ANALYSIS QUALIFYING EXAMINATION  
Spring 2015  
January 8, 2015

- Write your university ID number on every page you turn in. Do NOT write your name on any page you turn in.

- Your score will be based on a total of six problems. If you turn in more than six problems, all the problems will be graded and your scores will be based on the six lowest scoring problems. If you work fewer than six problems, you will receive a score of 0 for the missing problems.

- Work each problem on a separate sheet of paper, and clearly indicate the problem on each page.

- To pass, you must work on at least two problems from Part I and three problems from Part II, and receive substantial credits from both parts. In the grading, one completely correct solution will be counted as more than two “half correct” solutions.

- Every effort is made to proofread the exam, but misprints may occur. If you believe that a problem has been stated incorrectly, please check it with the proctor and indicate your interpretation in the solution. Do not interpret the problem in a way that it becomes trivial.
1. Let $a > 0$. Compute the value of the integral $\int_0^\infty \frac{x \sin(x)}{x^2 + a^2} \, dx$.

2. Let $f$ be an entire function with $|f(z)| = 1$ when $|z| = 1$. Prove that $f$ is of the form $f(z) = cz^N$ where $c$ is a complex constant with modulus one and $N$ is a non-negative integer.

3. Let $f$ be an analytic function on the open unit disk $B(0, 1)$ so that it satisfies $|f(z)| \leq \frac{1}{1 - |z|}$ for all $z$. Show that
   (i) $|f'(0)| \leq 4$ and
   (ii) $|f^{(n)}(0)| \leq (n + 1)!(1 + \frac{1}{n})^n$.

4. Let $D = \{z : |z| < 1\}$ be the unit disk in the complex plane. When $z \neq 0$, let $\arg(z)$ be the number in $[-\pi, \pi)$ such that $z = |z|e^{i\arg(z)}$. Prove that there is an analytic function $f : D \to \mathbb{C}$ such that $\lim_{z \to \zeta} \text{Im}(f(z)) = \arg(\zeta)$ for every $\zeta \in \partial D \setminus \{-1\}$. Is the function $f$ unique?

5. Let $f$ be an entire function on the complex plane and $g$ be an analytic function on the punctured unit disk $U = \{z : 0 < |z| < 1\}$. If $f(g(z))$ has a pole of order one at the origin, show that $f$ is a linear function and $g$ has a pole of order one at the origin.

6. Let $(a_n)$ and $(c_n)$ be two sequences of complex numbers. If $\sum_{n=0}^\infty a_n e^{c_n z}$ converges absolutely at the points $z_1, ..., z_M$, show that it converges absolutely at every point in the smallest convex polygon that contains the points $z_1, ..., z_M$. 
In the following problems, $\mu$ denotes the Lebesgue measure on $\mathbb{R}$.

1. Let $(E_n)$ be a sequence of Lebesgue measurable subsets of $\mathbb{R}$ with

$$\sum_{n=1}^{\infty} \mu(E_n) < \infty. \tag{1}$$

a) Show that

$$\mu(\limsup_{n \to \infty} E_n) = 0, \tag{2}$$

where $\limsup_{n \to \infty} E_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_n$.

b) Is the conclusion (2) still true if (1) is replaced by $\sum_{n=1}^{\infty} (\mu(E_n))^2 < \infty$?

2. \(a\) Let $E \subseteq [0,1]$ be a Borel set. Show that there exist a finite number of sub-intervals $I_1, I_2, \ldots, I_n$ of $[0,1]$ with $A = \bigcup_{j=1}^{n} I_j$ so that $\mu(A \Delta E) < \epsilon$. Here $A \Delta E = (A \setminus E) \cup (E \setminus A)$.

b) Show that there does not exist a Borel set $E \subseteq [0,1]$ so that $\mu(E \cap I) = \frac{1}{2} \mu(I)$ for every sub-interval $I \subseteq [0,1]$.

3. Let $1 < p < \infty$ and $q$ be conjugates so that $\frac{1}{p} + \frac{1}{q} = 1$. Fix a real-valued function $g$ in $L^q(\mathbb{R})$ and define the functional $F : L^p(\mathbb{R}) \to \mathbb{R}$ by $F(f) = \int_{\mathbb{R}} f(x)g(x)d\mu(x)$. Show that

i) $F$ is a linear functional defined on $L^p(\mathbb{R})$.

ii) Recall the norm $||F|| = \sup\{|F(f)| : ||f||_p = 1\}$. Show that $||F|| = ||g||_q$ where $||g||_q$ is the norm in $L^q(\mathbb{R})$.

4. Let $\{f_n\}$ be a sequence in $L^2(\mathbb{R})$ such that $\sum_{n=1}^{\infty} ||f_n||_2 < \infty$ and such that $\sum_{n=1}^{\infty} f_n(x) = 0$ for almost all $x$ in $\mathbb{R}$. Prove that for each $g$ in $L^2(\mathbb{R})$,

$$\sum_{n=1}^{\infty} \int_{\mathbb{R}} f_n(x)g(x)d\mu(x)$$

exists and is equal to zero.
5. a) Let $f : [0, 1] \rightarrow \mathbb{R}$ be an absolutely continuous function. Prove the following statement:

The function $f$ is Lipschitz continuous on $[0, 1]$ if and only if $\sup_{[0,1]} |f'(x)| < \infty$.

[ A function $f$ is Lipschitz continuous on $[0, 1]$ if there exists a constant $C > 0$ so that $|f(x) - f(y)| \leq C|x - y|$ for all $x, y$ in $[0, 1]$. ]

b) Is the above statement remains valid if it is assumed that the function $f$ is bounded variation on $[0, 1]$, but not necessarily absolutely continuous on $[0, 1]$?

6. Let $0 < a < b$. Show that the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$f(x, y) = \begin{cases} e^{-xy} & (x, y) \in [0, \infty) \times [a, b], \\ 0 & \text{otherwise} \end{cases}$$

is integrable with respect to the Lebesgue measure on $\mathbb{R}^2$. Compute the integral

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx.$$