Applied mathematics qualifying examination

Spring 2014
Friday, January 10, 9:00am-1:00pm
Room 305 Carver

Instructions:

• Write your student ID number on every page that you turn in. Do NOT write your name on any page you turn in.

• Turn in solutions to 6 problems. No credit will be given for additional problems.

• Start each problem on a separate sheet of paper, with the problem number clearly stated at the top. SHOW ALL WORK.

• In the event that you believe a problem has a misprint or is improperly stated, ask the proctor for a clarification. Problems are not to be interpreted trivially.

1. Define the integral operator

   \[ Tu(x) = \int_{-1}^{1} h(x,y)u(y) \, dy \]

   where

   \[ h(x,y) = \begin{cases} 
   1 - (x-y) & \text{if } x > y \\
   1 - (y-x) & \text{if } y > x 
   \end{cases} \]

   on \( L^2(-1,1) \). Find the spectrum of \( T \). Be sure to carefully distinguish the different parts of the spectrum. What is \( ||T|| \)? (Suggestion: Replace the integral equation \( Tu = \lambda u \) by an equivalent boundary value problem.)

2. Find the solution of the Cauchy problem

   \[ 2xu_x + (x+y)u_y = 2u, \quad u(x,-x) = \sqrt{x}, \quad x > 0. \]

   and state where it is valid.

3. Let \( f \in L^2(\mathbb{R}) \) and set

   \[ g(x) = \int_{x}^{x+1} f(t) \, dt \]

   (a) Find a relationship between the Fourier transform of \( f \) and the Fourier transform of \( g \).

   (b) Show that \( g \in H^1(\mathbb{R}) \).

   (c) Show that \( \hat{g} \in L^1(\mathbb{R}) \) and hence \( \lim_{|x| \to \infty} g(x) = 0 \).

4. Show that \( L^2(0,\infty) \) is separable. (Suggestion: consider the functions \( H(x-k)x^m \) for nonnegative integers \( k, m \), and use the Weierstrass approximation theorem.)
5. Let

\[ Lu = -\frac{d}{dx} \left( x \frac{du}{dx} \right) \]

on the domain

\[ D(L) = \{ u \in H^2(1, 2) : u(1) = u(2) = 0 \} \]

(a) Find the Green's function for the boundary value problem \( Lu = f \).

(b) State and prove a result about the continuous dependence of the solution on \( f \).

(c) Using an appropriate Rayleigh quotient, show that \( \lambda > \pi^2 \) if \( \lambda \) is an eigenvalue of \( L \).

6. Let \( T \) be a bounded linear operator on a Hilbert space \( H \) with \( \langle Tx, x \rangle \geq 0 \) for all \( x \in H \).

(a) If \( x + Tx = y \), show that \( ||x|| \leq ||y|| \).

(b) Show that \( N(T) = R(T^\perp) \). (Suggestion: Use \( \langle T(x + y), x + y \rangle \geq 0 \) for various choices of \( x, y \).)

7. Consider the sequence of functions \( f_k(x) = k^\alpha H(x)xe^{-kx} \), where \( H(x) \) denotes the Heaviside function and \( \alpha \geq 0 \). Determine the set of \( \alpha \)'s for which \( \lim_{k \to \infty} f_k(x) \) exists in the sense of distributions, and find this limit whenever it exists.

8. The ODE \( u'' - xu = 0 \) is known as Airy's equation, and solutions of it are called Airy functions.

(a) If \( u \) is an Airy function which is also a tempered distribution, use the Fourier transform to find a first order ODE for \( \hat{u}(y) \).

(b) Find the general solution of the ODE for \( \hat{u} \).

(c) Obtain the formal solution formula

\[ u(x) = C \int_{-\infty}^{\infty} e^{ixy+iy^3/3} dy \]

(d) Explain why this formula is not meaningful as an ordinary integral, and how it can be properly interpreted.

(e) Is this the general solution of the Airy equation?

9. Let

\[ J(u) = \int_{-1}^{0} 2(u'(x))^2 \, dx + \int_{0}^{1} [(u'(x))^2 + u(x)] \, dx \]

on \( H^1_0(-1, 1) \). Find the Euler-Lagrange condition for minimization of the functional \( J \). Determine the function that minimizes \( J \).