Instructions:

- **Write your student ID number on every page that you turn in.** Do **NOT** write your name on any page you turn in.
- **Turn in solutions to 6 problems.** No credit will be given for additional problems.
- **Start each problem on a separate sheet of paper, with the problem number clearly stated at the top.** **SHOW ALL WORK.**
- **In the event that you believe a problem has a misprint or is improperly stated, ask the proctor for a clarification.** Problems are not to be interpreted trivially.

1. Let $A$ denote the operator defined on $L^2(-1,1)$ by
   \[(Au)(x) = x^3 \int_{-1}^{1} yu(y) \, dy - x \int_{-1}^{1} y^2 u(y) \, dy\]
   Determine $N(A), N(A^*), R(A)$ and $R(A^*)$. What points are in the spectrum of $A$?

2. Let $\lambda_n > 0$, $\lambda_n \to +\infty$ and set
   
   \[f_n(x) = \sin(\lambda_n x) \quad \text{and} \quad g_n(x) = \frac{\sin(\lambda_n x)}{\pi x}.\]
   (a) Show that $f_n \to 0$ in $D'(\mathbb{R})$ as $n \to \infty$.
   (b) Show that $g_n \to \delta$ in $D'(\mathbb{R})$ as $n \to \infty$.
   (You may use the fact that the improper integral $\int_{-\infty}^{\infty} \frac{\sin(x)}{x} \, dx = \pi$.)

3. Let $T$ be a bounded linear operator on a Hilbert space.
   (a) Show that $T$ is compact if and only if $T^*$ is compact.
   (b) If $T$ commutes with $T^*$ show that $R(T) = R(T^*)$.

4. Let $V := \{ v \in H^1(-1,1) : v(-1) = v(1) \text{ and } \int_{-1}^{1} v(x) \, dx = 0 \}$. Determine
   \[\inf_{\{ v \in V : v \neq 0 \}} \frac{\int_{-1}^{1} |v'|^2 \, dx}{\int_{-1}^{1} |v|^2 \, dx}.\]

5. Use the method of eigenfunction expansion to find all solutions to
   \[-\Delta u = f, \quad \text{in } \Omega; \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma,\]
   where $\Omega$ is the unit square $(0,1) \times (0,1) \subset \mathbb{R}^2$, $\Gamma$ is the boundary of $\Omega$ and $\frac{\partial u}{\partial n}$ denotes the outward normal derivative. You should specify what the eigenfunctions are, and also indicate any necessary solvability conditions.
6. Let $H^1(0,1)$ be the usual Sobolev space $\{ f \in L^2(0,1) : f' \in L^2(0,1) \}$ equipped with the inner product
\[
\langle f, g \rangle = \int_0^1 f(x)g(x) + f'(x)g'(x) \, dx.
\]
(a) Show that $H^1(0,1) \subset C([0,1])$.

(b) Prove that for any $z \in (0,1)$ there exists a unique $g_z \in H^1(0,1)$ for which
\[
f(z) = \langle f, g_z \rangle \quad \forall f \in H^1(0,1).
\]

7. Define $Lu = -u'' + u$ with
\[
D(L) = \{ u \in H^2(0,1) : u(0) = u(\pi) = 0 \}.
\]
Use a fixed point argument in $L^2(0,\pi)$ to show that for all $f \in L^2(0,\pi)$,
\[
Lu + |u| \sin x = f
\]
has a unique solution in $D(L)$.

8. A sequence in a Banach space is said to be closed if the set of finite linear combinations of elements of the sequence is dense. Let $f_n(x) = x^n$ for $n = 0,1,\ldots$.

(a) Prove that the sequence $\{f_n\}_{n=0}^\infty$ is closed in $L^p(0,1)$ for any $1 \leq p < \infty$. (You may quote the Weierstrass approximation theorem, as long as you give a clear statement of it.)

(b) Is the closedness property true if $p = \infty$? Explain your answer.

(c) Find the best $L^2(0,1)$ approximation to $f_2$ in the span of $\{f_0, f_1\}$.

9. (a) Prove that for any bounded sequence of complex numbers $\{a_n\}_{n=1}^\infty$ the cosine series
\[
\sum_{n=1}^\infty a_n \cos nx
\]
is convergent in the sense of distributions on $\mathbb{R}$.

(b) What else can be said if $\{a_n\} \in \ell^2$?

(c) What else can be said if $\{a_n\} \in \ell^1$?