Directions: Write the problem number and the last four digits of your student ID number at the top of each page. Do not write your name on your paper. Write each solution on a separate page. Submit solutions in the same order as the questions. All the steps must be justified by computation or explanation. Greater weight will be given to one whole (correct) solution than to two error-free but incomplete solutions. To demonstrate adequate breadth, significant work must be done from each of Part I and Part II.

Part I

(1) Find the least positive integer $n$ such that $\alpha^n = 1$ for all $\alpha \in S_7$.

(2) Let $G$ be a group of order 105. Prove that if a Sylow 3-subgroup of $G$ is normal then $G$ is abelian.

(3) Suppose $G$ is a finite abelian group, and there are exactly $p - 1$ elements of order $p$ for each prime divisor $p$ of $|G|$. Prove that $G$ is cyclic.

(4) Let $R$ be a commutative ring with identity and suppose that every proper ideal of $R$ is prime. Show that $R$ is a field. (Hint: let $a \neq 0$ and consider $I = (a^2)$.)

(5) Let $R$ be a PID, $F$ the fraction field of $R$ and suppose that $S$ is a ring with $R \leq S \leq F$. Prove or disprove the statement: If $\alpha \in S$ then $\alpha = \frac{a}{b}$ with $a, b \in R$ and $\frac{1}{b} \in S$. 
Part II

Let \( \mathbb{R} \) and \( \mathbb{C} \) denote the field of real and complex respectively, and let \( M_n(F) \) denote the set of all \( n \times n \) matrices over a field \( F \).

(6) Prove that a real symmetric matrix \( A \) is positive definite if, and only if, there exists a real invertible matrix \( S \) such that \( A = SS^T \).

(7) Let \( A, B \in M_n(\mathbb{C}) \) such that \( A \) has distinct eigenvalues and \( AB = BA \). Prove or disprove the statement: \( A \) and \( B \) are simultaneously diagonalizable.

(8) Consider matrices over \( \mathbb{R} \), and answer the following questions.
   (a) Let \( A \) be a \( 8 \times 8 \) nilpotent matrix of index 3 (i.e., \( A^3 = 0 \) but \( A^2 \neq 0 \)), and let \( \text{rank}(A) = 5 \) and \( \text{rank}(A^2) = 2 \). Find the nilpotent matrix \( N \) in Jordan canonical form which is similar to \( A \).
   (b) Find all possible rational canonical forms of \( 7 \times 7 \) matrices with minimal polynomial \( m(t) = (t + 1)^3 \).

(9) Suppose \( A \in M_n(\mathbb{R}) \) and \( x \in \mathbb{R}^n \) such that \( A^k x = 0 \) and \( A^{k-1} x \neq 0 \) for some positive integer \( k \). Prove that \( S = \{ x, Ax, \ldots, A^{k-1}x \} \) spans an \( A \)-invariant subspace of dimension \( k \) in \( \mathbb{R}^n \).

(10) Let \( X \) be a finite set of \( N \) invertible matrices in \( M_n(\mathbb{C}) \) such that \( X \) is closed under matrix multiplication. For any \( A \in X \), prove that \( A \) is diagonalizable and the eigenvalues of \( A \) are \( N \)-th roots of unity.