

Levi Decomposition of Lie Algebras; Algorithms for its Computation

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ABSTRACT

In this creative component, we review the basic concepts of Lie algebras. The main focus is the Levi decomposition which says that a finite dimensional Lie algebra \mathcal{L} can be written as the direct sum of a semisimple Lie algebra and the solvable radical of \mathcal{L} . We present practical algorithms to compute the semisimple and solvable parts starting from a basis of a given finite dimensional Lie algebra, then we discuss some numerical examples.

CHAPTER 1. Introductory Definitions and Examples

In this chapter we give the basic definitions and examples that we will use in the rest of this work. We start with the definition of an algebra.

Definition 1.0.1 Let \mathcal{F} be a field and suppose that \mathcal{A} is a finite dimensional vector space over \mathcal{F} . A bilinear map $f : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ is called a *multiplication* on \mathcal{A} , \mathcal{A} is said to be an *algebra* with respect to f . A *subalgebra* \mathcal{B} of \mathcal{A} is a subspace of \mathcal{A} that is closed under the multiplication. An algebra does not necessarily need to be associative.

Definition 1.0.2 A *right* (left, two sided) *ideal* of \mathcal{A} is a subspace \mathcal{I} such that $xa \in \mathcal{I}$ ($ax \in \mathcal{I}, ax$ and $xa \in \mathcal{I}$) for all $a \in \mathcal{A}$ and for all $x \in \mathcal{I}$. \mathcal{A} is called *simple* if it has no ideals except $\{0\}$ and itself.

An algebra \mathcal{L} is called a *Lie algebra* if its multiplication $[\cdot, \cdot] : \mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{L}$ satisfies,

$$L_1 : [x, x] = 0$$

$$L_2 : [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

for all $x, y, z \in \mathcal{L}$. The bilinear map $(x, y) \longmapsto [x, y]$ is called the commutator or Lie bracket and $[x, y]$ is said to be the commutator of x and y . The property L_2 is called *Jacobi identity*. Let \mathcal{L} be a Lie algebra, and let $x, y \in \mathcal{L}$. Then,

$$[x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] = 0$$

which implies that

$$[x, y] = -[y, x] \quad (L'_1)$$

The property L'_1 is called *anti-commutativity* and if the characteristic of \mathcal{F} is not 2, then L'_1 implies that

$$[x, x] = -[x, x] \Rightarrow 2[x, x] = 0 \Rightarrow [x, x] = 0$$

In conclusion, if the characteristic of \mathcal{F} is not equal to 2, then the properties L_1 and L'_1 are equivalent. It immediately follows from the anti-commutativity property that a subalgebra \mathcal{I} is a two sided ideal if it is either a left ideal or a right ideal of \mathcal{L} .

In the rest of this work, we will deal with a finite dimensional Lie algebra over a field of characteristic 0.

Example 1.0.3 Let \mathcal{V} be a finite dimensional vector space over \mathcal{F} . We define $[x, y] = 0$ for all $x, y \in \mathcal{V}$, then obviously L_1 and L_2 hold, thus \mathcal{V} becomes a Lie algebra. In particular, all of the elements of \mathcal{V} commute. A Lie algebra where all the elements commute is called an *Abelian Lie algebra*.

Many standard facts of group theory and ring theory have direct analogous in Lie algebras. Let \mathcal{A} be an algebra and let \mathcal{I} be an ideal of \mathcal{A} , then the quotient vector space \mathcal{A}/\mathcal{I} becomes an algebra with respect to the well-defined multiplication

$$\bar{x}\bar{y} = \overline{xy}$$

where \bar{x} and \bar{y} denote the coset of x and $y \in \mathcal{A}$ respectively in the quotient vector space \mathcal{A}/\mathcal{I} . In fact, if \mathcal{A} is a Lie algebra, the quotient space \mathcal{A}/\mathcal{I} is also a Lie algebra whose multiplication is inherited from \mathcal{A} .

Definition 1.0.4 Let \mathcal{A} and \mathcal{B} be two algebras. A linear map $f : \mathcal{A} \longrightarrow \mathcal{B}$ is called a *homomorphism* of algebras if $f(xy) = f(x)f(y)$ for all $x, y \in \mathcal{A}$. Let \mathcal{L}_1 and \mathcal{L}_2 be two Lie algebras with commutators $[\cdot, \cdot]_{\mathcal{L}_1}$ and $[\cdot, \cdot]_{\mathcal{L}_2}$ respectively. A linear map $f : \mathcal{L}_1 \longrightarrow \mathcal{L}_2$ is called a *homomorphism of Lie algebras* if it satisfies

$$f([x, y]_{\mathcal{L}_1}) = [f(x), f(y)]_{\mathcal{L}_2}$$

i.e. f preserves the Lie bracket, for all $x, y \in \mathcal{L}_1$. A bijective homomorphism is called an *isomorphism*. If such a map exists, then \mathcal{L}_1 and \mathcal{L}_2 are said to be *isomorphic*.

An isomorphism from a Lie algebra \mathcal{L} onto itself is called an *automorphism* of \mathcal{L} . The set of all automorphism of \mathcal{L} forms a group denoted by $\text{Aut}(\mathcal{L})$ called the *automorphism group* of \mathcal{L} .

Let \mathcal{L}_1 and \mathcal{L}_2 be Lie algebras with a homomorphism $\varphi : \mathcal{L}_1 \longrightarrow \mathcal{L}_2$. The kernel of φ , $\text{Ker } \varphi$, is defined by

$$\text{Ker } \varphi = \{x \in \mathcal{L}_1 : \varphi(x) = 0\}$$

Note that $\text{Ker } \varphi$ is closed under $[\cdot, \cdot]$, i.e. if $x_1, x_2 \in \text{Ker } \varphi$, then $[x_1, x_2] \in \text{Ker } \varphi$. Let $x \in \text{Ker } \varphi$ and $y \in \mathcal{L}_1$. Since φ is an homomorphism and $\varphi(x) = 0$,

$$\varphi([x, y]) = [\varphi(x), \varphi(y)] = [0, \varphi(y)] = 0$$

which means that $[x, y] \in \text{Ker } \varphi$. Thus, $\text{Ker } \varphi$ is an ideal of \mathcal{L}_1 . The image of φ , $\text{Im } \varphi$, is defined by

$$\text{Im } \varphi = \{\varphi(x) : x \in \mathcal{L}_1\}$$

Let $u, v \in \text{Im } \varphi$, then there exist $x, y \in \mathcal{L}_1$ such that $\varphi(x) = u$ and $\varphi(y) = v$. Since φ is a homomorphism

$$[u, v] = [\varphi(x), \varphi(y)] = \varphi([x, y])$$

so $[u, v] \in \text{Im } \varphi$, and therefore $\text{Im } \varphi$ is a subalgebra of \mathcal{L}_2 .

Theorem 1 *Let \mathcal{L}_1 and \mathcal{L}_2 be two Lie algebras with a homomorphism $\varphi : \mathcal{L}_1 \longrightarrow \mathcal{L}_2$, then*

$$\mathcal{L}_1 / \text{Ker } \varphi \cong \text{Im } \varphi$$

Proof. We proved that $\text{Ker } \varphi$ is an ideal of \mathcal{L}_1 and $\text{Im } \varphi$ is a subalgebra of \mathcal{L}_2 . Now we define a map

$$\psi : \mathcal{L}_1 / \text{Ker } \varphi \longrightarrow \text{Im } \varphi; \quad x + \text{Ker } \varphi \longmapsto \varphi(x)$$

This map is well defined, one-to-one and onto. Since φ is a homomorphism it follows that ψ is a homomorphism, thus $\mathcal{L}_1/\text{Ker } \varphi \cong \text{Im } \varphi$ \square

Theorem 2 *Let \mathcal{L} be a Lie algebra and let \mathcal{I}, \mathcal{J} be ideals of \mathcal{L} , then*

i- If $\mathcal{I} \subset \mathcal{J}$ then \mathcal{J}/\mathcal{I} is an ideal of \mathcal{L}/\mathcal{I} and $(\mathcal{L}/\mathcal{I})/(\mathcal{J}/\mathcal{I}) \cong \mathcal{L}/\mathcal{J}$.

ii- $(\mathcal{I} + \mathcal{J})/\mathcal{J} \cong \mathcal{I}/\mathcal{I} \cap \mathcal{J}$.

The following example, taken from (5), concerns isomorphic Lie algebras.

Example 1.0.5 Let \mathcal{L}_1 be a 3-dimensional vector space over \mathbb{R} , and let $\{e_1, e_2, e_3\}$ be a basis of \mathcal{L}_1 . For $x = x_1e_1 + x_2e_2 + x_3e_3$, $y = y_1e_1 + y_2e_2 + y_3e_3 \in \mathcal{L}_1$ we define,

$$[x, y] = (x_2y_3 - x_3y_2)e_1 + (x_3y_1 - x_1y_3)e_2 + (x_1y_2 - x_2y_1)e_3$$

then \mathcal{L}_1 becomes a Lie algebra. We note that

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2$$

Let \mathcal{L}_2 be the vector space of 3×3 skew-symmetric matrices over \mathbb{R} . For $X, Y \in \mathcal{L}_2$, we define

$$[X, Y] = XY - YX$$

then \mathcal{L}_2 becomes a Lie Algebra. We choose a basis

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

of \mathcal{L}_2 , then

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2$$

Let $X, Y \in \mathcal{L}_2$, so X and Y can be written as

$$X = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} = x_1X_1 + x_2X_2 + x_3X_3$$

$$Y = \begin{pmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{pmatrix} = y_1 X_1 + X_2 X_2 + y_3 X_3$$

so

$$[X, Y] = (x_2 y_3 - x_3 y_2) X_1 + (x_3 y_1 - x_1 y_3) X_2 + (x_1 y_1 - x_2 y_1) X_3$$

We define a linear map $f : \mathcal{L}_1 \rightarrow \mathcal{L}_2$ by,

$$f(x) = f(x_1 e_1 + x_2 e_2 + x_3 e_3) = X = x_1 X_1 + x_2 X_2 + x_3 X_3$$

If $f(x) = X$ and $f(y) = Y$, then $f([x, y]) = [f(x), f(y)] = [X, Y]$, so that this map is a homomorphism from \mathcal{L}_1 to \mathcal{L}_2 . In particular, f is an isomorphism, thus \mathcal{L}_1 and \mathcal{L}_2 are isomorphic.

The following is one of the classical examples of isomorphisms of Lie algebras. (See eg.(2))

Example 1.0.6 Let \mathcal{V} be an n -dimensional vector space over \mathcal{F} . We consider the vector space $End(\mathcal{V})$ of all linear transformations from \mathcal{V} to itself. The multiplication on $End(\mathcal{V})$ is defined by

$$fg(v) = f(g(v))$$

for all $f, g \in End(\mathcal{V})$ and for all $v \in \mathcal{V}$. This multiplication of $End(\mathcal{V})$ makes $End(\mathcal{V})$ into an associative algebra. For $f, g \in End(\mathcal{V})$, we set

$$[f, g] = fg - gf$$

then

$$L_1 : [f, f] = ff - ff = 0$$

$$L_2 : [f, [g, h]] + [g, [h, f]] + [h, [f, g]]$$

$$= f(gh - hg) - (gh - hg)f + g(hf - fh) - (hf - fh)g + h(fg - gf) - (fg - gf)h$$

$$= fgh - fhg - ghf + hgf + ghf - gfh - hfg + fhg + hfg - hgf - fgh + gfh = 0$$

Thus $End(\mathcal{V})$ together with $[,]$ is a Lie algebra. It is denoted by $gl(\mathcal{V})$.

Let $M_n(\mathcal{F})$ be the vector space of all $n \times n$ matrices over \mathcal{F} . The usual matrix multiplication makes $M_n(\mathcal{F})$ into an associative algebra. For $F, G \in M_n(\mathcal{F})$, we set

$$[F, G] = FG - GF$$

then $M_n(\mathcal{F})$ becomes a Lie Algebra. We denote this Lie algebra by $gl_n(\mathcal{F})$.

Let $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$ be a fixed basis of \mathcal{V} so every linear transformation can be represented by a matrix relative to \mathcal{B} . The map which sends every linear transformation to its matrix,

$$gl(\mathcal{V}) \ni f \longmapsto \begin{pmatrix} f_{11} & \cdots & f_{1n} \\ f_{21} & \cdots & f_{2n} \\ \vdots & \cdots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{pmatrix} \in gl_n(\mathcal{F})$$

is an isomorphism from $gl(\mathcal{V})$ to $gl_n(\mathcal{F})$. Thus $gl(\mathcal{V})$ and $gl_n(\mathcal{F})$ are isomorphic Lie algebras.

Definition 1.0.7 Subalgebras of $gl_n(\mathcal{F})$ are called *linear Lie algebras*.

Example 1.0.8 Let

$$sl_n(\mathcal{F}) = \{A \in gl_n(\mathcal{F}) : \text{Tr}(A) = 0\}$$

where $\text{Tr}(A)$ denotes the trace of A . Let $A, B \in gl_n(\mathcal{F})$, then

$$\text{Tr}([A, B]) = \text{Tr}(AB - BA) = \text{Tr}(AB) - \text{Tr}(BA) = 0$$

thus $sl_n(\mathcal{F})$ is a subalgebra of $gl_n(\mathcal{F})$. This particular linear Lie algebra is called the *special Lie algebra*.

Example 1.0.9 The *orthogonal Lie algebra*

$$so_n(\mathcal{F}) = \{A \in gl_n(\mathcal{F}) : A + A^T = 0\}$$

is another example of a linear Lie algebra.

Example 1.0.10 Let

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

where I_n denote the $n \times n$ identity matrix. The *symplectic Lie algebra*,

$$sp_n(\mathcal{F}) = \{A \in gl_{2n}(\mathcal{F}) : JA + A^T J = 0\}$$

is also a linear Lie algebra.

Example 1.0.11 Let \mathcal{L}_1 and \mathcal{L}_2 be two Lie algebras with commutators $[\cdot, \cdot]_{\mathcal{L}_1}$ and $[\cdot, \cdot]_{\mathcal{L}_2}$ respectively, then

$$\mathcal{L}_1 \oplus \mathcal{L}_2 = \{(x, y) : x \in \mathcal{L}_1 \text{ and } y \in \mathcal{L}_2\}$$

becomes a Lie algebra with respect to the multiplication

$$[(x_1, y_1), (x_2, y_2)] = ([x_1, x_2]_{\mathcal{L}_1}, [y_1, y_2]_{\mathcal{L}_2})$$

Let $\{x_1, x_2, \dots, x_n\}$ be a basis of an n -dimensional Lie algebra \mathcal{L} over a field \mathcal{F} . The *structure constants* $c_{ij}^k \in \mathcal{F}$ of \mathcal{L} with respect to the basis $\{x_1, x_2, \dots, x_n\}$ are defined by

$$[x_i, x_j] = \sum_{k=1}^n c_{ij}^k x_k$$

There are n^3 structure constants of an n -dimensional Lie algebra. Let

$$x = \sum_{i=1}^n \alpha_i x_i, \quad y = \sum_{j=1}^n \beta_j x_j$$

be two elements of \mathcal{L} , then

$$[x, y] = \sum_{i,j,k=1}^n \alpha_i \beta_j c_{ij}^k x_k$$

so that the commutator of any given two elements can be determined by using n^3 structure constants. Because $[x_i, x_j] = -[x_j, x_i]$, $c_{ij}^k = -c_{ji}^k$ for $1 \leq i, j, k \leq n$. By using the Jacobi identity, it can be seen that

$$\sum_{t=1}^n (c_{ij}^t c_{tk}^u + c_{jk}^t c_{ti}^u + c_{ki}^t c_{tj}^u) = 0$$

for all $1 \leq i, j, k, u \leq n$.

Definition 1.0.12 Let \mathcal{S} be a subset of a Lie algebra \mathcal{L} . The *centralizer* of \mathcal{S} in \mathcal{L} is defined as the subset

$$C_{\mathcal{L}}(\mathcal{S}) = \{x \in \mathcal{L} : [x, s] = 0, \forall s \in \mathcal{S}\}$$

The *center* of \mathcal{L} is defined as the subset

$$C_{\mathcal{L}}(\mathcal{L}) = \{x \in \mathcal{L} : [x, y] = 0, \forall y \in \mathcal{L}\}$$

Let \mathcal{V} be a subspace of \mathcal{L} . The *normalizer* of \mathcal{V} in \mathcal{L} is defined as the subset

$$N_{\mathcal{L}}(\mathcal{V}) = \{x \in \mathcal{L} : [x, v] \in \mathcal{V}, \forall v \in \mathcal{V}\}$$

It is easy to show that the centralizer and the normalizer are subalgebras of \mathcal{L} and the center is an ideal of \mathcal{L} . It immediately follows from the definition of the center that \mathcal{L} is abelian if and only if $C_{\mathcal{L}}(\mathcal{L}) = \mathcal{L}$. If \mathcal{V} is a subalgebra, then \mathcal{V} is an ideal in the $N_{\mathcal{L}}(\mathcal{V})$.

Definition 1.0.13 Let \mathcal{L} be a Lie algebra. For $x \in \mathcal{L}$, we define

$$ad_{\mathcal{L}}x : \mathcal{L} \longrightarrow \mathcal{L}; y \longmapsto [x, y]$$

for all $y \in \mathcal{L}$. This map is a linear map and called *adjoint map* determined by x .

We consider the linear map $ad_{\mathcal{L}} : \mathcal{L} \longrightarrow gl(\mathcal{L})$. For all $x, y, z \in \mathcal{L}$, by using Jacobi identity, we have

$$ad_{\mathcal{L}}[x, y](z) = [[x, y], z] = [x, [y, z]] - [y, [x, z]] = (ad_{\mathcal{L}}x \cdot ad_{\mathcal{L}}y - ad_{\mathcal{L}}y \cdot ad_{\mathcal{L}}x)(z)$$

thus

$$ad_{\mathcal{L}}[x, y] = [ad_{\mathcal{L}}x, ad_{\mathcal{L}}y]$$

Hence, the linear map $ad_{\mathcal{L}}$ is a homomorphism of Lie algebras. We note that $ad_{\mathcal{L}}$ is not necessarily faithful. In order to see this, we consider the Heisenberg Lie algebra \mathcal{H} with

basis $\{x_1, x_2, x_3\}$ and multiplication table

	x_1	x_2	x_3
x_1	0	x_3	0
x_2	$-x_3$	0	0
x_3	0	0	0

$ad_{\mathcal{H}}x_3 : \mathcal{H} \longrightarrow \mathcal{H}$, for all $x \in \mathcal{H}$, $ad_{\mathcal{H}}x_3(x) = 0$ so that $ad_{\mathcal{H}}x_3 = 0$ which implies that $x_3 \in \text{Ker } ad_{\mathcal{H}}$. Therefore $ad_{\mathcal{H}}$ is not faithful.

Definition 1.0.14 Let \mathcal{A} be an algebra. A linear map $D : \mathcal{A} \longrightarrow \mathcal{A}$ is said to be a *derivation* if it satisfies

$$D(xy) = D(x)y + xD(y)$$

for all $x, y \in \mathcal{A}$. The set of all derivations $Der(\mathcal{A})$ forms a vector space. For $D_1, D_2 \in Der(\mathcal{A})$ define,

$$[,] : Der(\mathcal{A}) \times Der(\mathcal{A}) \longrightarrow Der(\mathcal{A})$$

$$[D_1, D_2](xy) = (D_1D_2 - D_2D_1)(xy)$$

so that

$$[D_1, D_2](xy) = [D_1, D_2](x)y + x[D_1, D_2](y)$$

thus $[D_1, D_2]$ is also a derivation of $Der(\mathcal{A})$. It is also easy to check that L_1 and L_2 hold for $Der(\mathcal{A})$. Hence $Der(\mathcal{A})$ has the structure of a Lie algebra.

Example 1.0.15 Let \mathcal{L} be a Lie algebra, and $x \in \mathcal{L}$. For all $y, z \in \mathcal{L}$,

$$ad_{\mathcal{L}}x([y, z]) = [x, [y, z]] \quad (\text{from Jacobi identity})$$

$$= -[y, [z, x]] - [z, [x, y]] \quad (\text{from } L_1)$$

$$= [[x, y], z] + [y, x, z]$$

$$= [ad_{\mathcal{L}}x(y), z] + [y, ad_{\mathcal{L}}x(z)]$$

Thus the adjoint map $ad_{\mathcal{L}}x : \mathcal{L} \longrightarrow \mathcal{L}$ is a derivation of \mathcal{L} .

Definition 1.0.16 Let \mathcal{L} be a Lie algebra and \mathcal{V} be a vector space over a field \mathcal{F} . A *representation* of \mathcal{L} on \mathcal{V} is a homomorphism $\rho : \mathcal{L} \longrightarrow gl(\mathcal{V})$, that is a representation of \mathcal{L} is a map ρ which satisfies

$$i - \rho(ax + by) = a\rho(x) + b\rho(y)$$

$$ii - \rho([x, y]) = [\rho(x), \rho(y)]$$

for all $x, y \in \mathcal{L}$, and $a, b \in \mathcal{F}$. ρ is called faithful if $\text{Ker } \rho = 0$. Suppose that there is a bilinear map $\mathcal{L} \times \mathcal{V} \longrightarrow \mathcal{V}$, $(x, v) \longmapsto x \cdot v$. If

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$$

for all $x, y \in \mathcal{L}$ and $v \in \mathcal{V}$, then \mathcal{V} is called an \mathcal{L} -module.

Let \mathcal{L} be a Lie algebra. We showed that the adjoint map $ad_{\mathcal{L}} : \mathcal{L} \longrightarrow gl(\mathcal{V})$ is a homomorphism of the Lie algebra \mathcal{L} , so it is a representation of \mathcal{L} . It is called the *adjoint representation*. Among the representations of a given Lie algebra, the adjoint representation plays an important role. It is the natural action of a Lie algebra \mathcal{L} on itself.

A subspace \mathcal{W} of \mathcal{V} is called an \mathcal{L} -submodule of \mathcal{V} if $x \cdot w \in \mathcal{W}$ for all $x \in \mathcal{L}$ and $w \in \mathcal{W}$. \mathcal{V} is said to be an *irreducible* \mathcal{L} -module if it has no submodules except $\{0\}$ and itself. \mathcal{V} is called a *completely reducible* \mathcal{L} -module if it is the direct sum of irreducible \mathcal{L} -modules.

Definition 1.0.17 Let \mathcal{V} be a finite dimensional \mathcal{L} -module. A finite sequence (\mathcal{V}_i) , $i \in \{1, 2, \dots, n\}$ of \mathcal{L} -submodules of \mathcal{V} is called a *composition series* for \mathcal{V} if

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \dots \subset \mathcal{V}_{n+1} = \mathcal{V}$$

such that $\mathcal{V}_{i+1}/\mathcal{V}_i$ are irreducible for all $0 \leq i \leq n$.

Theorem 3 Let \mathcal{V} be a finite dimensional \mathcal{L} -module, then there exists a composition series of \mathcal{V} .

Proof. We use induction on the dimension of \mathcal{V} . The theorem is trivial if the dimension of \mathcal{V} is 0 or if \mathcal{V} is irreducible. Now suppose that the theorem is true for all \mathcal{L} -modules of dimension less than n . Let \mathcal{W} be the proper submodule of maximal dimension. By the induction hypothesis, \mathcal{W} has a composition series

$$0 = \mathcal{W}_0 \subset \mathcal{W}_1 \subset \dots \subset \mathcal{W}_n = \mathcal{W}$$

Since \mathcal{W} is the proper maximal submodule, \mathcal{V}/\mathcal{W} is irreducible, so we add \mathcal{V} to the composition series of \mathcal{W} to obtain a composition series of \mathcal{V} . □

CHAPTER 2. Nilpotent and Solvable Lie Algebras

In this chapter, we will define *nilpotent Lie algebra*, *solvable Lie algebra*, *lower central series*, *derived series*, *nilradical*, and *solvable radical*, also we will give some basic theorems and related concepts.

Let \mathcal{L} be a Lie algebra and let \mathcal{I} and \mathcal{J} be subalgebras of \mathcal{L} . For $x \in \mathcal{I}$ and $y \in \mathcal{J}$, the linear span of the elements $[x, y]$ is called the *product space* of \mathcal{I} and \mathcal{J} and denoted by $[\mathcal{I}, \mathcal{J}]$.

Theorem 4 *Let \mathcal{L} be an n -dimensional Lie algebra and let \mathcal{I} and \mathcal{J} be ideals in \mathcal{L} . Then $[\mathcal{I}, \mathcal{J}]$ is an ideal in \mathcal{L} .*

Proof. Let $u \in [\mathcal{I}, \mathcal{J}]$, then u can be written as

$$u = \sum_{i=1}^n [x_i, y_i]$$

where $x_i \in \mathcal{I}$ and $y_i \in \mathcal{J}$. Let $z \in \mathcal{L}$, then

$$[z, u] = [z, \sum_{i=1}^n [x_i, y_i]] = \sum_{i=1}^n [z, [x_i, y_i]] = - \sum_{i=1}^n [x_i, [y_i, z]] - \sum_{i=1}^n [y_i, [z, x_i]]$$

Since \mathcal{I} and \mathcal{J} are ideals of \mathcal{L} , $[z, x_i] \in \mathcal{I}$ and $[y_i, z] \in \mathcal{J}$ so that

$$- \sum_{i=1}^n [x_i, [y_i, z]] - \sum_{i=1}^n [y_i, [z, x_i]] \in [\mathcal{I}, \mathcal{J}]$$

Thus, $[\mathcal{I}, \mathcal{J}]$ is an ideal in \mathcal{L} . □

Since \mathcal{L} is an ideal of itself, we define a sequence \mathcal{L}^n of ideals of \mathcal{L} inductively by,

$$\mathcal{L}^1 = \mathcal{L}, \mathcal{L}^2 = [\mathcal{L}^1, \mathcal{L}], \dots, \mathcal{L}^n = [\mathcal{L}^{n-1}, \mathcal{L}], \dots$$

It is obvious that $\mathcal{L}^n \subset \mathcal{L}^{n-1}$ for all $n \geq 1$, so that this is a nonincreasing sequence of ideals. The sequence

$$\mathcal{L} = \mathcal{L}^1 \supset \mathcal{L}^2 \supset \dots \supset \mathcal{L}^n \supset \dots$$

is called *lower central series* of \mathcal{L} . If there exists a positive integer $k > 0$ such that $\mathcal{L}^{k-1} \neq 0$ and $\mathcal{L}^k = 0$ then \mathcal{L} is called a *nilpotent Lie algebra*. The integer k is said to be the *nilpotency class* of \mathcal{L} . In particular, if $k = 2$, \mathcal{L} is an Abelian Lie algebra. Moreover, if $\mathcal{L}^{k-1} \neq 0$ and $\mathcal{L}^k = 0$, then $\mathcal{L}^{k-1} \subset C(\mathcal{L})$.

Theorem 5 *Let $\rho : \mathcal{L} \rightarrow gl(\mathcal{V})$ be a finite dimensional representation of a Lie algebra \mathcal{L} and let \mathcal{I} be an ideal in \mathcal{L} . Let $0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \dots \subset \mathcal{V}_{n+1} = \mathcal{V}$ be a composition series of \mathcal{V} , then the following are equivalent.*

- i-* $\rho(x)$ is nilpotent for all $x \in \mathcal{I}$ (i.e. $\rho(x)^k = 0$ for a $k > 0$)
- ii-* $\rho(x)\mathcal{V}_{i+1} \subset \mathcal{V}_i$ for all $x \in \mathcal{I}$, and $1 \leq i \leq n$.

Theorem 6 (Engel's Theorem) *Let \mathcal{L} be a finite dimensional Lie algebra, then \mathcal{L} is nilpotent if and only if $ad_{\mathcal{L}}x$ is a nilpotent endomorphism for all $x \in \mathcal{L}$.*

Proof. Suppose that \mathcal{L} is nilpotent. Since $ad_{\mathcal{L}}x(\mathcal{L}^i) \subset \mathcal{L}^{i+1}$, $ad_{\mathcal{L}}x$ is nilpotent for all $x \in \mathcal{L}$.

Now suppose that $ad_{\mathcal{L}}x$ is nilpotent for all $x \in \mathcal{L}$. We consider the composition series

$$0 = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \dots \subset \mathcal{L}_{n+1} = \mathcal{L}$$

of \mathcal{L} . The previous theorem implies that

$$ad_{\mathcal{L}}x(\mathcal{L}_{i+1}) \subset \mathcal{L}_i$$

for all $x \in \mathcal{L}$ and for all $1 \leq i \leq n$, so that $[\mathcal{L}, \mathcal{L}_{i+1}] \subset \mathcal{L}_i$. Take $i = n$ then $[\mathcal{L}, \mathcal{L}_{n+1}] \subset \mathcal{L}_n$, but $\mathcal{L}_{n+1} = \mathcal{L}$ and $[\mathcal{L}, \mathcal{L}] = \mathcal{L}^2$ so that $\mathcal{L}^2 \subset \mathcal{L}_n$. Now, take $i = n-1$, then $[\mathcal{L}, \mathcal{L}_n] \subset \mathcal{L}_{n-1}$ so that $[\mathcal{L}, \mathcal{L}^2] \subset [\mathcal{L}, \mathcal{L}_n]$ which implies $\mathcal{L}^3 \subset \mathcal{L}_{n-1}$. Thus for $i = k$ we have $\mathcal{L}^k \subset \mathcal{L}_{n+2-k}$, then we conclude that \mathcal{L} is nilpotent. \square

Engel's Theorem implies that in order to show \mathcal{L} is not nilpotent, it is sufficient to find at least one $x \in \mathcal{L}$ such that $ad_{\mathcal{L}}x$ is not a nilpotent endomorphism.

Example 2.0.18 Let \mathcal{L} be a 3-dimensional Lie algebra with a basis $\{x_1, x_2, x_3\}$ and multiplication table

	x_1	x_2	x_3
x_1	0	0	$x_1 + x_2$
x_2	0	0	$-x_2$
x_3	$-x_1 - x_2$	x_2	0

$ad_{\mathcal{L}}x_3(x_2) = x_2$ and then $ad_{\mathcal{L}}x_3^n(x_2) = x_2$ for all integers. Therefore $ad_{\mathcal{L}}x_3$ is not a nilpotent endomorphism. It follows from Engel's Theorem that \mathcal{L} is not a nilpotent Lie algebra.

Now we define another sequence $\mathcal{L}^{(n)}$ of \mathcal{L} inductively,

$$\mathcal{L}^{(1)} = \mathcal{L}, \mathcal{L}^{(2)} = [\mathcal{L}^{(1)}, \mathcal{L}^{(1)}], \dots, \mathcal{L}^{(n)} = [\mathcal{L}^{(n-1)}, \mathcal{L}^{(n-1)}], \dots$$

for all $n \geq 1$. By Theorem 4, $\mathcal{L}^{(n)}$ are ideals of \mathcal{L} and $\mathcal{L}^{(n)} \subset \mathcal{L}^{(n-1)}$ so that

$$\mathcal{L} = \mathcal{L}^{(1)} \supset \mathcal{L}^{(2)} \supset \dots \supset \mathcal{L}^{(n)} \supset \dots$$

is a nonincreasing sequence and is called the *derived series* of \mathcal{L} . If there exists a positive integer $k > 0$ such that $\mathcal{L}^{(k)} = 0$ and $\mathcal{L}^{(k-1)} \neq 0$, then \mathcal{L} is called *solvable Lie algebra*.

Given a Lie algebra \mathcal{L} , we use induction to prove that $\mathcal{L}^{(k)} \subset \mathcal{L}^k$. Indeed, by definition

$$\mathcal{L}^{(2)} = \mathcal{L}^2 \subset \mathcal{L}$$

Now if we suppose that $\mathcal{L}^{(k)} \subset \mathcal{L}^k \subset \mathcal{L}$ then,

$$\mathcal{L}^{(k+1)} = [\mathcal{L}^{(k)}, \mathcal{L}^{(k)}] \subset [\mathcal{L}^k, \mathcal{L}] = \mathcal{L}^{k+1}$$

Hence $\mathcal{L}^{(k)} \subset \mathcal{L}^k$ for all k . Thus every nilpotent Lie algebra is solvable, but in general the converse is not true. In order to see this, we consider the Lie algebra \mathcal{L} of the set of all 2×2 upper triangular matrices, i.e.,

$$\mathcal{L} = \{(x_{ij}) \in gl_2(\mathbb{R}) : x_{ij} = 0 \text{ for all } i > j\}$$

A basis for \mathcal{L} is

$$\left\{ x_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, x_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, x_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

so the multiplication table of \mathcal{L} is,

	x_1	x_2	x_3
x_1	0	x_2	0
x_2	$-x_2$	0	x_2
x_3	0	$-x_2$	0

It is easy to see that $\mathcal{L}^{(2)} = \text{span} \{x_2\}$ and $\mathcal{L}^{(3)} = 0$ so that \mathcal{L} is solvable, but $ad_{\mathcal{L}}x_2$ is not nilpotent and it follows from the Engel's Theorem that \mathcal{L} is not a nilpotent Lie algebra.

Here we show that the sum of two nilpotent ideals is also a nilpotent ideal, and conclude that an arbitrary Lie algebra contains a unique maximal nilpotent ideal.

Theorem 7 *Let \mathcal{L} be a Lie algebra. If \mathcal{L} is nilpotent, so are all subalgebras and homomorphic images of \mathcal{L} . If \mathcal{I} and \mathcal{J} are two nilpotent ideals, then $\mathcal{I} + \mathcal{J}$ is also a nilpotent ideal.*

Proof. Since \mathcal{I} and \mathcal{J} are nilpotent ideals, there exists an integer k such that $\mathcal{I}^k = \mathcal{J}^k = 0$ where \mathcal{I}^k and \mathcal{J}^k are the k -th terms of the lower central series of the ideals \mathcal{I} and \mathcal{J} respectively. Since $\mathcal{I}^k = \mathcal{J}^k = 0$, any m -fold $[x_1, [x_2, [\dots, [x_{m-1}, x_m] \dots]]]$ is equal to 0 if it contains at least k elements from either \mathcal{I} or \mathcal{J} . Let $x_i = y_i + z_i$ where $y_i \in \mathcal{I}$ and

$z_i \in \mathcal{J}$. Now, take $m = 2k$ then $[x_1, [x_2, [\dots, [x_{m-1}, x_m] \dots]]] = 0$ so that $(\mathcal{I} + \mathcal{J})^{2k} = 0$, thus $\mathcal{I} + \mathcal{J}$ is also a nilpotent ideal. \square

This theorem shows that the sum of nilpotent ideals of a Lie algebra \mathcal{L} is a nilpotent ideal, and so there exists a maximal nilpotent ideal of \mathcal{L} . The maximal nilpotent ideal of a Lie algebra is called the *nilradical* and is denoted by $\text{NR}(\mathcal{L})$.

Theorem 8 *Let \mathcal{L} be a Lie algebra then,*

i- *If \mathcal{L} is solvable, then the subalgebras and homomorphic images of \mathcal{L} are solvable (In particular, quotient algebras of solvable algebras are solvable).*

ii- *If \mathcal{I} is a solvable ideal of \mathcal{L} and \mathcal{L}/\mathcal{I} is solvable, then \mathcal{L} is solvable.*

iii- *If \mathcal{I} and \mathcal{J} are solvable ideals of \mathcal{L} , then $\mathcal{I} + \mathcal{J}$ is also a solvable ideal of \mathcal{L} .*

Proof. *i-* Let \mathcal{L}_1 be a subalgebra of a solvable algebra \mathcal{L} , then $\mathcal{L}_1^{(k)} \subset \mathcal{L}^{(k)}$ so that \mathcal{L}_1 is also solvable.

Let $f(\mathcal{L})$ be a homomorphic image of a solvable Lie algebra \mathcal{L} . By induction, we show $f(\mathcal{L}^{(n)}) = f(\mathcal{L})^{(n)}$. Since f is a homomorphism,

$$f(\mathcal{L})^{(2)} = f([\mathcal{L}, \mathcal{L}]) = [f(\mathcal{L}), f(\mathcal{L})] = f(\mathcal{L})^{(2)}$$

Now we suppose that $f(\mathcal{L}^{(n)}) = f(\mathcal{L})^{(n)}$, then

$$f(\mathcal{L}^{(n+1)}) = f([\mathcal{L}^{(n)}, \mathcal{L}^{(n)}]) = [f(\mathcal{L}^{(n)}), f(\mathcal{L}^{(n)})] = [f(\mathcal{L})^{(n)}, f(\mathcal{L})^{(n)}] = f(\mathcal{L})^{(n+1)}$$

Since \mathcal{L} is solvable, there exists an integer $k > 0$ such that $\mathcal{L}^{(k)} = 0$. But $f(\mathcal{L}^{(k)}) = f(0) = f(\mathcal{L})^{(k)} = 0$. Hence $f(\mathcal{L})$ is also solvable.

ii- Since \mathcal{L}/\mathcal{I} is solvable, $\mathcal{L}^k \subset \mathcal{I}$ for some k , but \mathcal{I} is solvable so that \mathcal{L} is solvable.

iii- $(\mathcal{I} + \mathcal{J})/\mathcal{J} \cong \mathcal{I}/\mathcal{I} \cap \mathcal{J}$ and since the right handside is solvable, the left handside is solvable, so that $\mathcal{I} + \mathcal{J}$ is solvable. \square

It follows from Theorem 8 that a finite dimensional Lie algebra \mathcal{L} contains a maximal solvable ideal, which is the sum of all solvable ideals of \mathcal{L} . This ideal is said to be the

solvable radical of \mathcal{L} , and it is denoted by $SR(\mathcal{L})$. Since every nilpotent ideal is solvable, $NR(\mathcal{L}) \subset SR(\mathcal{L})$.

Theorem 9 (Cartan's solvability criterion) *Let \mathcal{V} be a finite dimensional vector space over a field of characteristic 0 and let \mathcal{L} be a subalgebra of $gl(\mathcal{V})$, then \mathcal{L} is solvable if $\text{Tr}(xy) = 0$ for all $x \in \mathcal{L}$ and for all $y \in [\mathcal{L}, \mathcal{L}]$.*

Proposition 2.0.19 Let \mathcal{L} be a finite dimensional Lie algebra of characteristic 0 over a field \mathcal{F} . Then

$$SR(\mathcal{L}) = \{x \in \mathcal{L} : \text{Tr}(ad_{\mathcal{L}}x \cdot ad_{\mathcal{L}}y) = 0\}$$

for all $y \in [\mathcal{L}, \mathcal{L}]$.

We conclude that a finite dimensional Lie algebra \mathcal{L} of characteristic 0 is solvable if and only if $\text{Tr}(ad_{\mathcal{L}}x \cdot ad_{\mathcal{L}}y) = 0$ for all $x \in \mathcal{L}$ and for all $y \in [\mathcal{L}, \mathcal{L}]$.

Let $\rho : \mathcal{L} \longrightarrow gl(\mathcal{L})$ be a finite dimensional representation of a Lie algebra \mathcal{L} . For all $x, y \in \mathcal{L}$, a bilinear form $f_{\rho} : \mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{L}$ defined as

$$f_{\rho}(x, y) = \text{Tr}(\rho(x)\rho(y))$$

is called the trace form corresponding to the representation ρ . Let $\{x_1, \dots, x_n\}$ be a basis of \mathcal{L} , then $(f_{\rho}(x_i, x_j))_{n \times n}$ is the matrix of f_{ρ} with respect to $\{x_1, \dots, x_n\}$. The trace form corresponding to the adjoint representation is called the *Killing form*, and denoted by $\kappa_{\mathcal{L}}$. By the definition of a non-degenerate function, if $\kappa_{\mathcal{L}}$ is non-degenerate and if $\kappa_{\mathcal{L}}(x, y) = 0$ for $x \in \mathcal{L}$ and for all $y \in \mathcal{L}$, then $x = 0$. We define

$$R_p = \{x \in \mathcal{L} : f_p(x, y) = 0 \text{ for all } y \in \mathcal{L}\}$$

R_p is called the *radical* of f_p , and it can be shown that R_p forms a subspace. $R_p = 0$ if and only if f_p is non-degenerate.

Definition 2.0.20 A Lie algebra \mathcal{L} is said to be *semisimple* if \mathcal{L} does not contain any solvable ideal except $\{0\}$.

Theorem 10 (Cartan's semisimplicity criterion) *Let \mathcal{L} be a Lie algebra. If the Killing form $\kappa_{\mathcal{L}}$ is non-degenerate, then \mathcal{L} is semisimple, and if \mathcal{L} is a semisimple Lie algebra of characteristic 0, then the Killing form $\kappa_{\mathcal{L}}$ is non-degenerate.*

Proof. Suppose that the Killing form $\kappa_{\mathcal{L}}$ is non-degenerate. Assume that $SR(\mathcal{L}) \neq 0$. We consider the derived series of $SR(\mathcal{L})$, so that there exist an integer k such that $SR(\mathcal{L})^{(k)} \neq 0$ and $SR(\mathcal{L})^{(k+1)} = 0$. Let $x \in SR(\mathcal{L})^{(k)}$, and $y \in \mathcal{L}$, so $ad_{\mathcal{L}}x ad_{\mathcal{L}}y(\mathcal{L}) \subset SR(\mathcal{L})^{(k)}$. Since $ad_{\mathcal{L}}x SR(\mathcal{L})^{(k)} \subset SR(\mathcal{L})^{(k+1)}$, it follows that $(ad_{\mathcal{L}}x ad_{\mathcal{L}}y)^2 = 0$. Since $\kappa_{\mathcal{L}}$ is non-degenerate, $ad_{\mathcal{L}}x = 0$ and therefore $x = 0$ which contradicts the fact that $SR(\mathcal{L}) \neq 0$. Hence \mathcal{L} is semisimple.

Now suppose that \mathcal{L} is semisimple. Since $SR(\mathcal{L}) = 0$, $C_{\mathcal{L}}(\mathcal{L}) = 0$ so that adjoint representation is faithful, thus it follows that $\kappa_{\mathcal{L}}$ is non-degenerate. \square

Example 2.0.21 The multiplication table of $sl_2(\mathcal{L})$ is

	x_1	x_2	x_3
x_1	0	$2x_2$	$-2x_3$
x_2	$-2x_2$	0	x_1
x_3	$2x_3$	$-x_1$	0

$sl_2(\mathcal{L})$ is semisimple since the determinant of the matrix κ of the Killing form is -128.

Example 2.0.22 Let \mathcal{H} be the 3-dimensional Heisenberg Lie algebra with basis $\{x_1, x_2, x_3\}$ and multiplication table

	x_1	x_2	x_3
x_1	0	x_3	0
x_2	$-x_3$	0	0
x_3	0	0	0

\mathcal{H} is not a semisimple since the determinant of the matrix κ of the Killing form is 0.

Theorem 11 *Let \mathcal{L} be a semisimple Lie algebra of characteristic 0. Let $\rho : \mathcal{L} \rightarrow \mathfrak{gl}(\mathcal{V})$ be a finite dimensional representation of \mathcal{L} . If \mathcal{W}_1 is a submodule of \mathcal{V} , then there exists a submodule $\mathcal{W}_2 \subset \mathcal{V}$ such that $\mathcal{V} = \mathcal{W}_1 \oplus \mathcal{W}_2$.*

CHAPTER 3. Levi Decomposition

In this chapter, we describe Levi's theorem which states that an arbitrary finite dimensional Lie algebra is a direct sum of a semisimple algebra and the solvable radical. We have followed (2) and (3) for the proof of Levi's Theorem.

Theorem 12 *Let \mathcal{L} be a Lie algebra over a field \mathcal{F} of characteristic 0 and let \mathcal{L} be not solvable, then $\mathcal{L}/SR(\mathcal{L})$ is a semisimple Lie subalgebra.*

Proof. Let $\mathcal{I}/SR(\mathcal{L})$ be the solvable radical of $\mathcal{L}/SR(\mathcal{L})$. Since $SR(\mathcal{L})$ is solvable and $\mathcal{I}/SR(\mathcal{L})$ solvable, it follows from Theorem 8 that \mathcal{I} is solvable. But since $SR(\mathcal{L})$ is the maximal solvable ideal, $\mathcal{I} \subset SR(\mathcal{L})$. Hence $\mathcal{I} = SR(\mathcal{L})$ so the solvable radical of $\mathcal{L}/SR(\mathcal{L})$ is $\{0\}$ and therefore $\mathcal{L}/SR(\mathcal{L})$ is semisimple. \square

Theorem 13 (Levi) *Let \mathcal{L} be a finite dimensional Lie algebra over a field of characteristic 0. If \mathcal{L} is not solvable, then there exists a semisimple subalgebra \mathcal{S} of \mathcal{L} such that*

$$\mathcal{L} = \mathcal{S} \oplus SR(\mathcal{L})$$

In this decomposition, $\mathcal{S} \cong \mathcal{L}/SR(\mathcal{L})$ and we have the following commutation relations.

$$[\mathcal{S}, \mathcal{S}] = \mathcal{S}, \quad [\mathcal{S}, SR(\mathcal{L})] \subseteq SR(\mathcal{L}), \quad [SR(\mathcal{L}), SR(\mathcal{L})] \subseteq SR(\mathcal{L})$$

Proof. **Case I.** \mathcal{L} has no ideals \mathcal{I} such that $\mathcal{I} \neq 0$ and \mathcal{I} is properly contained in $SR(\mathcal{L})$. $[SR(\mathcal{L}), SR(\mathcal{L})]$ is an ideal of \mathcal{L} that is contained in $SR(\mathcal{L})$, so it follows that $[SR(\mathcal{L}), SR(\mathcal{L})] = 0$. $[\mathcal{L}, SR(\mathcal{L})]$ is an ideal of \mathcal{L} and $SR(\mathcal{L})$. But since $SR(\mathcal{L})$ does not

contain any of the ideals of \mathcal{L} properly, it follows that either $[\mathcal{L}, SR(\mathcal{L})] = SR(\mathcal{L})$ or $[\mathcal{L}, SR(\mathcal{L})] = 0$.

Subcase I. $[\mathcal{L}, SR(\mathcal{L})] \neq 0$, i.e. $[\mathcal{L}, SR(\mathcal{L})] = SR(\mathcal{L})$

Let \mathcal{V} be the set of all linear transformations from \mathcal{L} to itself, i.e. $\mathcal{V} = Hom(\mathcal{L}, \mathcal{L})$.

For all $x, y \in \mathcal{L}$ and $\varphi \in \mathcal{V}$, \mathcal{V} can be made into an \mathcal{L} -module by setting

$$(x \cdot \varphi)(y) = [x, \varphi(y)] - \varphi([x, y])$$

The following sets $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are \mathcal{L} -submodules of \mathcal{V} .

$$\mathcal{A} = \{ad_{\mathcal{L}}x : x \in SR(\mathcal{L})\}$$

$$\mathcal{B} = \{\varphi \in \mathcal{V} : \varphi(\mathcal{L}) \subset SR(\mathcal{L}) \text{ and } \varphi(SR(\mathcal{L})) = 0\}$$

$$\mathcal{C} = \{\varphi \in \mathcal{V} : \varphi(\mathcal{L}) \subset SR(\mathcal{L}), \text{ and } \varphi|_{SR(\mathcal{L})} \text{ is multiplication by a scalar}\}$$

Since $[\mathcal{L}, SR(\mathcal{L})] = SR(\mathcal{L})$, $ad_{\mathcal{L}}x(\mathcal{L}) \subset SR(\mathcal{L})$ for all $x \in SR(\mathcal{L})$, and since $[SR(\mathcal{L}), SR(\mathcal{L})] = 0$, $ad_{\mathcal{L}}x(SR(\mathcal{L})) = 0$, so $\mathcal{A} \subset \mathcal{B}$. From the definitions of \mathcal{B} and \mathcal{C} , we have that $\mathcal{B} \subset \mathcal{C}$, and therefore

$$\mathcal{A} \subset \mathcal{B} \subset \mathcal{C}$$

Let $\varphi_{\lambda} \in \mathcal{C}$ be the multiplication by the scalar λ , then for all $x \in \mathcal{L}$ and $y \in SR(\mathcal{L})$,

$$(x \cdot \varphi_{\lambda})(y) = [x, \varphi_{\lambda}(y)] - \varphi_{\lambda}[x, y] = [x, \lambda y] - \lambda[x, y] = \lambda[x, y] - \lambda[x, y] = 0$$

Therefore $x \cdot \varphi_{\lambda} \in \mathcal{B}$ which implies that

$$\mathcal{L} \cdot \mathcal{C} \subset \mathcal{B}$$

Since $\varphi_{\lambda}(x) \in SR(\mathcal{L})$ and $[SR(\mathcal{L}), SR(\mathcal{L})] = 0$, $[y, \varphi_{\lambda}(x)] = 0$ so that

$$(y \cdot \varphi_{\lambda})(x) = [y, \varphi_{\lambda}(x)] - \varphi_{\lambda}[y, x] = 0 - \lambda[y, x] = [-\lambda y, x] = ad_{\mathcal{L}}(-\lambda y)(x)$$

$$\Rightarrow y \cdot \varphi_{\lambda} = ad_{\mathcal{L}}(-\lambda y) \in \mathcal{A}$$

$$\Rightarrow SR(\mathcal{L}) \cdot \mathcal{C} \subset \mathcal{A}$$

$SR(\mathcal{L}) \cdot \mathcal{C} \subset \mathcal{A}$ so that \mathcal{C}/\mathcal{A} is an $\mathcal{L}/SR(\mathcal{L})$ -module, and \mathcal{B}/\mathcal{A} contains \mathcal{C}/\mathcal{A} as a proper submodule. Since $\mathcal{L}/SR(\mathcal{L})$ is a semisimple Lie algebra, Theorem 11 implies that there exists a complementary module \mathcal{D}/\mathcal{A} in \mathcal{C}/\mathcal{A} to \mathcal{B}/\mathcal{A} such that $\mathcal{B} + \mathcal{D} = \mathcal{C}$ and $\mathcal{B} \cap \mathcal{D} = \mathcal{A}$. $\mathcal{L} \cdot \mathcal{C} \subset \mathcal{B}$ implies that $\mathcal{L}/SR(\mathcal{L}) \cdot \mathcal{D}/\mathcal{A} \subset \mathcal{B}/\mathcal{A}$. Since \mathcal{D}/\mathcal{A} is complementary to \mathcal{B}/\mathcal{A} , \mathcal{D}/\mathcal{A} must be mapped to 0, so that $SR(\mathcal{L}) \cdot \mathcal{D} \subset \mathcal{A}$, and therefore $\mathcal{L} \cdot \mathcal{D} \subset \mathcal{A}$. Let $\varphi \in \mathcal{D}/\mathcal{A}$ such that $\varphi|_{SR(\mathcal{L})}$ is the identity on $SR(\mathcal{L})$. Since $\mathcal{L} \cdot \mathcal{D} \subset \mathcal{A}$, $\mathcal{L} \cdot \varphi \subset \mathcal{A}$. Define $\mathcal{S} = \{x \in \mathcal{L} : x \cdot \varphi = 0\}$. Let $x \in \mathcal{L}$ then $x \cdot \varphi \in \mathcal{A}$ so that $x \cdot \varphi = ad_{\mathcal{L}}y$ for some $y \in SR(\mathcal{L})$. For $z \in \mathcal{L}$,

$$(y \cdot \varphi)(z) = [y, \varphi(z)] - \varphi([y, z]) = -[y, z] = [-y, z] = ad_{\mathcal{L}}(-y)(z)$$

so that $y \cdot \varphi = ad_{\mathcal{L}}(-y)$.

$$(x + y) \cdot \varphi = x \cdot \varphi + y \cdot \varphi = ad_{\mathcal{L}}(y) + ad_{\mathcal{L}}(-y) = 0$$

therefore $x + y \in \mathcal{S}$. Hence

$$x = (x + y) - y \in \mathcal{S} + SR(\mathcal{L})$$

where $x \in \mathcal{L}$. In order to complete the proof, we need to show that $\mathcal{S} \cap SR(\mathcal{L}) = \{0\}$. Let $u \in \mathcal{S} \cap SR(\mathcal{L})$, then $u \in \mathcal{S}$ and $u \in SR(\mathcal{L})$. $u \in \mathcal{S}$ implies that $u \cdot \varphi = 0$ and $u \in SR(\mathcal{L})$ implies that $u \cdot \varphi = ad_{\mathcal{L}}(-u)$, so $ad_{\mathcal{L}}(-u) = 0$ and therefore u spans an ideal \mathcal{I} which is also a subset of $SR(\mathcal{L})$. By our assumption, since $SR(\mathcal{L})$ does not contain any proper ideal of \mathcal{L} , either $\mathcal{I} = SR(\mathcal{L})$ or $\mathcal{I} = 0$. If $\mathcal{I} = SR(\mathcal{L})$, since $ad_{\mathcal{L}}(-u) = 0$, it follows that $[\mathcal{L}, SR(\mathcal{L})] = 0$. This contradicts the fact that $[\mathcal{L}, SR(\mathcal{L})] \neq 0$. Hence $\mathcal{I} = 0$ so that $u = 0$ and therefore $\mathcal{S} \cap SR(\mathcal{L}) = \{0\}$. Hence

$$\mathcal{L} = \mathcal{S} \oplus SR(\mathcal{L})$$

Subcase II. $[\mathcal{L}, SR(\mathcal{L})] = 0$ then $SR(\mathcal{L}) \subset C_{\mathcal{L}}(\mathcal{L})$. But $C_{\mathcal{L}}(\mathcal{L})$ is an solvable ideal of \mathcal{L} ,

so $C_{\mathcal{L}}(\mathcal{L}) \subset SR(\mathcal{L})$ and therefore $SR(\mathcal{L}) = C_{\mathcal{L}}(\mathcal{L})$ which implies that $\text{Ker } ad_{\mathcal{L}} = SR(\mathcal{L})$.

By Theorem 1

$$\mathcal{L}/\text{Ker } ad_{\mathcal{L}} = \text{Im } ad_{\mathcal{L}}$$

i.e

$$\mathcal{L}/SR(\mathcal{L}) \cong ad_{\mathcal{L}}\mathcal{L}$$

so that the adjoint representation induces a representation

$$\rho : \mathcal{L}/SR(\mathcal{L}) \longrightarrow gl(\mathcal{L})$$

We know that $\mathcal{L}/SR(\mathcal{L})$ is semisimple and $SR(\mathcal{L})$ is a submodule, hence by Theorem 11 there exists a submodule $\mathcal{S} \subset \mathcal{L}$ such that

$$\mathcal{L} = \mathcal{S} \oplus SR(\mathcal{L})$$

Case II. Suppose that \mathcal{L} has an ideal \mathcal{I} that is properly contained in $SR(\mathcal{L})$.

We consider the minimal ideal \mathcal{S}_0 of \mathcal{L} that is contained in $SR(\mathcal{L})$. By induction on the dimension of $SR(\mathcal{L})$, we suppose that the theorem is true for all Lie algebras with a solvable radical of dimension less than $\dim SR(\mathcal{L})$. $SR(\mathcal{L})/\mathcal{S}_0$ is the solvable radical of $\mathcal{L}/\mathcal{S}_0$, and $\dim SR(\mathcal{L})/\mathcal{S}_0 < \dim SR(\mathcal{L})$, so there exists a subalgebra $\mathcal{S}_1/\mathcal{S}_0$ of $\mathcal{L}/\mathcal{S}_0$ such that $\mathcal{L}/\mathcal{S}_0 = \mathcal{S}_1/\mathcal{S}_0 + SR(\mathcal{L})/\mathcal{S}_0$, so $\mathcal{L} = \mathcal{S}_1 + SR(\mathcal{L})$. \mathcal{S}_0 is the solvable radical of \mathcal{S}_1 and since $\dim \mathcal{S}_0 < \dim SR(\mathcal{L})$, there exists a semisimple subalgebra \mathcal{S} so that $\mathcal{S}_1 = \mathcal{S} + \mathcal{S}_0$. Hence

$$\mathcal{L} = \mathcal{S} + \mathcal{S}_0 + SR(\mathcal{L})$$

$$\Rightarrow \mathcal{L} = \mathcal{S} + SR(\mathcal{L})$$

Since \mathcal{S} is semisimple $\mathcal{S} \cap SR(\mathcal{L}) = 0$ and therefore

$$\mathcal{L} = \mathcal{S} \oplus SR(\mathcal{L})$$

□

The semisimple subalgebra \mathcal{S} is called a *Levi subalgebra* of \mathcal{L} . In general the Levi subalgebra is not unique. On the other hand, the solvable radical $SR(\mathcal{L})$ is unique.

Theorem 14 (Malcev) *Let \mathcal{L} be a Lie algebra with solvable radical $SR(\mathcal{L})$. Suppose that \mathcal{S}_1 and \mathcal{S}_2 are semisimple subalgebras of \mathcal{L} with*

$$\mathcal{L} = \mathcal{S}_1 \oplus SR(\mathcal{L}) = \mathcal{S}_2 \oplus SR(\mathcal{L})$$

then there exists an automorphism f of \mathcal{L} such that $f(\mathcal{S}_1) = \mathcal{S}_2$.

Proof. Let $\mathcal{L} = \mathcal{S}_1 \oplus SR(\mathcal{L}) = \mathcal{S}_2 \oplus SR(\mathcal{L})$ then for every $x_1 \in \mathcal{S}_1$, there exists a unique $x_2 \in \mathcal{S}_2$ such that $x_1 = x_2 + SR(\mathcal{L})$ and $x_1 - x_2 \in SR(\mathcal{L})$. The mapping $x_1 \mapsto x_2$ is an isomorphism from \mathcal{S}_1 onto \mathcal{S}_2 . This isomorphism can be extended to an isomorphism of \mathcal{L} . For details see (3) pg. 156 □

CHAPTER 4. Algorithms

Levi's theorem states that an arbitrary finite dimensional Lie algebra over a field \mathcal{F} of characteristic 0 can be written as the direct sum of the Levi subalgebra \mathcal{S} and the solvable radical $SR(\mathcal{L})$. In this chapter we will give an algorithm to obtain a basis for the Levi subalgebra of a given finite dimensional Lie algebra over \mathcal{F} . This algorithm is taken from (1). Since we need to compute the product space $[\mathcal{L}, \mathcal{L}]$ and the solvable radical $SR(\mathcal{L})$ to obtain a basis for the Levi subalgebra, we first give simple algorithms to find the product space $[\mathcal{L}, \mathcal{L}]$ and the solvable radical $SR(\mathcal{L})$.

Algorithm: Product Space

Let \mathcal{L} be a finite dimensional Lie algebra and let \mathcal{I} and \mathcal{J} be two subspaces with basis $\{x_1, \dots, x_m\}$ and $\{y_1, \dots, y_n\}$ respectively. The algorithm for the product space is,

Input: A finite dimensional Lie algebra \mathcal{L} and subspaces \mathcal{I} and \mathcal{J} with basis $\{x_1, \dots, x_m\}$ and $\{y_1, \dots, y_n\}$ respectively.

Output: A basis of the product space $[\mathcal{I}, \mathcal{J}]$.

Step 1. Compute the set A of elements $[x_i, y_j]$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

Step 2. Calculate a maximal linearly independent subset B of A , return B .

Algorithm: Solvable Radical $SR(\mathcal{L})$

From Proposition 2.0.19, we have that $SR(\mathcal{L}) = \{x \in \mathcal{L} : \text{Tr}(ad_{\mathcal{L}}x \cdot ad_{\mathcal{L}}y) = 0\}$ for all $x \in \mathcal{L}$ and $y \in [\mathcal{L}, \mathcal{L}]$. By using this Proposition, we can find an algorithm to obtain

the solvable radical $SR(\mathcal{L})$. Let $\{x_1, x_2, \dots, x_m\}$ be a basis of \mathcal{L} and let $\{y_1, y_2, \dots, y_k\}$ be a basis of the product space $[\mathcal{L}, \mathcal{L}]$. Let $x = \sum_{i=1}^m \alpha_i x_i \in \mathcal{L}$, then Proposition 2.0.19 implies that

$$x = \sum_{i=1}^m \alpha_i x_i \in SR(\mathcal{L}) \Leftrightarrow \sum_{i=1}^m \alpha_i \operatorname{Tr}(ad_{\mathcal{L}} x_i \cdot ad_{\mathcal{L}} y_j) = 0$$

$1 \leq j \leq k$. Hence the algorithm for the solvable radical is,

Input. Finite dimensional Lie algebra \mathcal{L} over a field \mathcal{F} of characteristic 0 with a basis $\{x_1, x_2, \dots, x_m\}$

Output: A basis of the solvable radical $SR(\mathcal{L})$

Step 1. Find a basis $\{y_1, y_2, \dots, y_k\}$ of the product space $[\mathcal{L}, \mathcal{L}]$

Step 2. Calculate $ad_{\mathcal{L}} x_i$ and $ad_{\mathcal{L}} y_j$ for $1 \leq i \leq m, 1 \leq j \leq k$.

Step 3. $x = \sum_{i=1}^m \alpha_i x_i \in SR(\mathcal{L}) \Leftrightarrow \sum_{i=1}^m \alpha_i \operatorname{Tr}(ad_{\mathcal{L}} x_i \cdot ad_{\mathcal{L}} y_j) = 0$

where $1 \leq j \leq k$, then find α_i 's by solving linear equations.

Algorithm: Levi subalgebra

Let \mathcal{L} be a finite dimensional Lie algebra with a basis over a field \mathcal{F} of characteristic 0. First we obtain a basis $\{r_1, r_2, \dots, r_n\}$ of the solvable radical $SR(\mathcal{L})$.

Case 1. The solvable radical $SR(\mathcal{L})$ is abelian, i.e. $[r_i, r_j] = 0$ for all $1 \leq i, j \leq n$.

First we find a complementary basis $\{x_1, x_2, \dots, x_m\}$ to the $SR(\mathcal{L})$ in \mathcal{L} such that

$$\{x_1, x_2, \dots, x_m, r_1, r_2, \dots, r_n\}$$

span the Lie algebra \mathcal{L} . Let \bar{x}_i be the image of x_i in $\mathcal{L}/SR(\mathcal{L})$, so there corresponds a basis $\{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_m\}$ of $\mathcal{L}/SR(\mathcal{L})$ to the basis $\{x_1, x_2, \dots, x_m\}$ with $[\bar{x}_i, \bar{x}_j] = \sum_{k=1}^m c_{ij}^k \bar{x}_k$ where $1 \leq i, j, k \leq m$. By Levi's theorem, there exists a complementary basis to the $SR(\mathcal{L})$ which spans a Levi subalgebra. A basis for any subspace complementary to the $SR(\mathcal{L})$ can be written as

$$y_1 = x_1 + \sum_{j=1}^n \alpha_{1j} r_j, \quad y_2 = x_2 + \sum_{j=1}^n \alpha_{2j} r_j, \quad \dots \dots \dots y_m = x_m + \sum_{j=1}^n \alpha_{mj} r_j$$

By Levi's Theorem, we can choose α_{kj} 's $1 \leq k \leq m$, $1 \leq j \leq n$, so that $\{y_1, y_2, \dots, y_m\}$ span a semisimple Lie subalgebra which is isomorphic to $\mathcal{L}/SR(\mathcal{L})$. By our construction y_1, y_2, \dots, y_m have the same commutation relations as $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_m$. Therefore this Lie algebra is isomorphic to $\mathcal{L}/SR(\mathcal{L})$ via the map

$$y_i \longmapsto \bar{x}_i$$

$1 \leq k \leq m$. Since \bar{x}_i and y_i have the same commutation relations, for every $1 \leq i, j \leq m$,

$$[y_i, y_j] = \sum_{k=1}^m c_{ij}^k y_k$$

By replacing y_i with $x_i + \sum_{s=1}^n \alpha_{is} r_s$, we get

$$[x_i + \sum_{s=1}^n \alpha_{is} r_s, x_j + \sum_{t=1}^n \alpha_{jt} r_t] = \sum_{k=1}^m c_{ij}^k (x_k + \sum_{p=1}^n \alpha_{kp} r_p)$$

$$[x_i, x_j] + [x_i, \sum_{t=1}^n \alpha_{jt} r_t] + [\sum_{s=1}^n \alpha_{is} r_s, x_j] + [\sum_{s=1}^n \alpha_{is} r_s, \sum_{t=1}^n \alpha_{jt} r_t] = \sum_{k=1}^m c_{ij}^k x_k + \sum_{k,p=1}^{m,n} c_{ij}^k \alpha_{kp} r_p$$

$$[x_i, x_j] + \sum_{t=1}^n \alpha_{jt} [x_i, r_t] + \sum_{s=1}^n \alpha_{is} [r_s, x_j] + \sum_{s,t=1}^n \alpha_{is} \alpha_{jt} [r_s, r_t] = \sum_{k=1}^m c_{ij}^k x_k + \sum_{k,p=1}^{m,n} c_{ij}^k \alpha_{kp} r_p$$

Since $SR(\mathcal{L})$ is abelian, $[r_i, r_j] = 0$, $\forall 1 \leq i, j \leq n$, so that $\sum_{s,t=1}^n \alpha_{is} \alpha_{jt} [r_s, r_t] = 0$ and therefore

$$\begin{aligned} [x_i, x_j] + \sum_{t=1}^n \alpha_{jt} [x_i, r_t] + \sum_{s=1}^n \alpha_{is} [r_s, x_j] &= \sum_{k=1}^m c_{ij}^k x_k + \sum_{k,p=1}^{m,n} c_{ij}^k \alpha_{kp} r_p \\ \Rightarrow \sum_{t=1}^n \alpha_{jt} [x_i, r_t] + \sum_{s=1}^n \alpha_{is} [r_s, x_j] + \sum_{k,p=1}^{m,n} c_{ji}^k \alpha_{kp} r_p &= [x_j, x_i] + \sum_{k=1}^m c_{ij}^k x_k \end{aligned}$$

We have $\frac{(m-1)m}{2}$ equations with mn unknowns. It follows from Levi's theorem that this

linear equation system has a solution, thus we can find all the α_{kj} 's. Hence our algorithm for the abelian radical case is,

Input: Finite dimensional Lie algebra \mathcal{L} with a basis over a field \mathcal{F} of characteristic 0 with an Abelian solvable radical.

Output: A basis of a Levi subalgebra

Step 1. Compute a basis $\{r_1, r_2, \dots, r_n\}$ for the solvable radical $SR(\mathcal{L})$

Step 2. Find a complementary basis to $\{x_1, x_2, \dots, x_m\}$ to $SR(\mathcal{L})$ and compute the basis $\{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_m\}$ for the quotient algebra $\mathcal{L}/SR(\mathcal{L})$

Step 3. Set $y_i = x_i + \sum_{j=1}^m \alpha_{ij} r_j$ and require that y_i and \bar{x}_i have the same commutator relations, then find α_{ij} 's

Case II. The solvable radical $SR(\mathcal{L})$ is not abelian. Let

$$SR(\mathcal{L}) = \mathcal{R}_1 \supset \mathcal{R}_2 \supset \dots \supset \mathcal{R}_k \supset \mathcal{R}_{k+1} = 0$$

be a descending series of ideals of $SR(\mathcal{L})$ such that $[\mathcal{R}_i, \mathcal{R}_i] \subset \mathcal{R}_{i+1}$. Since the solvable radical is solvable, there exists an integer $k > 0$ such that $SR(\mathcal{L})^{(k)} = 0$, so we can choose $SR(\mathcal{L})^{(k)} = \mathcal{R}_k$. If the solvable radical is nilpotent, we can choose the lower central series of $SR(\mathcal{L})$. Note that if $SR(\mathcal{L})$ is not nilpotent, it is not necessary that $SR(\mathcal{L})^k$ is equal to 0 for some integer $k > 0$. Let $\{x_1, x_2, \dots, x_m\}$ be a complementary basis in \mathcal{L} to the $SR(\mathcal{L})$, then we have the following commutation relations.

$$[x_i, x_j] = \sum_{k=1}^m c_{ij}^k x_k \pmod{\mathcal{R}_1}$$

which means that $\{x_1, x_2, \dots, x_m\}$ span a Levi subalgebra module \mathcal{R}_1 . For $t = 1, 2, \dots$ we construct $u_i^t \in SR(\mathcal{L})$ such that $y_i^t = x_i + u_i^t$ span a Levi subalgebra modulo \mathcal{R}_t , $1 \leq i, j \leq m$. When we compute for $t = k+1$, we find that $\{y_i^{k+1}\}$ span a Levi subalgebra module $\mathcal{R}_{k+1} = 0$, i.e., $\{y_i^{k+1}\}$ span a Levi subalgebra of \mathcal{L} .

We start with $t = 1$, set $y_i^1 = x_i$ and write $\mathcal{R}_t = \mathcal{U}_t \oplus \mathcal{R}_{t+1}$ where \mathcal{U}_t is a complement to \mathcal{R}_{t+1} in \mathcal{R}_t . We write

$$y_i^{t+1} = y_i^t + u_i^t$$

and we require that $\{y_i^{t+1}\}$ span a Levi algebra modulo \mathcal{R}_{t+1} . Note that $u_i^t = \sum_k \alpha_k u_k^t \in \mathcal{U}_t$. Since $\{y_i^{t+1}\}$ span a Levi subalgebra modulo \mathcal{R}_{t+1} , we have the following commutation relations

$$[y_i^{t+1}, y_j^{t+1}] = \sum_k c_{ij}^k y_k^{t+1} \pmod{\mathcal{R}_{t+1}}$$

We know that $[\mathcal{R}_t, \mathcal{R}_t] \subset \mathcal{R}_{t+1}$, so by replacing y_i^{t+1} with $y_i^t + u_i^t$, we find that

$$[y_i^t, y_j^t] + [y_i^t, u_j^t] + [u_i^t, y_j^t] + [u_i^t, u_j^t] = \sum_k c_{ij}^k y_k^{t+1} \pmod{\mathcal{R}_{t+1}}$$

$$[y_i^t, u_j^t] + [u_i^t, y_j^t] + \sum_k c_{ij}^k u_k^t = -[y_i^t, y_j^t] + \sum_k c_{ij}^k y_k^{t+1} \pmod{\mathcal{R}_{t+1}}$$

These are linear equations for u_i^t . Since the equations are modulo \mathcal{R}_{t+1} , the left handside and the right handside can be viewed as elements of \mathcal{U}_t , we can work inside of this space when we are solving the equations. Also we note that working inside this space allow us to get rid of nonlinearity. We can apply Levi's theorem to the Lie subalgebra $\mathcal{L}/\mathcal{R}_{t+1}$, so that these linear equations have a solution. Hence our algorithm for the nonabelian radical case is,

Input: Finite dimensional Lie algebra \mathcal{L} with a basis over a field \mathcal{F} of characteristic 0 with a nonabelian solvable radical

Output: A basis of a Levi subalgebra

Step 1- Compute the solvable radical $SR(\mathcal{L})$

Step 2- Compute the descending series $SR(\mathcal{L}) = \mathcal{R}_1 \supset \dots \supset \mathcal{R}_k \supset \mathcal{R}_{k+1} = 0$ of

$$SR(\mathcal{L}) \text{ such that } [\mathcal{R}_i, \mathcal{R}_i] \subset \mathcal{R}_{i+1}$$

Step 3- Find a complementary basis $\{x_1, x_2, \dots, x_m\}$ to \mathcal{R}_1 and set $y_i^1 = x_i$

Step 4- For $1 \leq t \leq k$,

Step 4.1- Compute a complement \mathcal{U}_t in \mathcal{R}_t to \mathcal{R}_{t+1}

Step 4.2- Set $y_i^{t+1} = y_i^t + u_i^t$ where u_i^t are unknown elements of \mathcal{U}_t

Step 4.3- Compute the equations

$$[y_i^t, u_j^t] + [u_i^t, y_j^t] + \sum_k c_{ij}^k u_k^t = -[y_i^t, y_j^t] + \sum_k c_{ij}^k y_k^{t+1} \pmod{\mathcal{R}_{t+1}}$$

of u_i^t and solve them.

Step 5- Return the subalgebra spanned by $\{y_i^{k+1}\}$.

The subalgebra spanned by $\{y_i^{k+1}\}$ is the Levi subalgebra of \mathcal{L} .

Example 4.0.23 We find the Levi subalgebra of the Lie algebra \mathcal{L} spanned by

$$\left\{ x_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, x_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, x_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, x_4 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

The multiplication table of \mathcal{L} is,

	x_1	x_2	x_3	x_4
x_1	0	$4x_1 - 4x_3 - 2x_4$	$x_1 - x_2 - x_4$	0
x_2	$-4x_1 + 4x_3 + 2x_4$	0	$-2x_1 + 2x_3 + 2x_4$	$-4x_1 + 4x_3 + 2x_4$
x_3	$-x_1 + x_2 + x_4$	$2x_1 - 2x_3 - 2x_4$	0	$-x_1 + x_2 + x_4$
x_4	0	$4x_1 - 4x_3 - 2x_4$	$x_1 - x_2 - x_4$	0

Step 1. First, we find the solvable radical $SR(\mathcal{L})$.

The maximal linearly independent set in the multiplication table is spanned by $x_1 - x_2 - x_4$, $2x_1 - 2x_3 - x_4$ and $-x_1 + x_3 + x_4$, so these three linear combination of the elements span the product space $[\mathcal{L}, \mathcal{L}]$. Thus the product space is spanned by

$$\left\{ y_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, y_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, y_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

Proposition 2.0.19 implies that

$$x = \sum_{i=1}^4 \alpha_i x_i \in SR(\mathcal{L}) \Leftrightarrow \sum_{i=1}^4 \alpha_i \operatorname{Tr}(ad_{\mathcal{L}} x_i \cdot ad_{\mathcal{L}} y_j) = 0, \quad 1 \leq j \leq 3$$

Thus, by solving 3 equations with 4 unknowns, we find that $\alpha_1 = -\alpha_4$, $\alpha_2 = \alpha_3 = 0$, and therefore $SR(\mathcal{L}) = \operatorname{Span} \{ \alpha_1 x_1 - \alpha_1 x_4 \}$, say $x_1 - x_4 = r$ thus,

$$SR(\mathcal{L}) = \operatorname{Span} \{ r \} = \operatorname{Span} \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

Step 2. A complement in \mathcal{L} to the $SR(\mathcal{L})$ is spanned by $\{x_1, x_2, x_3\}$. The quotient algebra $\mathcal{L}/SR(\mathcal{L})$ is spanned by \bar{x}_1, \bar{x}_2 and \bar{x}_3 where \bar{x}_i is the image of x_i in $\mathcal{L}/SR(\mathcal{L})$.

We compute the commutation relations of \bar{x}_i for $1 \leq i \leq 3$, and find that

$$[\bar{x}_1, \bar{x}_2] = 2\bar{x}_1 - 4\bar{x}_3, \quad [\bar{x}_1, \bar{x}_3] = -\bar{x}_2, \quad [\bar{x}_2, \bar{x}_3] = 2\bar{x}_3$$

Step 3. Now we set

$$z_1 = x_1 + \alpha r, \quad z_2 = x_2 + \beta r, \quad z_3 = x_3 + \gamma r$$

and we require that z_1, z_2 and z_3 span a Levi subalgebra. Now, the problem is to find the three unknowns α, β and γ such that z_1, z_2, z_3 span a semisimple Lie algebra that is isomorphic to $\mathcal{L}/SR(\mathcal{L})$. Since z_1, z_2 and z_3 have the same commutation relations as \bar{x}_1, \bar{x}_2 and \bar{x}_3 , we have

$$[z_1, z_2] = 2z_1 - 4z_3, \quad [z_1, z_3] = -z_2, \quad [z_2, z_3] = 2z_3$$

These equalities imply that

$$[x_1 + \alpha r, x_2 + \beta r] = 2(x_1 + \alpha r) - 4(x_3 + \gamma r) \quad (1)$$

$$[x_1 + \alpha r, x_3 + \gamma r] = -(x_2 + \beta r)$$

$$[x_2 + \beta r, x_3 + \gamma r] = 2(x_3 + \gamma r)$$

From the first equation we find that $\alpha - 2\gamma = 1$ and from the third equation we find that $\gamma = -1$. Hence $\alpha = \gamma = -1$. The second equation implies that $\beta = -1$. Thus $z_1 = x_1 - r$, $z_2 = x_2 - r$, $z_3 = x_3 - r$ and therefore the Levi subalgebra is spanned by

$$\left\{ z_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, z_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, z_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

Example 4.0.24 Let \mathcal{L} be an 8-dimensional Lie algebra with basis $\{x_1, x_2, \dots, x_8\}$ and multiplication table

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
x_1	0	$2x_2$	$-2x_3$	0	$-x_5$	0	0	x_8
x_2	$-2x_2$	0	x_1	0	x_8	0	0	0
x_3	$2x_3$	$-x_1$	0	0	0	0	0	x_5
x_4	0	0	0	0	$-x_5$	$-2x_6$	$-x_6 - 2x_7$	$-x_8$
x_5	x_5	$-x_8$	0	x_5	0	0	0	$-x_6$
x_6	0	0	0	$2x_6$	0	0	0	0
x_7	0	0	0	$x_6 + 2x_7$	0	0	0	0
x_8	$-x_8$	0	$-x_5$	x_8	x_6	0	0	0

We compute a basis of the Levi subalgebra of \mathcal{L} . This Lie algebra is taken from (6).

Step 1. First, we compute the solvable radical $SR(\mathcal{L})$.

The product space is spanned by $\{x_1, x_2, x_3, x_5, x_6, x_7, x_8\}$. Now we can use the Proposition 2.0.19.

$$x = \sum_{i=1}^8 \alpha_i x_i \in SR(\mathcal{L}) \Leftrightarrow \sum_{i=1}^8 \alpha_i Tr(ad_{\mathcal{L}} x_i \cdot ad_{\mathcal{L}} x_j) = 0 \text{ for } j = 1, 2, 3, 5, 6, 7, 8$$

We find that the solvable radical $SR(\mathcal{L})$ is spanned by $\{x_4, x_5, x_6, x_7, x_8\}$. We denote the solvable radical by \mathcal{R} . A complement in \mathcal{L} to \mathcal{R} is spanned by x_1, x_2, x_3 . Let $\bar{x}_1, \bar{x}_2, \bar{x}_3$ be the images of x_1, x_2, x_3 in \mathcal{L}/\mathcal{R} , then we have the following commutation relations

$$[\bar{x}_1, \bar{x}_2] = 2\bar{x}_2, \quad [\bar{x}_1, \bar{x}_3] = -2\bar{x}_3, \quad [\bar{x}_2, \bar{x}_3] = \bar{x}_1$$

Step 2. We see that the solvable radical is not Abelian, then we calculate the derived series of the solvable radical \mathcal{R} .

$$\mathcal{R} = \mathcal{R}^{(1)} = \text{span} \{x_4, x_5, x_6, x_7, x_8\}$$

$$\mathcal{R}^{(2)} = [\mathcal{R}^{(1)}, \mathcal{R}^{(1)}] = \text{span} \{x_5, x_6, x_7, x_8\}$$

$$\mathcal{R}^{(3)} = [\mathcal{R}^{(2)}, \mathcal{R}^{(2)}] = \text{span} \{x_6\}$$

$$\mathcal{R}^{(4)} = [\mathcal{R}^{(3)}, \mathcal{R}^{(3)}] = 0$$

Hence

$$\mathcal{R} = \mathcal{R}^{(1)} \supset \mathcal{R}^{(2)} \supset \mathcal{R}^{(3)} \supset \mathcal{R}^{(4)} = 0$$

is the derived series of the solvable radical \mathcal{R} with $[\mathcal{R}^{(i)}, \mathcal{R}^{(i)}] \subset \mathcal{R}^{(i+1)}$.

Step 3. Since $\{x_1, x_2, x_3\}$ is a complement in \mathcal{L} to \mathcal{R} , so initially we set

$$y_1^1 = x_1, \quad y_2^1 = x_2, \quad y_3^1 = x_3$$

Step 4. x_4 is a complement in $\mathcal{R}^{(1)}$ to $\mathcal{R}^{(2)}$ so that we write

$$y_1^2 = y_1^1 + \alpha x_4 = x_1 + \alpha x_4$$

$$y_2^2 = y_2^1 + \beta x_4 = x_2 + \beta x_4$$

$$y_3^2 = y_3^1 + \gamma x_4 = x_3 + \gamma x_4$$

then we have the following commutation relations.

$$[y_1^2, y_2^2] = 2y_2^2 \pmod{\mathcal{R}^{(2)}}, \quad [y_1^2, y_3^2] = -2y_3^2 \pmod{\mathcal{R}^{(2)}}, \quad [y_2^2, y_3^2] = y_1^2 \pmod{\mathcal{R}^{(2)}}$$

The first commutation relation implies that

$$[x_1 + \alpha x_4, x_2 + \beta x_4] = 2(x_2 + \beta x_4) \pmod{\mathcal{R}^{(2)}}, \text{ then } \beta = 0.$$

The second commutation relation implies that

$$[x_1 + \alpha x_4, x_3 + \gamma x_4] = -2(x_3 + \gamma x_4) \pmod{\mathcal{R}^{(2)}}, \text{ then } \gamma = 0.$$

The third one implies that

$$[x_2 + \beta x_4, x_3 + \beta x_4] = x_1 + \alpha x_4 \pmod{\mathcal{R}^{(2)}}, \text{ so } \alpha = 0. \text{ Hence}$$

$$y_1^2 = x_1, \quad y_2^2 = x_2, \quad y_3^2 = x_3$$

Now, $\{x_5, x_7, x_8\}$ is a complement in $\mathcal{R}^{(2)}$ to $\mathcal{R}^{(3)}$, so that we write

$$y_1^3 = y_1^2 + a_1 x_5 + a_2 x_7 + a_3 x_8 = x_1 + a_1 x_5 + a_2 x_7 + a_3 x_8$$

$$y_2^3 = y_2^2 + b_1 x_5 + b_2 x_7 + b_3 x_8 = x_2 + b_1 x_5 + b_2 x_7 + b_3 x_8$$

$$y_3^3 = y_3^2 + c_1 x_5 + c_2 x_7 + c_3 x_8 = x_3 + c_1 x_5 + c_2 x_7 + c_3 x_8$$

and we have the following commutation relations,

$$[y_1^3, y_2^3] = 2y_2^3 \pmod{\mathcal{R}^{(3)}}, \quad [y_1^3, y_3^3] = -2y_3^3 \pmod{\mathcal{R}^{(3)}}, \quad [y_2^3, y_3^3] = y_1^3 \pmod{\mathcal{R}^{(3)}}$$

The first commutation relation implies that

$$[x_1 + a_1 x_5 + a_2 x_7 + a_3 x_8, x_2 + b_1 x_5 + b_2 x_7 + b_3 x_8] = 2(x_2 + b_1 x_5 + b_2 x_7 + b_3 x_8) \pmod{\mathcal{R}^{(3)}}$$

then we have $b_1 = b_2 = 0$, and $a_1 = -b_3$. The second one implies that

$$[x_1 + a_1 x_5 + a_2 x_7 + a_3 x_8, x_3 + c_1 x_5 + c_2 x_7 + c_3 x_8] = -2(x_3 + c_1 x_5 + c_2 x_7 + c_3 x_8) \pmod{\mathcal{R}^{(3)}}$$

then we find that $c_2 = c_3 = 0$, and $c_1 = a_3$. The last commutation relation implies that

$$[x_2 + b_1 x_5 + b_2 x_7 + b_3 x_8, x_3 + c_1 x_5 + c_2 x_7 + c_3 x_8] = x_1 + a_1 x_5 + a_2 x_7 + a_3 x_8 \pmod{\mathcal{R}^{(3)}}$$

then $a_2 = b_2 = 0$, and $c_1 = a_3$. Say $c_1 = a_3 = u$ and $a_1 = -b_3 = v$, then

$$y_1^3 = x_1 + vx_5 + ux_8, \quad y_2^3 = x_2 - vx_8, \quad y_3^3 = x_3 + ux_5$$

Now, $\{x_6\}$ is a complement in $\mathcal{R}^{(3)}$ to $\mathcal{R}^{(4)}$, so that we set

$$y_1^4 = y_1^3 + ex_6 = x_1 + vx_5 + ux_8 + ex_6$$

$$y_2^4 = y_2^3 + fx_6 = x_2 - vx_8 + fx_6$$

$$y_3^4 = y_3^3 + gx_6 = x_3 + ux_5 + gx_6$$

and we have the following commutation relations,

$$[y_1^4, y_2^4] = 2y_2^4 \pmod{\mathcal{R}^{(4)}}, \quad [y_1^4, y_3^4] = -2y_3^4 \pmod{\mathcal{R}^{(4)}}, \quad [y_2^4, y_3^4] = y_1^4 \pmod{\mathcal{R}^{(4)}}$$

Note that $\mathcal{R}^{(4)} = 0$, and $\{y_1^4, y_2^4, y_3^4\}$ span a Levi subalgebra modulo 0. From the first commutation relation we have that

$$[x_1 + vx_5 + ux_8 + ex_6, x_2 - vx_8 + fx_6] = 2(x_2 - vx_8 + fx_6) \pmod{\mathcal{R}^{(4)}}, \text{ then } f = \frac{v^2}{2}$$

The second commutation relation implies that

$$[x_1 + vx_5 + ux_8 + ex_6, x_3 + ux_5 + gx_6] = -2(x_3 + ux_5 + gx_6) \pmod{\mathcal{R}^{(4)}} \text{ then } g = \frac{-u^2}{2}$$

The last commutation relation implies that

$$[x_2 - vx_8 + fx_6, x_3 + ux_5 + gx_6] = x_1 + vx_5 + ux_8 + ex_6 \pmod{\mathcal{R}^{(4)}}, \text{ so } e = -uv.$$

Hence

$$\begin{aligned} y_1^4 &= x_1 + vx_5 + ux_8 - uvx_6 \\ y_2^4 &= x_2 - vx_8 + \frac{v^2}{2}x_6 \\ y_3^4 &= x_3 + ux_5 - \frac{u^2}{2}x_6 \end{aligned}$$

Step 5- We conclude that $\{y_1^4, y_2^4, y_3^4\}$ span a Levi subalgebra of \mathcal{L} .

By direct computations, it can be seen that the multiplication table of the Levi subalgebra is,

	y_1^4	y_2^4	y_3^4
y_1^4	0	$2y_2^4$	$-2y_3^4$
y_2^4	$-2y_2^4$	0	y_1^4
y_3^4	$2y_3^4$	$-y_1^4$	0

In fact, this Levi subalgebra is isomorphic to $sl_2(\mathcal{L})$. We recall that the Levi subalgebra is not unique. Different choices of u and v give the isomorphic Levi subalgebras.

CONCLUSION

Levi decomposition is a very general tool to study the structure of Lie algebras since it is applicable to all Lie algebras. It is also useful in analyzing the dynamics of the quantum control systems. In this creative component, we studied the general theory and some algorithms to compute the Levi decomposition starting from a basis of an arbitrary finite dimensional Lie algebra.

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