Start each solution on a separate page. At the top of each page, write the problem number and the last four digits of your student ID number. Do not write your name on your paper. Submit solutions in the same order as the questions.

Each step must be justified by computation or explanation. Greater weight will be given to one whole (correct) solution than to two error-free but incomplete solutions. To demonstrate adequate breadth, significant work must be done from each of Part I and Part II.

While every effort is made to proofread the exam, errors may occur. If you believe that a problem has been stated incorrectly, check with the proctor and indicate your interpretation in the solution. Do not interpret the problem in a way that it becomes trivial.

Part I

1. Let $G$ be a finite group acting on a finite set $X$. For $g \in G$ and $x \in X$, define
   
   $X^g := \{x \in X : gx = x\}$ and $G_x := \{g \in G : gx = x\}$.

   Show that $\sum_{g \in G} |X^g| = \sum_{x \in X} |G_x|$.

2. Prove that the symmetric group $S_5$ has no subgroup of order 15.

3. Prove that if $G$ is a group of order 385 then its center $Z(G)$ contains a Sylow 7-subgroup and $G$ has a normal Sylow 11-subgroup.
   (You may assume that if $p$ is an odd prime, then $\text{Aut}(\mathbb{Z}/p\mathbb{Z}) \simeq \mathbb{Z}/(p-1)\mathbb{Z}$.)

4. Let $R$ be a commutative ring. Given $r \in R$, define $\text{ann}(r) := \{x \in R : xr = 0\}$.
   (a) Verify that $\text{ann}(r)$ is an ideal in $R$.
   (b) Let $\mathcal{C} := \{\text{ann}(r) : r \in R, r \neq 0\}$. Let $Z$ be a maximal element of $\mathcal{C}$ with respect to inclusion. Show that $Z$ is a prime ideal of $R$.

5. Consider the Gaussian integers $\mathbb{Z}[i]$ and its norm to answer the following questions.
   (a) For a nonzero principal ideal $(a + bi)$ generated by $a + bi$ in $\mathbb{Z}[i]$, show that $\mathbb{Z}[i]/(a + bi)$ is a finite ring.
   (b) Find the order and characteristic of the field $\mathbb{Z}[i]/(1 + i)$. 
Let $\mathbb{R}$ and $\mathbb{C}$ denote the field of real and complex numbers, respectively, and let $\mathbb{F}^{n \times n}$ and $\mathbb{F}^n$ denote the set of all $n \times n$ matrices and that of all $n$-dimensional vectors over the field $\mathbb{F}$, respectively. In problems 7 – 9, $A^*$ and $u^*$ denote the (complex) conjugate transposes of $A \in \mathbb{C}^{n \times n}$ and the column vector $u \in \mathbb{C}^n$, respectively.

6. Let $A = \begin{bmatrix} 0 & -1 & -1 & -1 \\ 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$. Find the Jordan canonical form for $A$.

7. Let $V$ be the complex inner product space $(\mathbb{C}^2, \langle \cdot, \cdot \rangle_H)$ defined by $H = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $\langle x, y \rangle_H = y^* H x$. For $B \in \mathbb{C}^{2 \times 2}$, let $T_B : V \rightarrow V$ be the linear transformation defined by $T_B(v) = Bv$. Find the adjoint $T_B^H$ of $T_B$ in this inner product space for the matrix $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

8. Let $A$ and $B$ be $n \times n$ normal matrices. Show that there exists a unitary matrix $U$ such that $U^* A U$ and $U^* B U$ are both diagonal if and only if $AB = BA$.

9. Suppose $H, P \in \mathbb{C}^{n \times n}$ with $H$ Hermitian and $P$ positive definite.
   (a) Prove the eigenvalues of $HP$ are all real.
   (b) Denote the (real) eigenvalues of $HP$ by $\lambda_1(HP) \geq \lambda_2(HP) \geq \cdots \geq \lambda_n(HP)$. Prove that
   $$\lambda_k(HP) = \max_{\dim W = k} \min_{0 \neq w \in W} \frac{w^* H w}{w^* P^{-1} w}.$$ 

10. Let $V$ be a finite dimensional real vector space, and let $V^*$ be the dual space of $V$, that is, the (vector) space of all linear transformations from $V$ to $\mathbb{R}$. Given a subset $S^*$ of $V^*$, define $\text{Ann}(S^*) := \{v \in V : f(v) = 0 \text{ for all } f \in S^*\}$. Let $W_1^*$ and $W_2^*$ be subspaces of $V^*$. Show that
    $$\text{Ann}(W_1^* \cap W_2^*) = \text{Ann}(W_1^*) + \text{Ann}(W_2^*).$$
    (Hint: You may use the fact: $\dim(\text{Ann}(W^*)) + \dim(W^*) = \dim(V)$ without proof).