

NODAL POINT DATA FOR INVERSE STURM-LIOUVILLE
PROBLEMS

A project submitted by

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Chapter 1

Introduction

1.1 FORM OF STURM-LIOUVILLE PROBLEMS

In this project, we focus on the class of boundary value problems named after the French mathematicians Charles-François Sturm (1803-1855) and Joseph Liouville (1809-1882). In a series of papers in 1836 and 1837, they set forth many properties of the class of boundary value problems associated with their names. In this section, we briefly describe this class of boundary value problems.

Many eigenvalue problems in quantum mechanics as well as classical physics fit into the class of differential equations of the form:

$$[p(x)y'(x)]' + [\lambda\rho(x) - q(x)]y(x) = 0 \quad (1.1)$$

on the interval (a, b) together with boundary conditions:

$$a_1y(a) + a_2y'(a) = 0, \quad b_1y(b) + b_2y'(b) = 0 \quad (1.2)$$

Equation (1.1) is called the *Sturm-Liouville differential equation* while the equation (1.1) together with the boundary conditions (1.2) is called the *Sturm-Liouville boundary value problem*.

Two prime examples are:

1. A vibrating string with space dependent tension $p(x)$ and mass density $\rho(x)$ is a good example from classical physics. The amplitude $y(x)$ of

the associated wave equation satisfies the differential equation:

$$[p(x)y'(x)]' + \lambda\rho(x)y(x) = 0 \quad (1.3)$$

which is of the form (1.1).

2. In quantum mechanics, the one-dimensional time independent Schrödinger equation has the form:

$$y''(x) + [\lambda - q(x)]y(x) = 0 \quad (1.4)$$

which is a special case of (1.1) with $p(x) = \rho(x) = 1$. Here, q is the potential and the eigenvalue λ corresponds to the energy in the particular mode associated with the state y , where $y(x)$ represents the amplitude of the wave at the point x .

Indeed, any second order linear ordinary differential equation can be transformed into the form (1.1) using the Liouville transformation (see [5], [12]).

1.2 THE DIRECT PROBLEM

The Sturm-Liouville problem (1.1) , (1.2)

1. Has a trivial solution and infinitely many nontrivial solutions. The values of λ for which there exists nontrivial solutions are called *eigenvalues*. The corresponding nontrivial solutions $y(x)$ are called *eigenfunctions*.
2. Is called regular if:
 - (a) a_i, b_i , $i = 1, 2$ are real.
 - (b) The coefficients p, q, ρ are real and continuous everywhere including the end points.
 - (c) $p, \rho > 0$ everywhere including the endpoints.

Remarks 1. For any regular Sturm-Liouville problem (see [5]), the following theorems are valid:

- (a) All the eigenvalues λ are real.
- (b) The eigenvalues are infinite in number and can be arranged in increasing order
 $\lambda_1 < \lambda_2 < \lambda_3 < \dots$ such that $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$.
- (c) For each eigenvalue λ_n , there exists a function $y_n(x)$, called the eigenfunction of the Sturm-Liouville problem, which has $n-1$ zeros and is unique to within an arbitrary multiplicative constant.
- (d) The set $\{y_n(x)\}_{n=1}^{\infty}$ is orthogonal with respect to the weight function $\rho(x)$ on the interval $[a, b]$. That is, if y_1 and y_2 are two eigenfunctions of the Sturm-Liouville problem (1.1) , (1.2) corresponding to the eigenvalues λ_1 and λ_2 , respectively, and $\lambda_1 \neq \lambda_2$, then $\int_a^b \rho(x)y_1(x)y_2(x)dx = 0$.
- (e) The eigenfunctions $y_n(x)$ form a “complete” set: In other words, any piecewise smooth function $f(x)$ can be represented by a generalized Fourier series of eigenfunctions

$$f(x) \sim \sum_{n=1}^{\infty} a_n y_n(x)$$

and this infinite series converges to $[f(x+) + f(x-)]/2$ for $a < x < b$ (if the coefficients a_n are properly chosen).

- 2. For Sturm-Liouville eigenvalue problems that are not “regular”, some of the above assertions may be valid.

The direct problem is to determine the eigenvalues λ_n and the corresponding eigenfunctions $y_n(x)$ such that the Sturm-Liouville problem (1.1) , (1.2) is satisfied. This has been a venerable subject for the past 150 years or so.

In the case of the vibrating string for instance, the direct problem is to

determine the vibrational frequencies λ_n of the system, from the knowledge of the physical parameters $p(x)$ and $\rho(x)$ together with the associated boundary conditions.

1.3 THE INVERSE PROBLEM

In a typical inverse spectral problem, the potential is to be determined from spectral data (eg. two sets of eigenvalues (see [2], [15], [18]), or one set of eigenvalues and norming constants (defined in section 2.3). More information on norming constants can be found in (see [4], [11], [20]). This is referred to as the inverse eigenvalue problem.

An alternative is to take as data the nodes (i.e. zeros) of the eigenfunctions, which are just as experimentally observable as eigenvalues in some situations. This is generally referred to as the “Inverse Nodal Problem” and the main idea is discussed in detail in chapter three.

Chapter 2

History of Inverse Sturm-Liouville Problems

2.1 THE STARTING POINT

Inverse problems probably originated from Greece in the 19th century or even earlier. The modern starting point for Inverse Sturm-Liouville problems however, can only be traced from the beginning of early 20th century, in the work of the Russian Astronomist/physicist Ambartsumian.

In his 1929 paper (see [1]), he showed that if the eigenvalues of the Sturm-Liouville problem:

$$y''(x) + [\lambda - q(x)]y(x) = 0 \tag{2.1}$$

$$y'(0) = y'(\pi) = 0$$

are $\lambda_n = n^2\pi^2$, then $q \equiv 0$.

This gave motivation to the field and led to the speculation that the potential can be recovered from the knowledge of eigenvalues. It has since been proved that in general, a single spectrum, as in the case of Ambartsumian, is insufficient to determine the potential uniquely.

2.2 ADVANCEMENT OF THE EIGENVALUE INVERSE PROBLEM

Since the mid 1940's, the problem (most probably due to its wide range of applications) has attracted many researchers. In this section, we state some of the most important results obtained using sets of eigenvalues.

In 1946, the Swedish mathematician Borg (see [2]), considered the Sturm-Liouville equation:

$$y''(x) + [\lambda - q(x)]y(x) = 0. \quad (2.2)$$

and two different boundary conditions

$$y'(0) - hy(0) = 0, y'(1) + Hy(1) = 0. \quad (2.3)$$

$$y'(0) - hy(0) = 0, y'(1) + H'y(1) = 0. \quad (2.4)$$

where $H \neq H'$.

He proved that if $\{\lambda_n\}_{n=1}^{\infty}$ is the spectrum corresponding to (2.2) , (2.3) and $\{\mu_n\}_{n=1}^{\infty}$ is the spectrum corresponding to (2.2) , (2.4), then the pair $\{\lambda_n, \mu_n\}$ uniquely determines the potential $q(x)$, where $q \in L^2$.

He further showed that if $h = H$ in (2.3), and q is symmetric about the midline (i.e. $q(1 - x) = q(x)$), then a single spectrum $\{\lambda_n\}_{n=1}^{\infty}$ is sufficient to determine the potential uniquely.

In 1949, Levinson (see [15]) considered the same problem but shortened the proofs considerably using complex analysis techniques.

Shortly afterwards in 1952, Marčenko (see [17]) proved that the two spectra are sufficient to determine not only the potential, but also the constants h, H and H' in the boundary conditions (2.3) , (2.4).

In 1964 Levitan (see [16]) obtained a complete resolution of the necessary and sufficient conditions on spectra, for the existence of an L^2 potential giving rise to those spectral values.

2.3 OTHER POSSIBLE DATA COMBINATIONS

Gel'fand and Levitan (see [4]) seem to have been the first to seek an alternative to sets of eigenvalues as data. In 1951, they proved that the potential could be recovered uniquely from the pair $\{\lambda_n, \rho_n\}_{n=1}^{\infty}$ where $\rho_n(q) = \|y(\cdot, q, \lambda_n)\|_{L^2}^2 / |y'(0, q, \lambda_n)|^2$, $n = 1, 2, \dots$ is a set of norming constants.

Another set of norming constants, (see [11, 20]) could be $\kappa_n(q) = \log |(y'(1, q, \lambda_n)) / (y'(0, q, \lambda_n))|$, $n = 1, 2, \dots$. This can be used in the case of Dirichlet boundary conditions.

In 1973 Hochstadt (see [9]) considered the case of partial knowledge of spectral values. He showed that if a complete spectrum was provided, but the second spectrum was missing information from a certain indexing set Λ , then q could only be recovered modulo a certain sum over that index set of eigenfunctions of a given q .

In 1978, Hochstadt and Lieberman (see [10]) showed that if $q(x)$ is known on an interval $[\frac{1}{2}, 1]$, then a single spectra is sufficient to determine it on the remainder of the interval.

Chapter 3

NODAL POINT DATA AND THE UNIQUENESS RESULT

3.1 Definition

Nodal point data for the eigenvalue problem (1.1) , (1.2) is the set $\{x_n^j\}_{n \geq 2, j=1,2,\dots,n-1}$ of the roots of the eigenfunction $y_n(x)$. (i.e. $\{x_n^j\}$ = j^{th} root of the n^{th} eigenfunction $y_n(x)$) . The form of $\{x_n^j\}$ depends on the parameters $p(x)$, $\rho(x)$ and $q(x)$ in (1.1) and the values of a_1 , a_2 , b_1 , b_2 , a and b in (1.2). If for instance $p(x) = \rho(x) = 1$, $q(x) \equiv 0$, $a_1 = b_1 = 1$, $a_2 = b_2 = 0$, $a = 0$, $b = 1$, then $x_n^j = \frac{j}{n}$ where $n \geq 2$ (see details in a later section).

Each eigenvalue λ_n is the square of a natural frequency. The most natural experiment then for finding the nodal positions is to excite the vibrating system at a natural frequency and take measurements of the positions where the system does not vibrate. These positions are the zeros of the eigenfunctions.

3.2 The Starting Point

McLaughlin, J. R. (see [19]) seems to have been the first to consider the nodal point inverse problem for the one-dimensional Schrödinger equations

on an interval with Dirichlet boundary conditions.

The motivation in considering nodal point data was the desire to determine an alternative to norming constants, when this data is difficult to measure. There was also the desire to seek an alternative to measuring a second set of eigenvalues.

In 1986, she considered the problem:

$$y''(x) + [\lambda - q(x)]y(x) = 0 \quad (3.1)$$

$$y(0) = y(1) = 0 \quad (3.2)$$

where $q \in L^2(0, 1)$. Let the zeros of the eigenfunctions be labelled as

$$0 < x_n^1 < x_n^2 < \dots < x_n^{n-1} < 1$$

where $n \geq 2$.

A fundamental set of solutions $y_1(x, q, \lambda)$ and $y_2(x, q, \lambda)$ of (3.1) satisfying the conditions

$$y_1(0, q, \lambda) = y_2'(0, q, \lambda) = 1. \quad (3.3)$$

$$y_1'(0, q, \lambda) = y_2(0, q, \lambda) = 0. \quad (3.4)$$

is known (see [12], [18], [20]) to satisfy

$$y_1(x, q, \lambda) = \cos \sqrt{\lambda}x + O \left[\frac{\exp |Im \sqrt{\lambda}| \sqrt{x}}{\sqrt{|\lambda|}} \right] \quad (3.5)$$

$$y_2(x, q, \lambda) = \frac{\sin \sqrt{\lambda}x}{\sqrt{\lambda}} + O \left[\frac{\exp |Im \sqrt{\lambda}| \sqrt{x}}{|\lambda|} \right] \quad (3.6)$$

It is also known (see [12], [20]) that the eigenvalues λ_n , $n = 1, 2, 3, \dots$ satisfy

$$\lambda_n = n^2 \pi^2 + c_0 + \alpha_n$$

where

$$\sum_{n=1}^{\infty} (\alpha_n)^2 < \infty \quad [(i.e.) \{ \alpha_n \} \in \ell^2] \text{ and } c_0 = \int_0^1 q(x) dx$$

From this, we know the following:

1. Any solution $y_n(x)$ of (3.1) can be written as a linear combination of $y_1(x, q, \lambda)$ and $y_2(x, q, \lambda)$. For existence of nontrivial solutions, the boundary conditions require the arbitrary constant corresponding to $y_1(x, q, \lambda)$ to vanish identically. The eigenfunctions are therefore determined by only $y_2(x, q, \lambda)$.
2. $\int_0^1 q(x) dx$, the average of q on $[0, 1]$ is known from the spectral data.

The uniqueness result she proved is stated precisely as follows:

Theorem 1 (Uniqueness Theorem) *Consider the eigenvalue problem*

$$y'' + (\lambda - q_i)y = 0, i = 1, 2. \tag{3.7}$$

$$y(0) = y(1) = 0 \tag{3.8}$$

Let $q_1, q_2 \in L^2$ with $\int_0^1 q_1 dx = \int_0^1 q_2 dx$.

Suppose that for each $n = 1, 2, \dots$ we can find $j(n) \in \{1, 2, \dots, n-1\}$ such that $x_n^{j(n)}(q_1) = x_n^{j(n)}(q_2)$ and such that this common set of nodes is dense in $(0, 1)$. Then $q_1 = q_2$ a.e.

Based on this theorem, we first ask the following:

1. ($q \in L^2$) Why choose $q \in L^2$?
2. (Boundary conditions) Why consider the boundary conditions of the form (3.2)?
3. (Dense subset) Is it possible to construct a dense subset of the nodes?

4. (Approximation) Can the roots of the eigenfunctions for any $q \in L^2$ be approximated by the corresponding roots for $q \equiv 0$? If so, how “good” is the approximation $x_n^{j(n)}(q) \approx x_n^{j(n)}(0)$?

The answers to the first two questions are given in the remark below, while the answers to the other two questions are given in the Lemmas that follow.

Remark. $L^2(0, 1)$, the Hilbert space of all real-valued square integrable functions on $[0, 1]$, was considered because though it is possible to work with other spaces of functions or measures, the basic ideas and the construction are clearest in L^2 and that the boundary conditions (3.2), though apparently naive, exhibits many interesting features and displays the basic techniques in a transparent form.

Lemma 1 (Approximation) *Let $q \in L^2(0, 1)$ and consider the eigenvalue problem (3.1), (3.2). Let $y_2(x, q, \lambda_n)$ be the eigenfunction corresponding to $\lambda = \lambda_n, n = 1, 2, \dots$. Then $x_n^j(q) = x_n^j(0) + O(\frac{1}{n^2})$ where $x_n^j(0) = \frac{j}{n}$.*

Lemma 2 (Dense subset for $q \equiv 0$) *The set of numbers $\frac{(m+1)}{(2^{k+1}-m)}, k = 0, 1, 2, \dots, m = 0, 1, 2, \dots, 2^k - 1$ is dense in $[0, 1]$.*

We know that the eigenvalues for $q \equiv 0$ are $\lambda_n = n^2\pi^2$ and the corresponding eigenfunctions are $\sin n\pi x$. Hence the set of numbers $(2^{k+1} - m)^2\pi^2, k = 0, 1, 2, \dots, m = 0, 1, 2, \dots, 2^k - 1$ is the set of all eigenvalues for (3.1), (3.2) corresponding to $q \equiv 0$.

The eigenfunction corresponding to the eigenvalue $\lambda = (2^{k+1} - m)^2\pi^2$ is $\sin((2^{k+1} - m)\pi x)$, so that $\frac{(m+1)}{(2^{k+1}-m)}$ is a zero of the eigenfunction. The set of numbers given in Lemma 2 therefore represents a selection of one node from each eigenfunction except the eigenfunction corresponding to $\lambda = \lambda_1 = \pi^2$. From the point of view of the notation for $q \equiv 0$,

$$x_{2^{k+1}-m}^{m+1} = \frac{m+1}{2^{k+1}-m}$$

Lemma 3 (Dense Subset for any $q \in L^2$) Let $q \in L^2(0, 1)$. For each integer $n \geq 2$, find $k = 0, 1, 2, \dots$ and $m = 0, 1, 2, \dots, 2^k - 1$ such that $n = 2^{k+1} - m$. For each fixed n , define $j(n) = m + 1$. Then $\left\{x_n^{j(n)}\right\}_{n=2}^\infty$ is dense in $(0, 1)$.

Lemma 4 (Smoothness of x_n^j) Define

$$d_q x_n^j[w] = \lim_{\epsilon \rightarrow 0} \frac{x_n^j(q + \epsilon w) - x_n^j(q)}{\epsilon}$$

Then for $q \in L^2(0, 1)$ with $x_n^j(q)$ as earlier defined, we have

$$\begin{aligned} d_q x_n^j[w] &= \frac{1}{[y_2'(x_n^j, q, \lambda_n)]^2} \left\{ \int_0^{x_n^j} [y_2(t, q, \lambda_n)]^2 w(t) dt \right\} \\ &\quad - \frac{\dot{y}_2(x_n^j, q, \lambda_n)}{\dot{y}_2(1, q, \lambda_n)} \frac{1}{y_2'(x_n^j, q, \lambda_n) y_2'(1, q, \lambda_n)} \left\{ \int_0^1 [y_2(t, q, \lambda_n)]^2 w(t) dt \right\} \end{aligned} \quad (3.9)$$

which is a linear bounded operator on $L^2(0, 1)$, where

$$\dot{y}_2(x_n^j, q, \lambda_n) = \frac{d}{d\lambda} y_2(x, q, \lambda)|_{\lambda=\lambda_n, x=x_n^j}$$

$$\dot{y}_2(1, q, \lambda_n) = \frac{d}{d\lambda} y_2(1, q, \lambda)|_{\lambda=\lambda_n}$$

In other words, x_n^j is a differentiable function of q .

Remark. The proofs of the four Lemmas are outlined in Appendix B. To prove the uniqueness theorem, we shall make use of these lemmas along with the known results given in section 3.2.

Proof of Uniqueness Theorem.

Let $x \in [0, 1]$ be fixed but arbitrarily chosen. Since $\left\{x_n^{j(n)}\right\}_{n \geq 2}$ is dense in $(0, 1)$, there exists a subsequence $n_k, k = 1, 2, \dots$ such that

$$\lim_{k \rightarrow \infty} x_{n_k}^{j(n_k)} = x$$

For the ease of notation, let $x_{n_k}^{j(n_k)} = x_k$.

Let $y_2 = y_2(t, q_1, \lambda_{n_k}(q_1))$ and $\bar{y}_2 = y_2(t, q_2, \lambda_{n_k}(q_2))$ be two solutions of (3.7), (3.8). From (3.7), we have

$$[y_2' \bar{y}_2 - \bar{y}_2' y_2]' = [q_1 - q_2 + \lambda_{n_k}(q_1) - \lambda_{n_k}(q_2)] y_2 \bar{y}_2 \quad (3.10)$$

Integrating (3.10) from 0 to x_k , we have

$$[y_2' \bar{y}_2 - \bar{y}_2' y_2]_0^{x_k} = 0 = \int_0^{x_k} [\lambda_{n_k}(q_1) - \lambda_{n_k}(q_2) + q_1 - q_2] y_2(x) \bar{y}_2(x) dx$$

Next, define

$$J_k = n_k^2 \pi^2 \int_0^{x_k} [\lambda_{n_k}(q_1) - \lambda_{n_k}(q_2) + q_1 - q_2] [y_2(x, q_1, \lambda_{n_k}(q_1)) \times y_2(x, q_2, \lambda_{n_k}(q_2))] dx \quad (3.11)$$

We note that $J_k = 0$, $k = 1, 2, 3, \dots$. We are done if we can show

$$0 = \lim_{k \rightarrow \infty} J_k = \int_0^x [q_1(t) - q_2(t)] dt \quad (3.12)$$

To this end we recall that

$$\lambda_n(q) = n^2 \pi^2 + \int_0^1 q(x) dx + \alpha_n, \quad \sum_{n=1}^{\infty} (\alpha_n)^2 < \infty$$

Since the average of q on the interval is known, we get $\int_0^1 (q_1 - q_2) dt = 0$. Therefore $\lambda_{n_k}(q_1) - \lambda_{n_k}(q_2) = \alpha_{n_k}(q_1) - \alpha_{n_k}(q_2) = B_k$ (say) with

$$\sum_{k=1}^{\infty} (B_k)^2 < \infty$$

From this, we conclude that the term $\lambda_{n_k}(q_1) - \lambda_{n_k}(q_2) \rightarrow 0$ as $k \rightarrow \infty$ in (3.11).

For $q \equiv 0$ the eigenfunctions corresponding to the eigenvalues $\lambda = \lambda_{n_k}$ are $\sin(n_k \pi x)$. Hence $[y_2(x, 0, \lambda_{n_k}(0))]^2 = \sin^2(n_k \pi x) = \left[\frac{1 - \cos 2n_k \pi x}{2} \right]$. Now it can be shown that there exists a constant m such that

$$\left| (n_k \pi)^2 [y_2(x, q_1, \lambda_{n_k}(q_1)) y_2(x, q_2, \lambda_{n_k}(q_2))] - \left[\frac{1 - \cos 2n_k \pi x}{2} \right] \right| \leq \frac{m}{n_k}$$

so that

$$n_k^2 \pi^2 [y_2(x, q_1, \lambda_{n_k}(q_1)) y_2(x, q_2, \lambda_{n_k}(q_2))] \approx \left[\frac{1 - \cos 2n_k \pi x}{2} \right]$$

Then (3.11) reduces to

$$J_k = \int_0^{x_k} (-q_1 + q_2) \left(\frac{1 - \cos 2n_k \pi t}{2} \right) dt \quad (3.13)$$

Finally, we know that

$$\int_0^{x_k} (q_2 - q_1) \cos 2n_k t dt \rightarrow 0 \text{ as } k \rightarrow \infty \quad (3.14)$$

We are thus left with

$$J_k = \int_0^{x_k} (q_2(t) - q_1(t)) dt \quad (3.15)$$

Taking limits in (3.15), we are left with (3.12) as desired and we are done since x was chosen arbitrary.

CLOSING REMARKS. The main assumption in this theorem is that $x_n^j(q) = x_n^j(0) + O(\frac{1}{n^2})$. This has been proved via the Fréchet differentiability of $x_n^j(q)$ which is very technical.

I plan to attempt to prove the same assumption using simpler methods. In particular, I shall try to employ well known inequalities, which may also call for the change of the framework of the set up for the potential q .

Chapter 4

ADVANCEMENT OF THE INVERSE NODAL PROBLEMS

4.1 McLaughlin's work with collaborators

1988: With Hald (see [6]).

They surveyed other papers and generalized the uniqueness theorem to the case of general boundary conditions using McLaughlin's earlier method. In the same year, they established algorithms and bounds associated with the uniqueness result and presented some numerical results.

1989: With Hald (see [7]).

They considered the problem

$$(Py_x)_x + \lambda\rho y = 0, \quad 0 < x < L, \quad y_x(0) = y(L) = 0.$$

and proved that any dense subset of nodal positions determine P and ρ (simultaneously) uniquely (up to two multiplicative constants). They also established algorithms for the reconstruction.

1997 With Hald (see [8]).

They considered the problem of longitudinal vibration of a beam whose

governing equation is

$$(Py_x)_x + \lambda\rho y = 0, 0 \leq x \leq L$$

$$y(0) = y(L) = 0$$

where P is the modulus of elasticity and ρ is the density. They showed that density and modulus of elasticity are uniquely determined from a dense subset of the nodes when either of them is fixed provided both are of bounded variation. They also established an algorithm for the reconstruction.

4.2 Related Work by other Authors

1995: Shen and Tsai(see [21])

They considered the problem

$$y''(x) + \lambda\rho(x)y(x) = 0 \quad y(0) = y(1) = 0$$

Using eigenvalues and nodal points, they constructed a sequence of piecewise linear continuous functions which converge to $\rho^{(-1/2)}(x)$ uniformly. They also obtained a formula for derivatives of $\rho^{(-1/2)}(x)$ in terms of the same data.

1996:Yang, X. F. (see [22])

He considered the problem

$$y''(x) + [\lambda - q(x)]y(x) = 0$$

$$y(0) \cos \alpha + y'(0) \sin \alpha = 0$$

$$y(1) \cos \beta + y'(1) \sin \beta = 0$$

where $0 \leq \alpha, \beta < \pi, q \in L^1[0, 1]$ and proved the uniqueness theorem for the case where either α or $\beta = 0$ but not both, using the same method as McLaughlin. He also obtained an explicit solution for $\alpha, \beta \in [0, \pi)$ and

$q \in L^1[0, 1]$ from the nodes which eliminated the need of a numerical solution given earlier by McLaughlin and Hald.

1997: Law, C. K, Shen, C. L and Yang, C. F. (see [13]).

They considered the same problem as Yang, X. F. and extended the use of the nodal point data together with the integral average of q to reconstruction of not only the potential function and the boundary conditions, but also the derivatives of the potential function. Their algorithms have a first order convergence rate and seem to be useful in studying the smoothness of the potential function. In particular, they proved that given any non-negative integer N , when q is C^{N+1} , its derivatives $q^{(k)}(x)$ ($k = 1, 2, \dots, N$) can be approximated using nodal data and the convergence is order one.

2001: Law, C. K. and Tsay, J. (see [14])

Considered the same problem as Yang, X. F. (see [22]) and showed that the space of all (q, α, β) such that $\int_0^1 q = 0$ under certain metric is homeomorphic to the partition set of all asymptotically equivalent nodal sequences induced by an equivalence relation.

2001: Law, C. K, Shen, C. L and Yang, C. F. (see [13])

They considered Yang's 1996 problem and gave a simple and direct reconstruction formula for any $q \in L^1(0, 1)$ that improved Yang's earlier result.

2001: Yang, X. F. (see [23])

He considered his earlier problem of 1996 and showed that any dense subset of the nodal set in $(0, b)$, $b \in (\frac{1}{2}, 1]$ determines the potential and the boundary data uniquely.

2002: Law, C. K., Tsay, J., Ting, C. Y and Cheng, Y. H (see [3])

They considered the 2001 problem of Law and Tsay. They showed that q

can be constructed from the nodal data by a pointwise limit. Then they proved that the convergence is L^1 .

Appendix A

Fréchet Derivative

Let E and F be normed spaces over \mathbb{R} or \mathbb{C} , $U \subset E$ an open subset, $x \in U$ and $f:U \mapsto F$ be a (possibly nonlinear) mapping.

1. f is called continuous at x if $\forall \epsilon > 0, \exists \delta > 0$ such that $\|f(\hat{x}) - f(x)\| \leq \epsilon, \forall \hat{x} \in U$ with $\|x - \hat{x}\| \leq \delta$.
2. f is called (Fréchet) differentiable for $x \in U$ if there exists a linear bounded map $d_x f : E \mapsto F$ (depending on x) such that

$$\lim_{h \rightarrow 0} \frac{1}{\|h\|} \|f(x+h) - f(x) - d_x f(h)\| = 0$$

Or equivalently

$\|f(x+h) - f(x) - d_x f(h)\| = O(\|h\|)$. In other words $\forall \epsilon > 0, \exists \delta > 0$ such that

$$\|f(x+h) - f(x) - d_x f(h)\| \leq \epsilon \|h\| \quad \forall h \text{ with } \|h\| < \delta.$$

The linear map $d_x f$ is uniquely determined and is called the Fréchet derivative of f at x . The map f is differentiable on U if it is differentiable at each point x in U . We write $f'(x) = d_x f$. In particular, $f'(x) \in \mathcal{L}(E, F)$ (i.e. The set of bounded linear operators between E and F).

3. f is called continuously differentiable or of class C^1 on U if the map $x \mapsto d_x f$ is continuous.

To show that a map is (Fréchet) differentiable, we must therefore prove the existence of a bounded linear operator as described above above. In the next section, we prove this for the map $x_n^j(q)$.

Appendix B

Proof of Lemmas

B.1 Lemma 4

For all q and fixed $j = 1, 2, \dots, n - 1$ and $n = 2, 3, \dots$, we have

$$y_2(x_n^j, q, \lambda_n) = y_2(1, q, \lambda_n) = 0 \quad (\text{B.1})$$

Differentiating the expressions in (B.1) with respect to q , assuming these derivatives exist, we obtain

$$y_2'(x_n^j, q, \lambda_n) d_q x_n^j[w] + d_q y_2(x_n^j, q, \lambda_n)[w] + \dot{y}_2(x_n^j, q, \lambda_n) d_q \lambda_n[w] = 0 \quad (\text{B.2})$$

$$d_q y_2(1, q, \lambda_n)[w] + \dot{y}_2(1, q, \lambda_n) d_q \lambda_n[w] = 0 \quad (\text{B.3})$$

where

$$d_q y(x, q, \lambda)[w] = \lim_{\epsilon \rightarrow 0} \frac{y(x, q + \epsilon w, \lambda) - y(x, q, \lambda)}{\epsilon}$$

Now we know (see [12, 20]) that

$$\begin{aligned} d_q y_2(x, q, \lambda)[w] &= \int_0^x [y_2(x, q, \lambda) y_1(t, q, \lambda) - y_1(x, q, \lambda) y_2(t, q, \lambda)] \\ &\quad \times y_2(t, q, \lambda) w(t) dt \end{aligned} \quad (\text{B.4})$$

from which it follows

$$d_q y_2(1, q, \lambda)[w] = \int_0^1 [y_2(1, q, \lambda)y_1(t, q, \lambda) - y_1(1, q, \lambda)y_2(t, q, \lambda)] \times y_2(t, q, \lambda)w(t)dt \quad (\text{B.5})$$

In particular,

$$d_q y_2(x_n^j, q, \lambda_n)[w] = -y_1(x_n^j, q, \lambda_n) \int_0^{x_n^j} [y_2(t, q, \lambda_n)]^2 w(t)dt \quad (\text{B.6})$$

$$d_q y_2(1, q, \lambda_n)[w] = -y_1(1, q, \lambda_n) \int_0^1 [y_2(t, q, \lambda_n)]^2 w(t)dt \quad (\text{B.7})$$

The Wronskian of the fundamental solutions y_1 and y_2 is given by

$$[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

Differentiating, we obtain

$$\begin{aligned} [y_1, y_2]' &= (y_1 y_2' - y_2 y_1')' \\ &= y_1 y_2'' - y_2 y_1'' \\ &= y_1(q - \lambda)y_2 - y_2(q - \lambda)y_1 \\ &= 0 \text{ a.e.} \end{aligned}$$

Therefore, $[y_1, y_2] \equiv \text{constant a.e.}$ But $[y_1, y_2](0) = y_1(0)y_2'(0) - y_2(0)y_1'(0) = 1$ (see boundary conditions)

Hence $[y_1, y_2](x) = 1$ everywhere by continuity. This leads to the important result:

$$y_1(1, q, \lambda_n)y_2'(1, q, \lambda_n) = y_1(x_n^j, q, \lambda_n)y_2'(x_n^j, q, \lambda_n) = 1 \quad (\text{B.8})$$

The formula (3.9) in the lemma now follows easily upon using the results (B.6), (B.7) and (B.8) in (B.2) and (B.3)

B.2 Lemma 1

Since x_n^j is a continuous function of q (by virtue of its Fréchet differentiability), we can write

$x_n^j(q) - x_n^j(0) = \int_0^1 \frac{d}{dt} x_n^j(tq) dt$, where

$$\left(\frac{d}{dt}\right)x_n^j(tq) = \lim_{\epsilon \rightarrow 0} \left[\frac{x_n^j(tq + \epsilon q) - x_n^j(tq)}{\epsilon} \right] = d_{tq} x_n^j[q]$$

Now from Lemma 4 it is easy to see that

$$\begin{aligned} d_{tq} x_n^j[q] &= \frac{1}{[y_2'(x_n^j, tq, \lambda_n)]^2} \left\{ \int_0^{x_n^j} [y_2(t, tq, \lambda_n)]^2 q(t) dt \right\} \\ &\quad - \frac{\dot{y}_2(x_n^j, tq, \lambda_n)}{\dot{y}_2(1, tq, \lambda_n)} \frac{1}{y_2'(x_n^j, tq, \lambda_n) y_2'(1, tq, \lambda_n)} \left\{ \int_0^1 [y_2(t, tq, \lambda_n)]^2 q(t) dt \right\} \end{aligned} \quad (\text{B.9})$$

From (3.6), we have

$$y_2'(x, tq, \lambda_n) = \cos \sqrt{\lambda_n} x + O\left(\frac{1}{\sqrt{\lambda_n}}\right) \quad (\text{B.10})$$

$$[y_2(x, tq, \lambda_n)]^2 = \left[\frac{1 - \cos 2\sqrt{\lambda_n} x}{2\lambda_n} \right] + O\left(\frac{1}{(\lambda_n)^{\frac{3}{2}}}\right) \quad (\text{B.11})$$

Again from Lemma 1 we have

$$\dot{y}_2(x_n^j, q, \lambda_n) = \frac{d}{d\lambda} y_2(x, q, \lambda) \Big|_{\lambda=\lambda_n, x=x_n^j}$$

$$\dot{y}_2(1, q, \lambda_n) = \frac{d}{d\lambda} y_2(1, q, \lambda) \Big|_{\lambda=\lambda_n}$$

with $y_2(x, q, \lambda)$ as in (3.6).

A little calculus and algebra on the above expressions yield the crude bound

$$\frac{\dot{y}_2(x_n^j, tq, \lambda_n)}{\dot{y}_2(1, tq, \lambda_n)} = x_n^j + O(1) \quad (\text{B.12})$$

Use of (B.10), (B.11) and (B.12) in (B.9) yields the required result:

$$d_{tq}x_n^j[q] = \frac{1}{2n^2\pi^2} \left[\int_0^{x_n^j} q(s)ds - x_n^j \int_0^1 q(s)ds + O(1) \right]$$

Proving that $x_n^j(q) - x_n^j(0) = O(\frac{1}{n^2})$

B.3 Lemma 2

For fixed k , the set of numbers $\frac{m+1}{2^{k+1}-m}$ form the sequence

$0, \frac{1}{2^{k+1}}, \frac{2}{2^{k+1}-1}, \frac{3}{2^{k+1}-2}, \dots, \frac{2^k}{2^{k+1}}, 1. k = 0, 1, 2, \dots$

Now consider the differences

$$1 - \frac{2^k}{2^{k+1}} = \frac{1}{2^{k+1}}$$

$$\frac{1}{2^{k+1}} - 0 = \frac{1}{2^{k+1}}$$

and

$$\frac{m+2}{2^{k+1}-(m+1)} - \frac{m+1}{2^{k+1}-m} = \frac{2^{k+1}+1}{[2^{k+1}-(m+1)][2^{k+1}-m]} < \frac{1}{2^{k+1}}, m = 0, 1, 2, \dots, 2^k - 2$$

from which it is clear that the upper bound on the differences

$$\frac{1}{2^{k+1}} \rightarrow 0 \text{ as } k \rightarrow \infty$$

B.4 Lemma 3

This Lemma is straightforward since we have already established that $x_n^j(q) - x_n^j(0) = O(\frac{1}{n^2})$.

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