Numerical Analysis Qualify Exam

9am – 1pm, August 18th, 2014

Instructions

• Write your ISU ID number on every page that you turn in. DO not write your name on any sheet that you turn in.

• Work all 6 problems. Start each problem on a separate sheet of paper and clearly indicate the problem number.

1. Polynomial approximation.
   (a) Find the straight line that best fits the data points
   
   \[(0, 1), \ (1, 2), \ (2, 4), \ (3, 8)\]
   
   in the least squares sense.
   (b) Find the straight line that best fits the function \(f(x) = 2^x\) on the interval \([0, 3]\)
   in the least squares sense, using the inner product
   \[
   \langle f, g \rangle = \int_0^3 f(x)g(x) \, dx.
   \]

2. Let \(\omega : \mathbb{R} \rightarrow \mathbb{R}\) satisfy \(\omega \geq 0\) on \([a, b]\), \(\omega = 0\) only at isolated points in \([a, b]\), and
   \[
   \int_a^b x^p \omega(x) \, dx < \infty
   \]
   for any non-negative integer \(p\). Let
   \[
   \{P_0, P_1, P_2, P_3, \ldots\}
   \]
be a set of orthonormal polynomials where $P_n$ is a polynomial of degree exactly $n$ and
\[
\int_a^b P_m(x) P_n(x) \omega(x) \, dx = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}
\]

Consider the $N$-point quadrature rule
\[
\int_a^b f(x) \omega(x) \, dx \approx I_N := \sum_{k=1}^N w_k f(x_k),
\]
where the quadrature points $x_k$ for $k = 1, 2, 3, \ldots, N$ are the $N$ real distinct roots in $[a, b]$ of $P_N(x)$ and the quadrature weights $w_k$ for $k = 1, 2, 3, \ldots, N$ are
\[
w_k = \int_a^b L_{Nk}(x) \omega(x) \, dx,
\]
where $L_{Nk}(x)$ is the $N^{th}$ degree Lagrange polynomial associated to $x_k$.

(a) Prove that $I_N$ has degree of precision at least $N$.

(b) Prove that $I_N$ has degree of precision at least $2N - 1$.

(c) Prove that $I_N$ has degree of precision $2N - 1$.

3. A two-step method for solving the initial value problem
\[
y'(t) = f(t, y(t)), \\
y(t_0) = y_0
\]
is given by
\[
y_{n+1} = a_1 y_n + a_0 y_{n-1} + h \left[ b_1 f_n + b_0 f_{n-1} \right],
\]
where $f_k = f(t_k, y_k)$, and $h$ is the step size.

(a) Determine $a_0$, $b_0$, $b_1$ in terms of $a_1$, so that the method has order at least 2.

(b) For what values of $a_1$ is the method stable for small $h$?

(c) Can $a_1$ be chosen so that the method has order 3? If yes, can $a_1$ be chosen so that the method is also stable?

4. Gaussian elimination and LU factorization.

(a) Let $A \in \mathbb{R}^{m \times m}$ be a symmetric and positive definite matrix:
\[
A = \begin{bmatrix} a_{11} & a^T \\ a & B \end{bmatrix},
\]
where \( \mathbf{a} \in \mathbb{R}^{m-1} \) and \( \mathbf{B} \in \mathbb{R}^{(m-1) \times (m-1)} \). After one step of Gaussian elimination, \( \mathbf{A} \) is converted to
\[
\mathbf{A} = \begin{bmatrix} a_{11} & \mathbf{a}^T \\ 0 & \tilde{\mathbf{B}} \end{bmatrix},
\]
where \( \tilde{\mathbf{B}} = \mathbf{B} - (\mathbf{a} \mathbf{a}^T) / a_{11} \in \mathbb{R}^{(m-1) \times (m-1)} \). Prove that \( \tilde{\mathbf{B}} \) is symmetric and positive definite.

(b) Use the above result to show that if \( \mathbf{A} \in \mathbb{R}^{m \times m} \) is symmetric and positive definite matrix, then there exists a unique unit lower triangular matrix \( \mathbf{L} \in \mathbb{R}^{m \times m} \) and a unique upper triangular matrix \( \mathbf{U} \in \mathbb{R}^{m \times m} \) such that
\[
\mathbf{A} = \mathbf{LU}.
\]

(c) If \( \mathbf{A} \in \mathbb{R}^{m \times m} \) is nonsingular (not necessarily symmetric and positive definite), then there exists a permutation matrix \( \mathbf{P} \in \mathbb{R}^{m \times m} \), a unit lower triangular matrix \( \mathbf{L} \in \mathbb{R}^{m \times m} \), and an upper triangular matrix \( \mathbf{U} \in \mathbb{R}^{m \times m} \) such that
\[
\mathbf{PA} = \mathbf{LU}.
\]

5. Consider matrix
\[
\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.
\]

(a) Using any method you like, determine the reduced and full QR factorizations \( \mathbf{A} = \hat{\mathbf{Q}} \hat{\mathbf{R}} \) and \( \mathbf{A} = \mathbf{QR} \).

(b) Using the QR factorization above, solve the linear least square problem
\[
\min_x \| \mathbf{A} \mathbf{x} - \mathbf{b} \|_2^2 \quad \text{with} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.
\]

6. Rayleigh quotient iteration for the eigenvalue/eigenvector pair of a symmetric matrix \( \mathbf{A} \in \mathbb{R}^{m \times m} \).

(a) Show that if \( \mathbf{A} \in \mathbb{R}^{m \times m} \) is symmetric, then the eigenvalues of \( \mathbf{A} \) are all real.

(b) Write a MATLAB function to implement the Rayleigh quotient iteration scheme starting from an initial unit vector:
\[
\mathbf{v}^{(0)} \in \mathbb{R}^m \quad \text{such that} \quad \| \mathbf{v}^{(0)} \|_2 = 1.
\]
This function must have stopping criteria to guarantee that the function terminates in a finite number of steps.
(c) Prove that the Rayleigh quotient iteration scheme convergence cubically:

\[ \| \mathbf{v}^{(k+1)} - q_J \|_2 = \mathcal{O} \left( \| \mathbf{v}^{(k)} - q_J \|_2^3 \right), \]

\[ | \lambda^{(k+1)} - \lambda_J | = \mathcal{O} \left( | \lambda^{(k)} - \lambda_J |^3 \right). \]

for some eigenvalue/eigenvector pair \( \lambda_J \) and \( q_J \) of \( A \in \mathbb{R}^{m \times m} \).