Applied Mathematics Qualifying Exam
Spring 2012

Friday, January 6th, 9:00 am – 1:00 pm; Room: Carver 305

Instructions

• Write your student ID number on every page that you turn in. Do NOT write your name on any page you turn in.
• Turn in solutions to 6 problems. No credit will be given for additional problems.
• Start each problem on a separate sheet of paper, with the problem number clearly stated at the top. SHOW ALL WORK.
• In the event that you believe a problem has a misprint or is improperly stated, explain your concerns to the proctor. Problems should not be interpreted trivially.

Problems

(1) Let $K : L^2(0, \pi) \rightarrow L^2(0, \pi)$ be the integral operator defined by

$$Ku(x) = \int_0^\pi \sin^2(x-y)u(y)\,dy.$$ 

(a) Determine the eigenvalues of $K$ and describe the eigenfunctions associated with them.

(b) Find the solution of the nonhomogeneous integral equation $Ku - u = f$, where $f(x) = x$.

(2) Let $G$ be a multiplication operator on $L^2(\mathbb{R})$ defined by

$$(Gf)(x) = g(x)f(x), \quad x \in \mathbb{R},$$

where $g(x)$ is a continuous bounded function on $\mathbb{R}$.

(a) Prove that $G : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is a bounded linear operator.

(b) Let $F = \{g(x) : x \in \mathbb{R}\}$ be the closure of the range of $g$ and $F^c$ its complement. Show that if $\lambda \in F^c$ then $\lambda$ is in the resolvent set of $G$.

(c) Assuming that $g$ is a strictly monotone function, show that the spectrum $\sigma(G) = F$. Hint: You may want to use the fact that $\sigma(G)$ is closed.

(3) Use the method of characteristics to solve:

$$yu_x - xy_y + u_z = 1, \quad u(x,y,0) = x + y.$$
(4) Assuming that $\lambda \in (-1, 1)$, show there exists a unique solution $u \in C[0,1]$ to the non-local boundary value problem
\[-u''(x) + \lambda \int_0^1 \sin u(x) \, dx = h(x), \quad u(0) = 0, \ u'(1) = 0,
\]
for every $h \in C[0,1]$.

(5) Find the Green’s function for the boundary value problem
\[x^2u'' - 2xu' + 2u = f, \quad 1 < x < 2, \quad u(1) = u(2) = 0.\]
Hint: Solutions of a homogeneous differential equation of Cauchy-Euler type can be found in the form $u = x^p$.

(6) Find the minimizer of the functional
\[J(u) = \int_0^1 (u'^2 + u^2 - 2u) \, dx\]
over the class of admissible functions $A = \{u \in C^1[0,1] : u(0) = 0\}$ by solving the Euler-Lagrange equations associated with the minimization problem.

(7) Let $X$ be a Banach space and $B(X)$ denote the Banach space of bounded operators on $X$. Show that the set $\{A \in B(X) : A^{-1} \in B(X)\}$ of operators that have bounded inverses is an open set in $B(X)$.

(8) Find the solution of Laplace’s equation on the half disk
\[
\begin{align*}
\Delta u &= u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, \quad 0 < r < 1, \ 0 < \theta < \pi, \\
 u(r, 0) &= u(r, \pi) = 0, \ 0 < r < 1, \quad u(1, \theta) = f(\theta).
\end{align*}
\]
Show that if the boundary data $f$ satisfies the symmetry condition $f(\theta) = f(\pi - \theta), \ \theta \in (0, \pi/2)$, then so does the solution of the boundary value problem.

(9) Let $f \in C^\infty(\mathbb{R})$ be periodic, with period $2\pi$ and have mean value zero, so that $\int_0^{2\pi} f(x) \, dx = 0$. Show that for any positive integer $p$ the sequential limit
\[
\lim_{n \to \infty} n^p f(nx) = 0
\]
is valid in the sense of distributions on $\mathbb{R}$. 