac{k}{2} \left[ f(y^n) + f(y^{n+1}) \right] + \frac{k^2}{12} \left[ f(y^n) f_y(y^n) - f(y^{n+1}) f_y(y^{n+1}) \right],
\]

where \( k \) is the step size.
(a) Is this method explicit or implicit? Explain your answer.
(b) Compute the local truncation error for this numerical method.
(c) Give a definition for the region of absolute stability. Show that the region of absolute stability for
the numerical method given above contains the entire negative real axis.
(d) Is this method convergent? Explain your answer.

4. Let \( f : [a, b] \to \mathbb{R} \) be continuously differentiable on \([a, b]\) and let \( x^* \in (a, b) \) be the unique root of \( f \) on
\([a, b]\). Consider the following iteration scheme:
\[
x_{n+1} = x_n - \frac{f(x_n)}{g_n},
\]
for \( n = 0, 1, 2, \ldots \) and for some \( g_n := g(x_n) \).
(a) Under some appropriate assumptions (which you should mention), prove that the iteration scheme
with \( g_n := f'(x_n) \)
will converge to \( x^* \). What is the rate of convergence?
(b) Under some appropriate assumptions (which you should mention), prove that the iteration scheme
with \( g_n := \frac{f(x_n + f(x_n)) - f(x_n)}{f(x_n)} \)
will converge to \( x^* \). What is the rate of convergence?
(c) Given your results to the previous two questions, which method would be “better” to use in practice?
Explain your answer.

5. Let \( A \in \mathbb{R}^{m \times m} \) be a matrix and \( b, c \in \mathbb{R}^m \) be two vectors. Let \( x_{n+1} = Mx_n + c \) be an iterative method
for solving \( Ax = b \), and \( x = Mx + c \) if and only if \( Ax = b \).
(a) Use the contraction mapping theorem to show that \( \{x_n\} \) converges to \( x \) if \( \|M\| < 1 \).
(b) Write down the Jacobi method for solving \( Ax = b \). Assume that \( A \) is strong row diagonal dominant.
Prove the convergence of the method.
(c) Write down the Gauss-Seidel method for solving \( Ax = b \). Assume that \( A \) is strong row diagonal
dominant. Prove the convergence of the method.

6. Let \( A \in \mathbb{R}^{m \times m} \) be a symmetric nonsingular matrix with real eigenvalues and orthogonal eigenvectors.
Recall the following two schemes for computing the eigenvalues and eigenvectors of \( A \):

**Simultaneous Iteration:**
\[
Q^{(0)} = I
\]
\[
Z = AQ^{(k-1)}, \quad Z = Q^{(k)}R^{(k)}, \quad k \geq 1
\]

**Unshifted QR Iteration:**
\[
A^{(0)} = A
\]
\[
A^{(k-1)} = Q^{(k)}R^{(k)}, \quad A^{(k)} = R^{(k)}Q^{(k)}, \quad k \geq 1
\]
Let \( R^{(k)} = R^{(k)}R^{(k-1)} \cdots R^{(1)} \). Prove by induction that for all \( k \geq 0 \),
(a) \( Q^{(k)} = Q^{(1)} \cdots Q^{(k-1)}Q^{(k)} \);
(b) \( A^{(k)} = Q^{(k)}A Q^{(k)} \);
(c) \( A^k = Q^{(k)}R^{(k)} \).