The hyperbolic metric and two-point distortion theorems for univalent functions

by

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CHAPTER 1. INTRODUCTION

Univalent function theory has been around since the early 1900's. In 1916, Bieberbach gave an estimate for the second Taylor coefficient of a normalized univalent function. Some important consequences of this coefficient estimate are the growth and distortion theorems discussed in section 2.4. The classical growth theorem (Theorem 2.5) discussed in chapter 2 gives necessary but not sufficient conditions for univalence. In 1978, Blatter published a paper stating a two-point distortion theorem that gives both necessary and sufficient conditions for univalence. Since then, Blatter's method has been used to extend his result to a one-parameter family of theorems containing Blatter's original theorem as a special case. These theorems are stated in terms of hyperbolic distance and are invariant under composition with automorphisms of the unit disk.

In chapter 2 we discuss general properties of univalent functions and give some classical results, including the classical growth and distortion theorems. As the hyperbolic metric plays a prominent role in the two-point distortion theorems, we give a construction of the hyperbolic metric in chapter 3. Finally, in chapter 4 we discuss two-point distortion theorems, using the methods employed by Blatter [2], Kim and Minda [6], and Ma and Minda [8, 9]. We conclude chapter 4 by using the two-point distortion theorems to make some comparisons between hyperbolic and Euclidean geometries on simply connected regions.

CHAPTER 2. PRELIMINARIES

2.1 Basic Concepts and Definitions

Definition 2.1. Let $D \subseteq \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ an analytic function. We say that f is *univalent* in D if $f(z_1) \neq f(z_2)$ whenever z_1 and z_2 are points in D with $z_1 \neq z_2$.

We will mainly be concerned with functions analytic and univalent on the unit disk $\{z : |z| < 1\}$, which we denote throughout this paper by $\mathbb{D}$. In particular, we will consider normalized univalent functions. The *schlicht class* S is the class of functions f univalent on $\mathbb{D}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. As a result of these normalizations, if $f \in S$, then f has a Taylor expansion of the form

$$f(z) = z + a_2z^2 + a_3z^3 + a_4z^4 + \cdots, \ |z| < 1.$$

Example 2.1. The most important example of a function in class S is the *Koebe function*, given by

$$k(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n.$$

It is easy to verify that $k(z)$ is univalent. In fact, if $z_1, z_2 \in \mathbb{D}$ and $k(z_1) = k(z_2)$, then:

$$\begin{aligned}
\frac{z_1}{(1-z_1)^2} = \frac{z_2}{(1-z_2)^2} &\Rightarrow z_1(1-z_2)^2 = z_2(1-z_1)^2 \\
&\Rightarrow z_1(1-2z_2+z_2^2) = z_2(1-2z_1+z_1^2) \\
&\Rightarrow z_1 - 2z_1z_2 + z_1z_2^2 = z_2 - 2z_1z_2 + z_2z_1^2 \\
&\Rightarrow z_1 + z_1z_2^2 = z_2 + z_2z_1^2 \\
&\Rightarrow z_1(1-z_1z_2) = z_2(1-z_1z_2) \\
&\Rightarrow z_1 = z_2.
\end{aligned}$$

A function of the form

$$e^{-i\beta}k(e^{i\beta}z) = \frac{z}{(1-e^{i\beta}z)^2} = \sum_{n=1}^{\infty} ne^{i(n-1)\beta}z^n, \quad \beta \in \mathbb{R},$$

is called a rotation of the Koebe function. The Koebe function is important because it and its rotations exhibit many extremal properties within the class S.

Univalence is not preserved under addition, but is preserved under other elementary operations such as rotation and dilation. In particular, if $f \in S$ and F is defined by

$$F(z) = \frac{f\left(\dfrac{z+t}{1+\bar{t}z}\right) - f(t)}{(1-|t|^2)f'(t)}$$

for some $t \in \mathbb{D}$, then $F \in S$. To see this, notice that the function $T(z) = \dfrac{z+t}{1+\bar{t}z}$ is a conformal mapping of $\mathbb{D}$ onto $\mathbb{D}$. Therefore, $f \circ T$ is univalent on $\mathbb{D}$. Since $F(0) = 0$ and $F'(0) = 1$, we have $F \in S$. A function F defined this way is called a Koebe transform of f.

Another related class of functions that will be of interest to us is the class Σ. A function g that is analytic (except for a simple pole at ∞) and univalent on $\Delta = \{z : |z| > 1\}$ is in Σ if g is of the form

$$g(z) = z + b_0 + b_1z^{-1} + b_2z^{-2} + b_3z^{-3} + \cdots, \ |z| > 1.$$

2.2 Subclasses of Univalent Functions

Here we will define some important subclasses of S and Σ.

We first define what it means for a set to be starlike. If $A \subset \mathbb{C}$, then we say that A is *starlike with respect to the point* $w_0 \in A$ if the line segment joining w_0 to any other point of A is contained in A. A function is said to be *starlike* if it maps the unit disk conformally onto a set that is starlike with respect to the origin. The subclass of S consisting of starlike functions is denoted by S^*.

If A is starlike with respect to each point of A, then we say that A is *convex*. In other words, a set is convex if the line segments joining any two points of the set is entirely contained in the set. A function is said to be *convex* if it maps the unit disk conformally onto a convex set. The subclass of S consisting of convex functions is denoted by C. Note that $C \subset S^* \subset S$.

Closely related to the class of convex function is the class of close-to-convex functions. We say that a function f is *close-to-convex* if there is a convex function g (not necessarily in C) such that $\mathrm{Re}\left\{\frac{f'(z)}{g'(z)}\right\} > 0$ for all $z \in \mathbb{D}$. Note that the definition of a close-to-convex function does not require univalence; however, as shown by Duren in [3], every close-to-convex function is univalent.

The subclass of functions that are close-to-convex with the normalizations $f(0) = 0$ and $f'(0) = 1$ is denoted K. The subclass K_0 of K is the set of all functions f such that $\mathrm{Re}\left\{\frac{f'(z)}{g'(z)}\right\} > 0$ for all $z \in \mathbb{D}$ and some $g \in C$.

We are also interested in some subclasses of Σ. Define Σ' to be the subclass of functions $g \in \Sigma$ such that $g(z) \neq 0$ in Δ. The subclass Σ_0 is defined to be the class of functions $g \in \Sigma$ such that $b_0 = 0$. Finally, we define $\widetilde{\Sigma}$ to be the subclass of functions $g \in \Sigma$ such that $\mathbb{C} \backslash g(\Delta)$ has two-dimensional Lebesgue measure zero.

2.3 Some Classical Results for Univalent Functions

We now consider some classical results for univalent functions. These are standard results that can be found in books such as Duren [3] and Pommerenke [10]. These theorems are useful in proving the results of section 2.4.

Theorem 2.1 (Area Theorem)**.** *If* $g(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots \in \Sigma$, *then* $\sum_{n=1}^{\infty} n|b_n|^2 \leq 1$. *Equality holds if and only if* $g \in \widetilde{\Sigma}$.

Proof. Let $E = \mathbb{C} \backslash g(\Delta)$, C_r be the image under g of the circle $|z| = r > 1$, $E_r = \mathbb{C} \backslash \{g(z) : |z| > r\}$, and $D_r = g^{-1}(E_r)$. Note that the C_r is given by $w(t) = g(re^{it}) = u(t) + iv(t)$, $0 \leq t \leq 2\pi$.

By Green's Theorem, the area of E_r is given by

$$\begin{aligned}
\operatorname{area}(E_r) &= \frac{1}{2}\int_{C_r}(udv - vdu)\\
&= \frac{1}{2i}\int_{C_r}(iudv - ivdu)\\
&= \frac{1}{2i}\int_{C_r}(iudv - ivdu) + \frac{1}{2i}\int_{C_r}(udu + vdv)\\
&= \frac{1}{2i}\int_{C_r}(udu + iudv - ivdu + vdv)\\
&= \frac{1}{2i}\int_{C_r}(u - iv)(du + idv)\\
&= \frac{1}{2i}\int_{C_r}\overline{w}\ dw\\
&= \frac{1}{2i}\int_{|z|=r}\overline{g(z)}g'(z)\ dz
\end{aligned}$$

Letting $z = re^{it}, 0 \leq t \leq 2\pi$, we get

$$\begin{aligned}
\operatorname{area}(E_r) &= \frac{1}{2}\int_0^{2\pi} re^{it}\overline{g(re^{it})}g'(re^{it})\ dt\\
&= \frac{1}{2}\int_0^{2\pi} re^{it}\overline{\left[re^{it} + \sum_{n=0}^{\infty} b_n r^{-n}e^{-int}\right]}\left[1 + \sum_{n=1}^{\infty} -nb_n r^{-n-1}e^{i(-n-1)t}\right]\ dt\\
&= \frac{1}{2}\int_0^{2\pi}\left[re^{-it} + \sum_{n=0}^{\infty}\overline{b_n}r^{-n}e^{int}\right]\left[re^{it} - \sum_{n=1}^{\infty} nb_n r^{-n}e^{-int}\right]\ dt\\
&= \frac{1}{2}\int_0^{2\pi}\left(r^2 - \sum_{n=1}^{\infty} nb_n r^{-n+1}e^{-i(n+1)t} + \sum_{n=0}^{\infty}\overline{b_n}r^{-n+1}e^{i(n+1)t} - \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n}\right)\ dt\\
&= \pi\left(r^2 - \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n}\right)
\end{aligned}$$

Letting $r \to 1^+$, we get $m^*(E) = \pi\left(1 - \sum_{n=1}^{\infty} n|b_n|^2\right)$, where m^* denotes two-dimensional Lebesgue outer measure. Since $m^*(E) \geq 0$, we must have $\sum_{n=1}^{\infty} n|b_n|^2 \leq 1$.

For the equality case, note that $\sum_{n=1}^{\infty} n|b_n|^2 = 1$ if and only if $m^*(E) = 0$, i.e. $g \in \widetilde{\Sigma}$. $\square$

Corollary 2.1. *If* $g \in \Sigma$*, then* $|b_1| \leq 1$*. Equality holds if and only if* g *has the form* $g(z) = z + b_0 + b_1/z$ *where* $|b_1| = 1$*.*

Proof. We have $|b_1|^2 \leq \sum_{n=1}^{\infty} n|b_n|^2 \leq 1$. Therefore, $|b_1|^2 \leq 1$ and it follows that $|b_1| \leq 1$.

If $|b_1| = 1$, then we have $1 \geq \sum_{n=1}^{\infty} n|b_n|^2 = |b_1| + \sum_{n=2}^{\infty} n|b_n|^2 = 1 + \sum_{n=2}^{\infty} n|b_n|^2$. Therefore, $b_n = 0$ for all $n \geq 2$, and so $g(z) = z + b_0 + b_1/z$ with $|b_1| = 1$. Conversely, if g is of the given form, then $g \in \Sigma$ and $|b_1| = 1$. ☐

Theorem 2.2 (Bieberbach's Theorem)**.** *If* $f \in S$ *with* $f(z) = z + a_2z^2 + a_3z^3 + \cdots$*, then* $|a_2| \leq 2$*. Equality holds if and only if* f *is a rotation of the Koebe function.*

Proof. Define $g(z) = [f(1/z^2)]^{-1/2} = z - \frac{a_2}{2}z^{-1} + \cdots \in \Sigma$.
The corollary to Theorem 2.1 implies $\left|\frac{a_2}{2}\right| \leq 1$, and so we must have $|a_2| \leq 2$.

Equality holds if and only if $g(z) = z - b_1/z$, with $|b_1| = 1$. Let $b_1 = e^{i\alpha}$ for some $\alpha \in \mathbb{R}$. Then $f(z) = [g(z^{-1/2})]^{-2} = (z^{-1/2} - e^{i\alpha}z^{1/2})^{-2} = [z^{-1/2}(1 - e^{i\alpha}z)]^{-2} = z(1 - e^{i\alpha}z)^{-2}$, which is a rotation of the Koebe function. ☐

Theorem 2.3 (Koebe 1/4-Theorem)**.** *If* $f \in S$*, then the range of* f *contains the disk* $\{w : |w| < 1/4\}$*.*

Proof. Suppose $w \notin \{f(z) : |z| < 1\}$. Then $g(z) = \dfrac{wf(z)}{w - f(z)}$ is analytic and univalent in $\mathbb{D}$. Note that

$$\begin{aligned} g(0) &= \frac{wf(0)}{w - f(0)} = 0, \\ g'(z) &= \frac{(w - f(z))wf'(z) - wf(z)(-f'(z))}{(w - f(z))^2} = \frac{w^2f'(z)}{(w - f(z))^2}, \\ g'(0) &= \frac{w^2}{w^2} = 1, \\ g''(z) &= \frac{(w - f(z))^2w^2f''(z) - w^2f'(z)2(w - f(z))(-f'(z))}{(w - f(z))^4}, \end{aligned}$$

and

$$g''(0) = \frac{w^4f''(0) + 2w^3}{w^4} = f''(0) + \frac{2}{w} = 2a_2 + \frac{2}{w}.$$

Note in particular that $g(0) = 0$ and $g'(0) = 1$, so that $g \in S$.

Thus, $g(z) = \displaystyle\sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n = z + \frac{1}{2}\left(2a_2 + \frac{2}{w}\right) z^2 + \cdots = z + \left(a_2 + \frac{1}{w}\right) z^2 + \cdots$. Because $g \in S$, Theorem 2.2 implies that $\left|a_2 + \dfrac{1}{w}\right| \leq 2$, and, because $f \in S$, we also have $|a_2| \leq 2$. Thus, we have $\left|\dfrac{1}{w}\right| = \left|\dfrac{1}{w} + a_2 - a_2\right| \leq \left|\dfrac{1}{w} + a_2\right| + |a_2| \leq 2 + 2 = 4$. Therefore, $|w| \geq \dfrac{1}{4}$. Hence, if $|w| < \dfrac{1}{4}$, then $w \in \{f(z) : |z| < 1\}$, i.e. w is in the range of f. □

Lemma 2.1. *If $f \in S$, then $\left|\dfrac{zf''(z)}{f'(z)} - \dfrac{2|z|^2}{1-|z|^2}\right| \leq \dfrac{4|z|}{1-|z|^2}$ for all $z \in \mathbb{D}$.*

Proof. Let $f \in S$ and fix $t \in \mathbb{D}$. Let F be defined by

$$F(z) = \frac{f\left(\dfrac{z+t}{1+\bar{t}z}\right) - f(t)}{(1-|t|^2)f'(t)} = z + A_2(t)z^2 + A_3(t)z^3 + \cdots .$$

Note that F is a Koebe transform, so $F \in S$. Therefore, Theorem 2.2 implies that $|A_2(t)| \leq 2$ for all $t \in \mathbb{D}$. A calculation shows that $F''(0) = (1-|t|^2)\dfrac{f''(t)}{f'(t)} - 2\bar{t}$. Since $F''(0) = 2A_2(t)$, we must have $\left|(1-|t|^2)\dfrac{f''(t)}{f'(t)} - 2\bar{t}\right| \leq 4$. Multiplying both sides by $\dfrac{|t|}{1-|t|^2}$ results in $\left|\dfrac{tf''(t)}{f'(t)} - \dfrac{2|t|^2}{1-|t|^2}\right| \leq \dfrac{4|t|}{1-|t|^2}$, which is the desired result. □

2.4 Classical Growth and Distortion Theorems

We are especially interested in the classical growth and distortion theorems. The two-point distortion theorems that we will explore later are closely related to these.

Theorem 2.4. *If $f \in S$ and $z \in \mathbb{D}$, then $\dfrac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \dfrac{1+|z|}{(1-|z|)^3}$. Equality holds if and only if f is a suitable rotation of the Koebe function.*

Proof. Let $f \in S$ and $z \in D$. From Lemma 2.1, we have $\left|\dfrac{zf''(z)}{f'(z)} - \dfrac{2|z|^2}{1-|z|^2}\right| \leq \dfrac{4|z|}{1-|z|^2}$. Therefore:

$$\begin{aligned}
&-\frac{4|z|}{1-|z|^2} \leq \operatorname{Re}\left\{\frac{zf''(z)}{f'(z)} - \frac{2|z|^2}{1-|z|^2}\right\} \leq \frac{4|z|}{1-|z|^2} \\
&\Rightarrow -\frac{4|z|}{1-|z|^2} \leq \operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} - \frac{2|z|^2}{1-|z|^2} \leq \frac{4|z|}{1-|z|^2} \\
&\Rightarrow \frac{2|z|^2-4|z|}{1-|z|^2} \leq \operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} \leq \frac{2|z|^2+4|z|}{1-|z|^2}.
\end{aligned}$$

Let $\log f'(z)$ denote a single-valued branch of the logarithm of f' with $\log f'(0) = 0$. Note that we can choose such a branch of the logarithm since $f'(z) \neq 0$ and $f'(0) = 1$.

Now, $\operatorname{Re}\left\{\dfrac{zf''(z)}{f'(z)}\right\} = r\dfrac{\partial}{\partial r}\operatorname{Re}\{\log f'(z)\}$ (where $z = re^{i\theta}$). Therefore, we have

$$\frac{2r^2-4r}{1-r^2} \leq r\frac{\partial}{\partial r}\operatorname{Re}\{\log f'(z)\} \leq \frac{2r^2+4r}{1-r^2},$$

which implies that

$$\frac{2r-4}{1-r^2} \leq \frac{\partial}{\partial r}\operatorname{Re}\{\log f'(z)\} \leq \frac{2r+4}{1-r^2}.$$

Holding θ fixed and integrating with respect to r from 0 to $|z|$ results in

$$\log\left|\frac{1-|z|}{(1+|z|)^3}\right| \leq \log|f'(z)| \leq \log\left|\frac{1+|z|}{(1-|z|)^3}\right|.$$

Exponentiating this expression gives the desired result.

Suitable rotations of the Koebe function provide cases of equality. Conversely, if equality holds in either the upper bound or the lower bound for some $z = Re^{i\theta}$, then $\operatorname{Re}\left\{e^{i\theta}\dfrac{f''(0)}{f'(0)}\right\} = \pm 4$. In this case, we must have $|a_2| = 2$, which implies that f is a rotation of the Koebe function. □

Theorem 2.5. *If $f \in S$ and $z \in \mathbb{D}$, then $\dfrac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \dfrac{|z|}{(1-|z|)^2}$. Equality holds if and only if f is a suitable rotation of the Koebe function.*

Proof. Let $f \in S$ and $z = re^{i\theta}$ with $0 < r < 1$.

To obtain the upper bound, let Γ denote the line segment from 0 to z and write

$$f(z) = \int_\Gamma f'(\zeta)\, d\zeta = \int_0^r f'(\rho e^{i\theta})e^{i\theta}\, d\rho.$$

Then Theorem 2.4 implies that:

$$|f(z)| \leq \int_0^r |f'(\rho e^{i\theta})|\, d\rho \leq \int_0^r \frac{1+\rho}{(1-\rho)^3}\, d\rho = \frac{r}{(1-r)^2}.$$

This establishes the upper bound.

To obtain the lower bound, first notice that $\dfrac{r}{(1+r)^2} < \dfrac{1}{4}$ for $0 < r < 1$. Therefore, if $|f(z)| \geq \dfrac{1}{4}$, then the result holds trivially. If $|f(z)| < \dfrac{1}{4}$, Theorem 2.3 implies that the range

of f contains the line segment from 0 to $f(z)$. Let this line segment be denoted Λ and let $\lambda = f^{-1}(\Lambda)$ be its preimage under f. Then $f(z) = \displaystyle\int_\Lambda dw = \int_\lambda f'(\zeta)\, d\zeta$. Because $f'(\zeta)d\zeta$ has constant argument along λ, Theorem 2.4 implies:
$|f(z)| = \displaystyle\int_\lambda |f'(\zeta)||d\zeta| \geq \int_\lambda \frac{1-|\zeta|}{(1+|\zeta|)^3} d|\zeta| = \int_0^r \frac{1-\rho}{(1+\rho)^3}\, d\rho = \frac{r}{(1+r)^2}$.

The statement about equality follows from the case of equality in Theorem 2.4. □

Theorem 2.6. *If $f \in S$ and $z \in \mathbb{D}$, then $\dfrac{1-|z|}{1+|z|} \leq \left|\dfrac{zf'(z)}{f(z)}\right| \leq \dfrac{1+|z|}{1-|z|}$. Equality holds if and only if f is a suitable rotation of the Koebe function.*

Proof. Let $f \in S$ be given and define $F(z) = \dfrac{f\left(\dfrac{z+\zeta}{1+\overline{\zeta}z}\right) - f(\zeta)}{(1-|\zeta|^2)f'(\zeta)} \in S$. By Theorem 2.5, we have

$$\frac{|\zeta|}{(1+|\zeta|)^2} \leq |F(-\zeta)| \leq \frac{|\zeta|}{(1-|\zeta|)^2}.$$

But note that $F(-\zeta) = \dfrac{f(0) - f(-\zeta)}{(1-|\zeta|^2)f'(-\zeta)} = \dfrac{-f(\zeta)}{(1-|\zeta|^2)f'(-\zeta)}$.

Therefore:

$$\begin{aligned}
&\frac{|\zeta|(1-|\zeta|^2)}{(1+|\zeta|)^2} \leq \left|\frac{f(\zeta)}{f'(\zeta)}\right| \leq \frac{|\zeta|(1-|\zeta|^2)}{(1-|\zeta|)^2}\\
&\Rightarrow \frac{1-|\zeta|^2}{(1+|\zeta|)^2} \leq \left|\frac{f(\zeta)}{\zeta f'(\zeta)}\right| \leq \frac{1-|\zeta|^2}{(1-|\zeta|)^2}\\
&\Rightarrow \frac{(1-|\zeta|)(1+|\zeta|)}{(1+|\zeta|)^2} \leq \left|\frac{f(\zeta)}{\zeta f'(\zeta)}\right| \leq \frac{(1-|\zeta|)(1+|\zeta|)}{(1-|\zeta|)^2}\\
&\Rightarrow \frac{1-|\zeta|}{1+|\zeta|} \leq \left|\frac{f(\zeta)}{\zeta f'(\zeta)}\right| \leq \frac{1+|\zeta|}{1-|\zeta|}\\
&\Rightarrow \frac{1-|\zeta|}{1+|\zeta|} \leq \left|\frac{\zeta f'(\zeta)}{f(\zeta)}\right| \leq \frac{1+|\zeta|}{1-|\zeta|}.
\end{aligned}$$

Cases of equality are provided by suitable rotations of the Koebe function. We next show that rotations of the Koebe function provide the only cases of equality. To this end, suppose $f \in S$ satisfies

$$\left|\frac{\zeta f'(\zeta)}{f(\zeta)}\right| = \frac{1-|\zeta|}{1+|\zeta|}$$

for some $\zeta \in \mathbb{D}$. Note then that $\left|\dfrac{f(\zeta)}{f'(\zeta)}\right| = |\zeta|\dfrac{1+|\zeta|}{1-|\zeta|}$.

Now, consider $F \in S$ defined by

$$F(z) = \frac{f\left(\dfrac{z+\zeta}{1+\overline{\zeta}z}\right) - f(\zeta)}{(1-|\zeta|^2)f'(\zeta)}.$$

Then we have

$$\begin{aligned}
|F(-\zeta)| &= \left|\frac{f(0)-f(\zeta)}{(1-|\zeta|^2)f'(\zeta)}\right| \\
&= \left|\frac{-f(\zeta)}{(1-|\zeta|^2)f'(\zeta)}\right| \\
&= \frac{1}{(1-|\zeta|^2)}\left|\frac{f(\zeta)}{f'(\zeta)}\right| \\
&= \frac{1}{(1-|\zeta|)(1+|\zeta|)}|\zeta|\frac{1+|\zeta|}{1-|\zeta|} \\
&= \frac{|\zeta|}{(1-|\zeta|)^2}.
\end{aligned}$$

By Theorem 2.5, F is a rotation of the Koebe function. It then can be shown that f must also be a rotation of the Koebe function.

A similar argument shows that rotations of the Koebe function provide the only cases of equality in the upper bound as well. □

2.5 Motivation for Using the Hyperbolic Metric

The theorems we have just seen give conditions that are necessary, but not sufficient, for univalence. In fact, there are many functions that satisfy the theorems but fail to be univalent. Blatter [2] wondered if there were similar theorems that would give sufficient conditions for univalence. Blatter's research led to the proof of a two-point distortion theorem that gives both necessary and sufficient conditions for univalence.

Blatter's theorem and related results are given in terms of hyperbolic distance. In particular, Blatter related $|f(a) - f(b)|$ to the hyperbolic distance between a and b ($a, b \in \mathbb{D}$). As we will see, Blatter's result is invariant under suitable compositions with automorphisms of $\mathbb{D}$ and $\mathbb{C}$. The hyperbolic metric is useful in these results because it is invariant under conformal automorphisms of $\mathbb{D}$.

CHAPTER 3. THE HYPERBOLIC METRIC

In this chapter, we will follow the construction of the hyperbolic metric as given by Anderson [1], with some results given by Krantz [7]. The goal is to construct a metric that preserves distances under conformal automorphisms of the unit disk. To do so we will first construct the hyperbolic metric on the upper-half plane model for the hyperbolic plane. We will then transform this model to the unit disk.

3.1 The Upper-Half Plane Model of the Hyperbolic Plane

The underlying space in the upper-half plane model is $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. To construct this model of the hyperbolic plane, we will first define a hyperbolic line.

Definition 3.1. A *hyperbolic line in* $\mathbb{H}$ is:

1. the intersection of $\mathbb{H}$ with a Euclidean line in $\mathbb{C}$ that is orthogonal to $\mathbb{R}$, or
2. the intersection of $\mathbb{H}$ with a Euclidean circle centered on $\mathbb{R}$

Remark. If L is a Euclidean line in $\mathbb{C}$, we can view $L \cup \{\infty\}$ as a circle in $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (the extended complex plane). Therefore, every hyperbolic line in $\mathbb{H}$ is contained in a circle in $\widehat{\mathbb{C}}$ that is orthogonal to the extended real line $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.

Proposition 3.1. *Every circle in* $\widehat{\mathbb{C}}$ *can be expressed as the set of solutions in* $\widehat{\mathbb{C}}$ *to an equation of the form* $\alpha z\overline{z} + \beta z + \overline{\beta}\overline{z} + \gamma = 0$*, where* $\alpha, \gamma \in \mathbb{R}$ *and* $\beta \in \mathbb{C}$*.*

Proof. Since a circle in $\widehat{\mathbb{C}}$ is a Euclidean line in $\mathbb{C}$ or a Euclidean circle in $\mathbb{C}$, we will consider these two cases.

Case 1: First consider a Euclidean line in $\mathbb{C}$ given by $ax + by + c = 0$. Let $z = x + iy$. Since $x = \text{Re}(z) = \frac{1}{2}(z + \overline{z})$ and $y = \text{Im}(z) = -\frac{i}{2}(z - \overline{z})$, we have:

$$\begin{aligned} ax + by + c &= \frac{a}{2}(z + \overline{z}) - i\frac{b}{2}(z - \overline{z}) + c \\ &= \frac{1}{2}(a - ib)z + \frac{1}{2}(a + ib)\overline{z} + c \end{aligned}$$

Taking $\alpha = 0$, $\beta = \frac{1}{2}(a - ib)$, and $\gamma = c$, we see that the line is given by the set of points $\{z : \alpha z\overline{z} + \beta z + \overline{\beta}\overline{z} + \gamma = 0\}$.

Case 2: Next consider the Euclidean circle given by $(x-h)^2 + (y-k)^2 = r^2$. Letting $z_0 = h + ik$, we have:

$$\begin{aligned} r^2 &= (x - h)^2 + (y - k)^2 \\ &= |z - z_0|^2 \\ &= z\overline{z} - \overline{z_0}z - z_0\overline{z} + |z_0|^2. \end{aligned}$$

Hence, $z\overline{z} - \overline{z_0}z - z_0\overline{z} + |z_0|^2 - r^2 = 0$, and so the circle is given by the set of points $\{z : \alpha z\overline{z} + \beta z + \overline{\beta}\overline{z} + \gamma = 0\}$, where $\alpha = 1$, $\beta = -\overline{z_0}$, and $\gamma = |z_0|^2 - r^2$. □

Definition 3.2. We say that two hyperbolic lines in $\mathbb{H}$ are *parallel* if they are disjoint.

This construction of the hyperbolic plane does satisfy the axioms of hyperbolic geometry. In particular, given a hyperbolic line ℓ in $\mathbb{H}$ and any point P in $\mathbb{H}$ not on ℓ, there are infinitely many hyperbolic lines through P parallel to ℓ.

3.2 Möbius Transformations and the General Möbius Group

The metric that we construct on $\mathbb{H}$ needs to be invariant under certain transformations taking $\mathbb{H}$ to itself. Since a hyperbolic line in $\mathbb{H}$ is contained in a circle in $\widehat{\mathbb{C}}$, we will consider transformations taking circles in $\widehat{\mathbb{C}}$ to circles in $\widehat{\mathbb{C}}$.

We will begin by considering the set $\mathcal{L}$ of linear fractional transformations (also called Möbius transformations). These are functions of the form $T(z) = \dfrac{az + b}{cz + d}$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. If $T \in \mathcal{L}$ is of this form, then define:

1. $T(\infty) = \lim_{z\to\infty} T(z) = \dfrac{a}{c}$
2. $T\left(-\dfrac{d}{c}\right) = \lim_{z\to -d/c} T(z) = \infty$

With these conventions, every function $T \in \mathcal{L}$ is a continuous, bijective mapping of $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$. Note that $\mathcal{L}$ is a group under composition.

Proposition 3.2. *The group $\mathcal{L}$ is generated by elements of the form $m(z) = \alpha z + \beta$ $(\alpha, \beta \in \mathbb{C})$ and the function $J(z) = -\dfrac{1}{z}$.*

Proof. Let $T(z) = \dfrac{az+b}{cz+d} \in \mathcal{L}$. We will consider two cases.
Case 1: If $c = 0$, then $T(z) = \dfrac{a}{d}z + \dfrac{b}{d}$, which is an element of the form $m(z) = \alpha z + \beta$ with $\alpha = \dfrac{a}{d}$ and $\beta = \dfrac{b}{d}$.

Case 2: If $c \neq 0$, then we can write

$$\begin{aligned} T(z) &= \frac{az+b}{cz+d} \\ &= \frac{(az+b)c}{(cz+d)c} \\ &= \frac{acz+bc}{c^2z+cd} \\ &= \frac{acz+ad-(ad-bc)}{c^2z+cd} \\ &= \frac{a(cz+d)}{c(cz+d)} - \frac{ad-bc}{c^2z+cd} \\ &= \frac{a}{c} - \frac{ad-bc}{c^2z+cd} \\ &= f(J(g(z)), \end{aligned}$$

where $g(z) = c^2z + cd$ and $f(z) = (ad-bc)z + \dfrac{a}{c}$.

In either case, we have written T as a composition of elements of the form $m(z) = \alpha z + \beta$ and the function $J(z) = -\dfrac{1}{z}$. □

Proposition 3.3. *Every element $T \in \mathcal{L}$ maps circles in $\widehat{\mathbb{C}}$ to circles in $\widehat{\mathbb{C}}$.*

Proof. By Proposition 3.2, it is enough to consider functions of the form $m(z) = az + b$ $(a, b \in \mathbb{C})$ and the function $J(z) = -\dfrac{1}{z}$.

Recall that a circle in $\widehat{\mathbb{C}}$ can be expressed as the set of solutions to an equation of the form $\alpha z\overline{z} + \beta z + \overline{\beta}\overline{z} + \gamma = 0$, where $\alpha, \gamma \in \mathbb{R}$ and $\beta \in \mathbb{C}$ (Proposition 3.1).
Let $A = \{z \in \mathbb{C} : \alpha z\overline{z} + \beta z + \overline{\beta}\overline{z} + \gamma = 0\}$ be a circle in $\widehat{\mathbb{C}}$.

First we consider $m(z) = az + b$. Let $w = az + b$. Then $z = \dfrac{1}{a}(w - b)$. If $z \in A$, then we have

$$\begin{aligned}
0 &= \alpha z\overline{z} + \beta z + \overline{\beta}\overline{z} + \gamma \\
&= \frac{\alpha}{|a|^2}(w-b)\overline{(w-b)} + \beta\frac{1}{a}(w-b) + \overline{\beta}\overline{\frac{1}{a}(w-b)} + \gamma \\
&= \frac{\alpha}{|a|^2}(w\overline{w} - \overline{b}w - b\overline{w} + |b|^2) + \beta\frac{1}{a}(w-b) + \overline{\beta}\overline{\frac{1}{a}(w-b)} + \gamma \\
&= \frac{\alpha}{|a|^2}w\overline{w} + \left(\frac{\beta}{a} - \frac{\alpha\overline{b}}{|a|^2}\right)w + \left(\frac{\overline{\beta}}{\overline{a}} - \frac{\alpha b}{|a|^2}\right)\overline{w} + \left(\frac{\alpha|b|^2}{|a|^2} - 2\mathrm{Re}\Big(\frac{\beta b}{a}\Big) + \gamma\right) \\
&= \frac{\alpha}{|a|^2}w\overline{w} + \left(\frac{\beta}{a} - \frac{\alpha\overline{b}}{|a|^2}\right)w + \overline{\left(\frac{\beta}{a} - \frac{\alpha\overline{b}}{|a|^2}\right)}\overline{w} + \left(\frac{\alpha|b|^2}{|a|^2} - 2\mathrm{Re}\Big(\frac{\beta b}{a}\Big) + \gamma\right),
\end{aligned}$$

which gives the equation of a circle in $\widehat{C}$. Therefore, the image of A under m is a circle in $\widehat{\mathbb{C}}$.

Next consider $J(z) = -\dfrac{1}{z}$. Let $w = J(z) = -\dfrac{1}{z}$. Then $z = -\dfrac{1}{w}$ and so for any $z \in A$, we have

$$\begin{aligned}
0 &= \alpha z\overline{z} + \beta z + \overline{\beta}\overline{z} + \gamma \\
&= \alpha\frac{1}{w}\frac{1}{\overline{w}} + \beta\frac{1}{w} + \overline{\beta}\frac{1}{\overline{w}} + \gamma.
\end{aligned}$$

Multiplying through by $w\overline{w}$ results in

$$\gamma w\overline{w} + \overline{\beta}w + \beta\overline{w} + \alpha = 0.$$

Since this is again an equation of a circle in $\widehat{C}$, we see that the image of A under J is a circle in $\widehat{C}$. □

We now extend $\mathcal{L}$ to a larger group, the *general Möbius group* $\mathcal{M}$. The group $\mathcal{M}$ is generated by $\mathcal{L}$ and C, where $C : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is defined by $C(z) = \overline{z}$ for $z \in \mathbb{C}$ and $C(\infty) = \infty$. We have seen that circles in $\widehat{\mathbb{C}}$ are mapped by elements of $\mathcal{L}$ to circles in $\widehat{\mathbb{C}}$. This is also true of $\mathcal{M}$.

Proposition 3.4. *Every element of $\mathcal{M}$ maps circles in $\widehat{\mathbb{C}}$ to circles in $\widehat{\mathbb{C}}$.*

Proof. Consider a circle in $\widehat{\mathbb{C}}$ given by the equation $\alpha z\overline{z} + \beta z + \overline{\beta}\overline{z} + \gamma = 0$. Let $w = C(z) = \overline{z}$. Then $z = \overline{w}$, and so we have $\alpha w\overline{w} + \beta\overline{w} + \overline{\beta}w + \gamma = 0$. Note that this is also the equation of a circle in $\widehat{\mathbb{C}}$. Therefore, C maps circles in $\widehat{\mathbb{C}}$ to circles in $\widehat{\mathbb{C}}$. Now, since every element of a generating set for $\mathcal{M}$ takes circles in $\widehat{\mathbb{C}}$ to circles in $\widehat{\mathbb{C}}$, the property must also hold for $\mathcal{M}$. □

Since our goal is to construct a metric that is invariant under conformal automorphisms, we are interested in conformal mapping properties.

Proposition 3.5. *Every element of $\mathcal{M}$ preserves magnitudes of angles.*

Proof. It is well known that the elements of $\mathcal{L}$ are conformal mappings, and so preserve magnitudes of angles. Thus, we need only consider the function $C(z) = \overline{z}$. Since the geometric interpretation of C is reflection across the real axis, it is easy to see that C preserves magnitudes of angles. □

Since we are concerned with elements of $\mathcal{M}$ that map $\mathbb{H}$ to itself, we define the subgroup $\mathcal{M}(\mathbb{H}) := \{m \in \mathcal{M} : m(\mathbb{H}) = \mathbb{H}\}$. It can be shown that a generating group for $\mathcal{M}(\mathbb{H})$ is given by elements of the form $m(z) = az + b$ $(a > 0, b \in \mathbb{R})$ and the functions $J(z) = -\dfrac{1}{z}$ and $B(z) = -\overline{z}$. We will also make use of the subgroup $\mathcal{L}(\mathbb{H}) := \{T \in \mathcal{L} : T(\mathbb{H}) = \mathbb{H}\}$. Note that $\mathcal{L}(\mathbb{H}) \subset \mathcal{M}(\mathbb{H}) \subset \mathcal{M}$. Additionally, $\mathcal{L}(\mathbb{H}) \subset \mathcal{L} \subset \mathcal{M}$. We have the following results for these subgroups.

Proposition 3.6. *Every element of $\mathcal{M}(\mathbb{H})$ maps hyperbolic lines in $\mathbb{H}$ to hyperbolic lines in $\mathbb{H}$.*

Proof. Apply Propositions 3.4 and 3.5 and the fact that every hyperbolic line in $\mathbb{H}$ is contained in a circle in $\widehat{\mathbb{C}}$ orthogonal to $\widehat{\mathbb{R}}$. □

3.3 Hyperbolic Length and Distance in $\mathbb{H}$

If $f : [a,b] \to \mathbb{C}$ is a piecewise C^1 path with $f(t) = x(t) + iy(t)$, then we have $f'(t) = x'(t) + iy'(t)$ and $|f'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}$. Therefore, the Euclidean length of f is given by

$$\text{length}(f) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\ dt = \int_a^b |f'(t)|\ dt = \int_f |dz|.$$

We say that the standard Euclidean element of arc length in $\mathbb{C}$ is $|dz| = |f'(t)|\ dt$. For any continuous function $\rho : \mathbb{C} \to \mathbb{R}$, the path integral of ρ along f is given by

$$\int_f \rho(z)|dz| = \int_a^b \rho(f(t))|f'(t)|\ dt.$$

Viewing $\rho(z)|dz|$ as a new element of arc length motivates the following definition.

Definition 3.3. Let $f : [a,b] \to \mathbb{C}$ be a piecewise C^1 path and $\rho : \mathbb{C} \to \mathbb{R}$ a continuous function. The *length of f with respect to the element of arc length $\rho(z)|dz|$* is given by

$$\text{length}_\rho(f) = \int_f \rho(z)|dz|.$$

To measure hyperbolic arc length, we must find an appropriate element of arc length. In view of the fact that we want our metric to be invariant under composition with elements of $\mathcal{M}(\mathbb{H})$, we will look for an element of arc length $\rho(z)|dz|$ so that $\text{length}_\rho(f) = \text{length}_\rho(\gamma \circ f)$ holds for all $\gamma \in \mathcal{M}(\mathbb{H})$ and all piecewise C^1 functions $f : [a,b] \to \mathbb{H}$.

We begin by investigating the conditions imposed on ρ by elements of $\mathcal{L}(\mathbb{H}) \subset \mathcal{M}(\mathbb{H})$. To this end, let $\gamma \in \mathcal{L}(\mathbb{H})$ and let $f : [a,b] \to \mathbb{H}$ be a piecewise C^1 path. Then we have:

$$\begin{aligned}\text{length}_\rho(f) = \text{length}_\rho(\gamma \circ f) &\Rightarrow \int_a^b \rho(f(t))|f'(t)|\ dt = \int_a^b \rho\big(\gamma(f(t))\big)|\gamma'(f(t))||f'(t)|\ dt \\ &\Rightarrow \int_a^b \Big(\rho(f(t)) - \rho\big(\gamma(f(t))\big)|\gamma'(f(t))|\Big)|f'(t)|\ dt = 0\end{aligned}$$

Now, define $\mu_\gamma(z) = \rho(z) - \rho(\gamma(z))|\gamma'(z)|$. Then the above condition becomes

$$\int_f \mu_\gamma(z)|dz| = \int_a^b \mu_\gamma(f(t))|f'(t)|\ dt = 0$$

for every piecewise C^1 path $f : [a,b] \to \mathbb{H}$ and every $\gamma \in \mathcal{L}(\mathbb{H})$.

Lemma 3.1. *Let $D \subseteq \mathbb{C}$ be open and $\mu : D \to \mathbb{R}$ continuous. If $\int_f \mu(z)|dz| = 0$ for every piecewise C^1 path $f : [a,b] \to D$, then $\mu \equiv 0$ on D.*

Proof. Suppose by way of contradiction that $\mu \not\equiv 0$. Then there is some $z_0 \in D$ with $\mu(z_0) \neq 0$. Without loss of generality, assume $\mu(z_0) > 0$. Let $\epsilon = \frac{1}{2}|\mu(z_0)| > 0$. By the continuity of μ and openness of D, there is a $\delta > 0$ so that $\mu(w) \in B_\epsilon(\mu(z_0))$ whenever $w \in B_\delta(z_0) \subset D$. Since $\mu(w) \in B_\epsilon(\mu(z_0))$, we have: $|\mu(z_0)| - |\mu(w)| \leq |\mu(w) - \mu(z_0)| < \frac{1}{2}|\mu(z_0)|$. Therefore, $|\mu(w)| > |\mu(z_0)| - \frac{1}{2}|\mu(z_0)| = \frac{1}{2}|\mu(z_0)| > 0$. Hence, $\mu(w) > 0$ for all $w \in B_\delta(z_0)$.
Since $\int_f \mu(z)|dz| = 0$ for all C^1 paths $f : [a,b] \to D$, it must hold for the path $f : [0,1] \to D$ defined by $f(t) = z_0 + \frac{1}{2}\delta t$. Note that $f'(t) = \frac{\delta}{2} > 0$. Also, for all $t \in [0,1]$, we have $f(t) \in B_\delta(z_0)$. Then by the above argument, we have $\mu(f(t)) > 0$ for all $t \in [0,1]$.
In that case, $\int_f \mu(z)|dz| = \int_0^1 \mu(f(t))|f'(t)|\, dt > 0$, which contradicts our hypothesis.
Therefore, we must have $\mu \equiv 0$ on D. □

Applying Lemma 3.1 to the analysis for the conditions imposed on ρ by elements of $\mathcal{L}(\mathbb{H})$, we see that $0 = \mu_\gamma(z) = \rho(z) - \rho(\gamma(z))|\gamma'(z)|$, and this is true for all $z \in \mathbb{H}$.
Now if $\gamma, \varphi \in \mathcal{L}(\mathbb{H})$, we have:

$$\begin{aligned}
\mu_{\gamma\circ\varphi}(z) &= \rho(z) - \rho\Big((\gamma\circ\varphi)(z)\Big)|(\gamma\circ\varphi)'(z)| \\
&= \rho(z) - \rho\Big(\gamma(\varphi(z))\Big)|\gamma'(\varphi(z))||\varphi'(z)| \\
&= \rho(z) - \rho(\varphi(z))|\varphi'(z)| + \big(\rho(\varphi(z)) - \rho\big(\gamma(\varphi(z))\big)|\gamma'(\varphi(z))|\big)\,|\varphi'(z)| \\
&= \mu_\varphi(z) + \mu_\gamma(\varphi(z))|\varphi'(z)|
\end{aligned}$$

Notice that if $\mu_\gamma \equiv 0$ and $\mu_\varphi \equiv 0$, then $\mu_{\gamma\circ\varphi} \equiv 0$. Therefore, instead of considering $\gamma \in \mathcal{L}(\mathbb{H})$, we may consider γ in a generating set for $\mathcal{L}(\mathbb{H})$. It can be shown that a generating set for $\mathcal{L}(\mathbb{H})$ is given by elements of the form $m(z) = az + b$ $(a > 0, b \in \mathbb{R})$ and the function $J(z) = -\frac{1}{z}$. We will consider the conditions imposed on ρ by the functions of this generating set.

First consider $\gamma(z) = z + b$ $(a = 1)$. Then we have $\gamma'(z) = 1$ so that

$$\mu_\gamma(z) = \rho(z) - \rho(\gamma(z))|\gamma'(z)| = \rho(z) - \rho(z+b).$$

Now, if $\mu_\gamma \equiv 0$, then we must have $\rho(z) = \rho(z+b)$ for all $z \in \mathbb{C}$, $b \in \mathbb{R}$. In particular, this implies that ρ depends only on $\mathrm{Im}(z)$. Let $y = \mathrm{Im}(z)$ and consider a function $r : (0, \infty) \to (0, \infty)$ defined by $r(y) = \rho(iy)$. Note that since ρ depends only on its imaginary part, we have

$$\rho(z) = \rho(i\mathrm{Im}(z)) = r(\mathrm{Im}(z)) = r(y).$$

Now consider $\gamma(z) = az$ $(a > 0, b = 0)$. Then $\gamma'(z) = a$ so that

$$\mu_\gamma(z) = \rho(z) - \rho(\gamma(z))|\gamma'(z)| = \rho(z) - a\rho(az) = r(y) - ar(ay).$$

The condition $\mu_\gamma \equiv 0$ implies that $r(y) = ar(ay)$ for all $y > 0$. Letting $y = 1$, we get $r(1) = ar(a)$; therefore, $r(a) = \dfrac{r(1)}{a}$ for all $a > 0$. Thus, $\rho(z) = r(\mathrm{Im}(z)) = \dfrac{r(1)}{\mathrm{Im}(z)} = \dfrac{c}{\mathrm{Im}(z)}$ for some constant c. The value of c is arbitrary, so we will take $c = \frac{1}{2}$ for later convenience.

In our construction of ρ, we only used elements of the generating set for $\mathcal{L}(\mathbb{H})$ of the form $m(z) = az + b$, where $a > 0$ and $b \in \mathbb{R}$. Therefore, $\mathrm{length}_\rho(f) = \mathrm{length}_\rho(m \circ f)$ for any m of this form. The question we have yet to answer is: Does this ρ produce a length function that is invariant under composition with elements of $\mathcal{M}(\mathbb{H})$? This question is answered by the following proposition.

Proposition 3.7. *If $f : [a, b] \to \mathbb{H}$ is a piecewise C^1 path, then*

$$\mathit{length}_\rho(f) = \mathit{length}_\rho(\gamma \circ f)$$

for any $\gamma \in \mathcal{M}(\mathbb{H})$.

Proof. We have already seen that $\mathrm{length}_\rho(f) = \mathrm{length}_\rho(m \circ f)$ for any $m = az+b$ $(a > 0, b \in \mathbb{R})$.

Next consider $\text{length}_\rho(J \circ f)$, where $J(z) = -\dfrac{1}{z}$. Note that $J'(z) = \dfrac{1}{z^2}$, and so we have

$$\begin{aligned}
\mu_J(z) &= \rho(z) - \rho(J(z))|J'(z)| \\
&= \rho(z) - \rho\left(-\frac{1}{z}\right)\frac{1}{|z|^2} \\
&= \rho(z) - \rho\left(-\frac{\overline{z}}{|z|^2}\right)\frac{1}{|z|^2} \\
&= \frac{1}{2\text{Im}(z)} - \frac{1}{2\text{Im}\left(-\overline{z}/|z|^2\right)}\frac{1}{|z|^2} \\
&= \frac{1}{2\text{Im}(z)} - \frac{1}{2\text{Im}(-\overline{z})/|z|^2}\cdot\frac{1}{|z|^2} \\
&= \frac{1}{2\text{Im}(z)} - \frac{1}{2\text{Im}(-\overline{z})} \\
&= \frac{1}{2\text{Im}(z)} - \frac{1}{2\text{Im}(z)} = 0
\end{aligned}$$

Therefore, $\text{length}_\rho(f) - \text{length}_\rho(J \circ f) = \displaystyle\int_f \mu_J(z)|dz| = \int_f 0|dz| = 0$.
Hence, $\text{length}_\rho(f) = \text{length}_\rho(J \circ f)$.

Finally, consider $\text{length}_\rho(B \circ f)$, where $B(z) = -\overline{z}$. Write $f(t) = x(t) + iy(t)$, where $x : [a,b] \to \mathbb{R}$ and $y : [a,b] \to \mathbb{R}$. Note then that $B \circ f(t) = -x(t) + iy(t)$,
$|(B \circ f)'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2} = |f'(t)|$, and $\text{Im}((B \circ f)(t)) = y(t) = \text{Im}(f(t))$.
Therefore:
$\text{length}_\rho(B \circ f) = \displaystyle\int_a^b \frac{1}{2\text{Im}(B \circ f)(t)}|(B \circ f)'(t)|\ dt = \int_a^b \frac{1}{2\text{Im}(f(t))}|f'(t)|\ dt = \text{length}_\rho(f)$.

Since the elements considered above form a generating set for $\mathcal{M}(\mathbb{H})$, we must have $\text{length}_\rho(f) = \text{length}_\rho(\gamma \circ f)$ for every element of $\mathcal{M}(\mathbb{H})$. □

Motivated by these results, we make the following definitions.

Definition 3.4. The *hyperbolic element of arc length* $\lambda_{\mathbb{H}}|dz|$ *on* $\mathbb{H}$ is defined to be

$$\lambda_{\mathbb{H}}|dz| = \frac{1}{2\text{Im}(z)}|dz|.$$

Definition 3.5. Let $f : [a,b] \to \mathbb{H}$ be a piecewise C^1 path. The *hyperbolic length of* f is defined to be

$$\text{length}_{\mathbb{H}}(f) = \int_f \lambda_{\mathbb{H}}(z)|dz| = \int_f \frac{1}{2\text{Im}(z)}|dz| = \int_a^b \frac{1}{2\text{Im}(f(t))}|f'(t)|\ dt.$$

Definition 3.6. For $x, y \in \mathbb{H}$, define the *hyperbolic distance between x and y* by

$$\mathrm{d}_{\mathbb{H}}(x, y) = \inf\{\mathrm{length}_{\mathbb{H}}(f) : f \in \Gamma[x, y]\},$$

where $\Gamma[x, y] = \{f : [a, b] \to \mathbb{H} : f \text{ is piecewise } C^1, f(a) = x, \text{ and } f(b) = y\}$.

Note that we have now constructed length and distance functions that are invariant under composition with elements of $\mathcal{M}(\mathbb{H})$.

3.4 The Poincaré Disk Model, $\mathbb{D}$

Recall that our goal is to construct the hyperbolic metric on the Poincaré disk model of the hyperbolic plane. The underlying space for this model is the unit disk $\mathbb{D} = \{z : |z| < 1\}$. To transform the model $\mathbb{H}$ of the hyperbolic plane into the Poincaré disk model $\mathbb{D}$, consider an element $m \in \mathcal{M}$ taking $\mathbb{D}$ to $\mathbb{H}$. Define a hyperbolic line in $\mathbb{D}$ to be the image under m^{-1} of a hyperbolic line in $\mathbb{H}$. We will use the following facts:

1. Every hyperbolic line in $\mathbb{H}$ is contained in a circle in $\widehat{\mathbb{C}}$ orthogonal to $\mathbb{R}$ (by the remark following Definition 3.1);
2. Every element of $\mathcal{M}$ takes circles in $\widehat{\mathbb{C}}$ to circles in $\widehat{\mathbb{C}}$ (by Proposition 3.4);
3. Every element of $\mathcal{M}$ preserves magnitudes of angles (by Proposition 3.5); and
4. $\mathbb{R}$ is mapped by m^{-1} to the unit circle, $\{z : |z| = 1\}$ since the boundary of $\mathbb{H}$ must be mapped to the boundary of $\mathbb{D}$ by m^{-1}.

Combined, these facts imply that a hyperbolic line in $\mathbb{D}$ is the intersection of $\mathbb{D}$ with a circle in $\widehat{\mathbb{C}}$ orthogonal to the unit circle.

If $f : [a, b] \to \mathbb{D}$ is a piecewise C^1 path in $\mathbb{D}$ and ξ is any element of $\mathcal{L}$ taking $\mathbb{D}$ to $\mathbb{H}$, then the composition $\xi \circ f : [a, b] \to \mathbb{H}$ is a piecewise C^1 path in $\mathbb{H}$. Now, if ξ and η are any two elements of $\mathcal{M}$ taking $\mathbb{D}$ to $\mathbb{H}$, then $q = \eta \circ \xi^{-1}$ is an element of $\mathcal{M}(\mathbb{H})$. Since hyperbolic length is invariant under composition with elements of $\mathcal{M}(\mathbb{H})$, we have

$$\mathrm{length}_{\mathbb{H}}(\xi \circ f) = \mathrm{length}_{\mathbb{H}}(q \circ \xi \circ f) = \mathrm{length}_{\mathbb{H}}(\eta \circ f).$$

Therefore, we make the following definition.

Definition 3.7. Let ξ be any element of $\mathcal{L}$ taking $\mathbb{D}$ to $\mathbb{H}$ and $f : [a, b] \to \mathbb{D}$ be a piecewise C^1 path into $\mathbb{D}$. The *hyperbolic length of f in $\mathbb{D}$* is given by

$$\text{length}_{\mathbb{D}}(f) = \text{length}_{\mathbb{H}}(\xi \circ f)$$

By the above work, we see that this definition is independent of the choice of ξ. Therefore, we can find an integral expression for $\text{length}_{\mathbb{D}}(f)$ using any $\xi \in \mathcal{L}$ taking $\mathbb{D}$ to $\mathbb{H}$. In particular, we will use the element ξ defined by $\xi(z) = \dfrac{\frac{i}{\sqrt{2}}z + \frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}z - \frac{i}{\sqrt{2}}}$. Note that $\text{Im}(\xi(z)) = \dfrac{1 - |z|^2}{|-z-i|^2}$ and $|\xi'(z)| = \dfrac{2}{|z+i|^2}$. Hence,

$$\frac{1}{2\text{Im}(\xi(z))}|\xi'(z)| = \frac{1}{1-|z|^2}.$$

Therefore,

$$\begin{aligned}\text{length}_{\mathbb{D}}(f) = \text{length}_{\mathbb{H}}(\xi \circ f) &= \int_{\xi \circ f} \frac{1}{2\text{Im}(z)}|dz| = \int_a^b \frac{1}{2\text{Im}(\xi(f(t)))}|\xi'(f(t))||f'(t)|\, dt \\ &= \int_f \frac{1}{2\text{Im}(\xi(z))}|\xi'(z)||dz| = \int_f \frac{1}{1-|z|^2}|dz|.\end{aligned}$$

Thus, the hyperbolic element of arc length on $\mathbb{D}$ is given by $\lambda_{\mathbb{D}}(z)|dz| = \dfrac{1}{1-|z|^2}|dz|$. Therefore, we make the following definition.

Definition 3.8. The hyperbolic length of any piecewise C^1 path $f : [a, b] \to \mathbb{D}$ is given by

$$\text{length}_{\mathbb{D}}(f) = \int_f \lambda_{\mathbb{D}}(z)|dz| = \int_f \frac{1}{1-|z|^2}|dz|.$$

This length function is invariant under composition with conformal automorphisms of $\mathbb{D}$.

Lemma 3.2. *Every conformal automorphism of $\mathbb{D}$ is the composition of maps of the form*

(i) $g_\tau(z) = e^{i\tau}z$, $\tau \in \mathbb{R}$, *and*

(ii) $\phi_a(z) = \dfrac{z-a}{1-\overline{a}z}$, $a \in \mathbb{D}$.

Proof. We first show that any function of form (ii) is a conformal automorphism of $\mathbb{D}$. To that end, first consider $\{z : |z| = 1\}$. Now, if $|z| = 1$, we have

$$|\phi_a(z)| = \left|\frac{z-a}{1-\overline{a}z}\right| = \left|\frac{1}{\overline{z}}\right|\left|\frac{z-a}{1-\overline{a}z}\right| = \left|\frac{z-a}{\overline{z}-\overline{a}|z|^2}\right| = \left|\frac{z-a}{\overline{z}-\overline{a}}\right| = 1.$$

Thus, ϕ_a maps $\{z : |z| = 1\}$ to $\{z : |z| = 1\}$. Moreover, $\phi_a(a) = 0$. Therefore, ϕ_a maps $\mathbb{D}$ to $\mathbb{D}$. Also, a calculation shows that $(\varphi_a)^{-1} = \varphi_{-a}$. Thus, φ_a is one-to-one and onto. Hence, φ_a is a conformal automorphism of $\mathbb{D}$.

Now, suppose that h is a conformal automorphism of $\mathbb{D}$. Let $a = h(0)$ and consider the function $G = \varphi_a \circ h$. Note that G is a composition of conformal automorphisms of $\mathbb{D}$, so G itself must be a conformal automorphism of $\mathbb{D}$. Also, $G(0) = \varphi_a(h(0)) = \varphi_a(a) = 0$. Then the Schwarz Lemma implies that $|G'(0)| \leq 1$. Applying the Schwarz Lemma to G^{-1} results in $\left|\frac{1}{G'(0)}\right| = |(G^{-1})'(0)| \leq 1$. Hence, we may conclude that $|G'(0)| = 1$, which implies that G is of the form $G(z) = e^{i\tau}z = g_\tau(z)$ $(\tau \in \mathbb{R})$. Since $G = \varphi_a \circ h = g_\tau$, we must have $h = g_\tau \circ (\varphi_a)^{-1} = g_\tau \circ \varphi_{-a}$. Thus, h can be written as a composition of functions of the form (i) and (ii). □

Proposition 3.8. *Let $h : \mathbb{D} \to \mathbb{D}$ be a conformal automorphism of the disk and $f : [a, b] \to \mathbb{D}$ a piecewise C^1 path. Then $\text{length}_{\mathbb{D}}(h \circ f) = \text{length}_{\mathbb{D}}(f)$.*

Proof. By Lemma 3.2, we need to consider only two cases.

<u>Case 1:</u> If h is a rotation, then $h(z) = e^{i\tau}z$ for some $\tau \in \mathbb{R}$. Therefore, $|h'(z)| = 1$, so we have

$$\lambda_{\mathbb{D}}(h(z))|h'(z)| = \lambda_{\mathbb{D}}(e^{i\tau}z) = \frac{1}{1 - |e^{i\tau}z|^2} = \frac{1}{1 - |z|^2} = \lambda_{\mathbb{D}}(z).$$

Hence:

$$\begin{aligned} \text{length}_{\mathbb{D}}(h\circ f) &= \int_{h\circ f} \lambda_{\mathbb{D}}(z)|dz| = \int_a^b \lambda_{\mathbb{D}}(h\circ f(t))|h'(f(t))||f'(t)|\,dt = \int_f \lambda_{\mathbb{D}}(h(z))|h'(z)||dz| \\ &= \int_f \lambda_{\mathbb{D}}(z)|dz| = \text{length}_{\mathbb{D}}(f) \end{aligned}$$

<u>Case 2:</u> If $h(z) = \dfrac{z - a}{1 - \overline{a}z}$ for some $a \in \mathbb{D}$, then $|h'(z)| = \dfrac{1 - |a|^2}{|1 - \overline{a}z|^2}$. Therefore:

$$\begin{aligned} \lambda_{\mathbb{D}}(h(z))|h'(z)| &= \lambda_{\mathbb{D}}\left(\frac{z - a}{1 - \overline{a}z}\right) \cdot \frac{1 - |a|^2}{|1 - \overline{a}z|^2} = \frac{1}{1 - \left|\dfrac{z - a}{1 - \overline{a}z}\right|^2} \cdot \frac{1 - |a|^2}{|1 - \overline{a}z|^2} \\ &= \frac{1 - |a|^2}{|1 - \overline{a}z|^2 - |z - a|^2} = \frac{1 - |a|^2}{1 - |z|^2 - |a|^2 + |a|^2|z|^2} \\ &= \frac{1 - |a|^2}{(1 - |z|^2)(1 - |a|^2)} = \frac{1}{1 - |z|^2} = \lambda_{\mathbb{D}}(z). \end{aligned}$$

Then, as above, we must have $\text{length}_{\mathbb{D}}(h \circ f) = \text{length}_{\mathbb{D}}(f)$.

Since any conformal automorphism of $\mathbb{D}$ is the composition of maps of the above forms, we have $\text{length}_{\mathbb{D}}(h \circ f) = \text{length}_{\mathbb{D}}(f)$ for any conformal automorphism h. $\square$

We are finally ready to define the hyperbolic distance between two points in $\mathbb{D}$.

Definition 3.9. For $P, Q \in \mathbb{D}$, define the *hyperbolic distance between P and Q* by

$$\mathrm{d}_{\mathbb{D}}(P,Q) = \inf\{\text{length}_{\mathbb{D}}(f) : f \in \Gamma[P,Q]\},$$

where $\Gamma[P,Q] = \{f : [a,b] \to \mathbb{D} : f \text{ is piecewise } C^1, f(a) = P, \text{ and } f(b) = Q\}$.

In fact, we can find an explicit formula for the hyperbolic distance between two points.

Proposition 3.9. *If $P, Q \in \mathbb{D}$, then the hyperbolic distance from P to Q is given by*

$$d_{\mathbb{D}}(P,Q) = \frac{1}{2}\log\left(\frac{1+\left|\dfrac{P-Q}{1-\overline{P}Q}\right|}{1-\left|\dfrac{P-Q}{1-\overline{P}Q}\right|}\right).$$

Proof. <u>Case 1:</u> First consider the case where $P = 0$ and $Q \in \mathbb{R}$.

We claim that among all C^1 curves of the form $f(t) = t + iw(t)$, $0 \le t \le 1-\epsilon$, that satisfy $f(0) = 0$ and $f(1-\epsilon) = 1-\epsilon$, the one of least length is $\mu(t) = t$. To see that this is the case, consider any such f. We have:

$$\begin{aligned}\text{length}_{\mathbb{D}}(f) &= \int_f \lambda_{\mathbb{D}}(z)|dz| = \int_f \frac{1}{1-|z|^2}|dz| = \int_0^{1-\epsilon} \frac{1}{1-|f(t)|^2}|f'(t)|\,dt \\ &= \int_0^{1-\epsilon} \frac{1}{1-t^2-[w(t)]^2}|1+iw'(t)|\,dt = \int_0^{1-\epsilon} \frac{1}{1-t^2-[w(t)]^2}(1+[w'(t)]^2)^{1/2}\,dt.\end{aligned}$$

Note that $\dfrac{1}{1-t^2-[w(t)]^2} \ge \dfrac{1}{1-t^2}$ and $(1+[w'(t)]^2)^{1/2} \ge 1$.

Thus,

$$\text{length}_{\mathbb{D}}(f) \ge \int_0^{1-\epsilon} \frac{1}{1-t^2}\,dt = \text{length}_{\mathbb{D}}(\mu).$$

Note that with small modifications, this result can be extended to piecewise C^1 curves.

Moreover, if a piecewise C^1 curve connecting 0 to $1-\epsilon$ is not of the form

$$f(t) = t + iw(t), \qquad (*)$$

it may cross itself. A shorter curve can be obtained by eliminating the loops. If the resulting curve is not of the form $(*)$, it can be shown that it will be longer than a curve of the form

$(*)$. Hence, the shortest curve connecting 0 to $1-\epsilon$ is $\mu(t)=t$.

Note that

$$\begin{aligned}\text{length}_{\mathbb{D}}(\mu) &= \int_0^{1-\epsilon} \frac{1}{1-t^2}\,dt = \frac{1}{2}\int_0^{1-\epsilon}\left(\frac{1}{1+t}-\frac{1}{1-t}\right)\,dt\\ &= \frac{1}{2}\left[\log\left|\frac{1+t}{1-t}\right|\right]_0^{1-\epsilon} = \frac{1}{2}\log\left|\frac{2-\epsilon}{\epsilon}\right|.\end{aligned}$$

Taking $1-\epsilon=Q$, we get $\mathrm{d}_{\mathbb{D}}(P,Q)=\mathrm{d}_{\mathbb{D}}(0,Q)=\dfrac{1}{2}\log\left|\dfrac{1+Q}{1-Q}\right|$.

<u>Case 2:</u> Now we consider the general case where $P,Q\in\mathbb{D}$.

Define $\phi(z)=\dfrac{z-P}{1-\overline{P}z}$. Then Proposition 3.8 implies $\mathrm{d}_{\mathbb{D}}(P,Q)=\mathrm{d}_{\mathbb{D}}(\phi(P),\phi(Q))=\mathrm{d}_{\mathbb{D}}(0,\phi(Q))$. Since we can find $\tau\in\mathbb{R}$ so that $|\phi(Q)|=e^{i\tau}\phi(Q)$, Proposition 3.8 also implies $\mathrm{d}_{\mathbb{D}}(0,\phi(Q))=\mathrm{d}_{\mathbb{D}}(0,|\phi(Q)|)$. Finally, note that $|\phi(Q)|=\left|\dfrac{Q-P}{1-\overline{P}Q}\right|=\left|\dfrac{P-Q}{1-\overline{P}Q}\right|$. Therefore, Case 1 implies $\mathrm{d}_{\mathbb{D}}(P,Q)=\dfrac{1}{2}\log\left(\dfrac{1+\left|\dfrac{P-Q}{1-\overline{P}Q}\right|}{1-\left|\dfrac{P-Q}{1-\overline{P}Q}\right|}\right)$. □

CHAPTER 4. TWO-POINT DISTORTION THEOREMS

We will now discuss two-point distortion theorems for univalent functions. The theorems on which we will focus were proved by Kim and Minda in [6] and Ma and Minda in [8]. These theorems are extensions of the work done by Blatter in [2].

4.1 An Invariant Koebe Distortion Theorem

We will begin by looking at a special case of Kim and Minda's distortion theorem. The general theorem will be discussed in section 4.2. This result is given in [6] and is also proved by Graham and Kohr in [4].

Theorem 4.1. *Suppose f is univalent on $\mathbb{D}$. Then for $a, b \in \mathbb{D}$,*

$$|f(a) - f(b)| \geq \frac{\sinh(2d_{\mathbb{D}}(a,b))}{2\exp(2d_{\mathbb{D}}(a,b))} \max\left\{(1-|a|^2)|f'(a)|, (1-|b|^2)|f'(b)|\right\}.$$

Conversely, if a nonconstant analytic function f satisfies this inequality, then f is univalent on $\mathbb{D}$.

Proof. If g is a normalized univalent function, then Theorem 2.5 implies that

$$|g(z)| \geq \frac{|z|}{(1+|z|)^2}.$$

Now, since $d_{\mathbb{D}}(y,z) = \dfrac{1}{2}\log\left(\dfrac{1+\left|\frac{y-z}{1-\overline{y}z}\right|}{1-\left|\frac{y-z}{1-\overline{y}z}\right|}\right)$ for any $y, z \in \mathbb{D}$, we have

$$d_{\mathbb{D}}(0,z) = \frac{1}{2}\log\left(\frac{1+\left|\frac{-z}{1-\overline{0}z}\right|}{1-\left|\frac{0-z}{1-\overline{0}z}\right|}\right) = \frac{1}{2}\log\left(\frac{1+|z|}{1-|z|}\right).$$

Therefore, $\exp(2d_{\mathbb{D}}(0,z)) = \dfrac{1+|z|}{1-|z|}$.

Also, because $\sinh(t) = \dfrac{\exp(t)+\exp(-t)}{2}$, we have

$$\begin{aligned}
\sinh(2d_{\mathbb{D}}(0,z)) &= \frac{1}{2}\Bigg(\exp\bigg(\log\Big(\frac{1+|z|}{1-|z|}\Big)\bigg) - \exp\bigg(-\log\Big(\frac{1+|z|}{1-|z|}\Big)\bigg)\Bigg) \\
&= \frac{1}{2}\left(\frac{1+|z|}{1-|z|} - \frac{1-|z|}{1+|z|}\right) \\
&= \frac{1}{2}\left(\frac{(1+|z|)^2-(1-|z|)^2}{(1-|z|)(1+|z|)}\right) \\
&= \frac{1}{2}\left(\frac{1+2|z|+|z|^2-1+2|z|-|z|^2}{(1-|z|)(1+|z|)}\right) \\
&= \frac{2|z|}{(1-|z|)(1+|z|)}
\end{aligned}$$

Therefore,

$$\frac{\sinh(2d_{\mathbb{D}}(0,z))}{2\exp(2d_{\mathbb{D}}(0,z))} = \frac{2|z|}{(1-|z|)(1+|z|)} \cdot \frac{1-|z|}{2(1+|z|)} = \frac{|z|}{(1+|z|)^2}.$$

Thus, the lower bound in Theorem 2.5 can be restated as

$$|g(z)| \geq \frac{\sinh(2d_{\mathbb{D}}(0,z))}{2\exp(2d_{\mathbb{D}}(0,z))}$$

for any normalized univalent function g.

Now let f be any function univalent in $\mathbb{D}$. Note in particular that f need not be normalized. Let $a, b \in \mathbb{D}$ and $T(z) = \dfrac{z+a}{1+\overline{a}z}$. Then T is a conformal automorphism of $\mathbb{D}$ mapping 0 to a. Consider the function g defined by

$$g(z) = \frac{f\circ T(z) - f\circ T(0)}{(f\circ T)'(0)} = \frac{f\circ T(z) - f(a)}{(1-|a|^2)f'(a)}.$$

Let $z_0 = \dfrac{b-a}{1-\overline{a}b}$. Then $T(z_0) = b$ and so we have:

$$|g(z_0)| = \left|\frac{f\circ T(z_0) - f(a)}{(1-|a|^2)f'(a)}\right| = \left|\frac{f(b)-f(a)}{(1-|a|^2)f'(a)}\right|$$

Also, since g is a normalized univalent function and $d_{\mathbb{D}}$ is invariant under conformal automorphisms of $\mathbb{D}$, Theorem 2.5 implies that

$$|g(z_0)| \geq \frac{\sinh(2d_{\mathbb{D}}(0,z_0))}{2\exp(2d_{\mathbb{D}}(0,z_0))} = \frac{\sinh(2d_{\mathbb{D}}(T(0),T(z_0)))}{2\exp(2d_{\mathbb{D}}(T(0),T(z_0)))} = \frac{\sinh(2d_{\mathbb{D}}(a,b))}{2\exp(2d_{\mathbb{D}}(a,b))}.$$

Hence:

$$\left|\frac{f(b)-f(a)}{(1-|a|^2)f'(a)}\right| \geq \frac{\sinh(2d_{\mathbb{D}}(a,b))}{2\exp(2d_{\mathbb{D}}(a,b))},$$

which implies that

$$|f(b)-f(a)| \geq \frac{\sinh(2d_{\mathbb{D}}(a,b))}{2\exp(2d_{\mathbb{D}}(a,b))}(1-|a|^2)|f'(a)|.$$

By a similar argument, we also have

$$|f(b)-f(a)| \geq \frac{\sinh(2d_{\mathbb{D}}(a,b))}{2\exp(2d_{\mathbb{D}}(a,b))}(1-|b|^2)|f'(b)|.$$

Taking the maximum of these two lower bounds results in

$$|f(b)-f(a)| \geq \frac{\sinh(2d_{\mathbb{D}}(a,b))}{2\exp(2d_{\mathbb{D}}(a,b))}\max\left\{(1-|a|^2)|f'(a)|,(1-|b|^2)|f'(b)|\right\},$$

which is the desired inequality.

Next, suppose f is a nonconstant analytic function on $\mathbb{D}$ which satisfies the inequality. If f is not univalent, then we can find some $a, b \in \mathbb{D}$ with $a \neq b$ and $f(a) = f(b)$. Since $f(a) = f(b)$, we have $\dfrac{\sinh(2d_{\mathbb{D}}(a,b))}{2\exp(2d_{\mathbb{D}}(a,b))}\max\left\{(1-|a|^2)|f'(a)|,(1-|b|^2)|f'(b)|\right\} = 0$. Note that $\dfrac{\sinh(2d_{\mathbb{D}}(a,b))}{2\exp(2d_{\mathbb{D}}(a,b))} \neq 0$ since $a \neq b$. Hence, $(1-|a|^2)|f'(a)| = (1-|b|^2)|f'(b)| = 0$. Because $a, b \in \mathbb{D}$, we have $|a| < 1$ and $|b| < 1$, so that $(1-|a|^2) > 0$ and $(1-|b|^2) > 0$. Hence, $f'(a) = f'(b) = 0$, and so f is not univalent in any neighborhood of a or b. Since f is not univalent in any neighborhood of a, we can find two sequences $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ of distinct points in $\mathbb{D}$ so that $\lim_{n\to\infty} c_n = \lim_{n\to\infty} d_n = a$ and $f(c_n) = f(d_n)$ for all $n \in \mathbb{N}$. Apply the inequality again to conclude that $f'(c_n) = 0$ for all $n \in \mathbb{N}$, which implies that f must be constant. But this contradicts the hypothesis that f is nonconstant. Therefore, f must be univalent on $\mathbb{D}$. $\square$

4.2 Lower Bounds for $|f(a) - f(b)|$

The work that led to many of the results regarding two-point distortion theorems was done by Blatter in [2]. Blatter's distortion theorem is as follows.

Theorem 4.2. *Suppose f is univalent in $\mathbb{D}$ and $a, b \in \mathbb{D}$. Then*

$$|f(a)-f(b)|^2 \geq \frac{\sinh^2(2d_{\mathbb{D}}(a,b))}{8\cosh(4d_{\mathbb{D}}(a,b))}\Big(\big((1-|a|^2)|f'(a)|\big)^2+\big((1-|b|^2)|f'(b)|\big)^2\Big).$$

Conversely, if a nonconstant analytic function f satisfies this inequality, then f is univalent on $\mathbb{D}$.

Notice in particular that Blatter's theorem gives necessary and sufficient conditions for univalence. Moreover, the theorem requires no normalization on f. Since Blatter's distortion theorem is a special case of the theorem proved by Kim and Minda, we will not give the proof here. The method used by Kim and Minda to prove their distortion theorem is an extension of the method used by Blatter. Before we can prove Kim and Minda's distortion theorem, we need some more preliminary results.

The first result that we need, the Minimum Principle, is an extension of a result proved by Blatter in [2]. The other results are from [6].

Lemma 4.1 (Minimum Principle)**.** *Suppose that a function $u : [-L, L] \to \mathbb{R}$ satisfies the following two conditions:*

(i) $|u'| \leq q$,

(ii) $u'' \leq p(q^2-(u')^2)$,

where p and q are positive constants. If v satisfies

(i) $|v'| \leq q$,

(ii) $v'' = p(q^2-(v')^2)$,

(iii) $v(L) = u(L)$, *and*

(iv) $v(-L) = u(-L)$,

then $u(s) \geq v(s)$ for all $s \in [-L, L]$.

Proof. First note that condition (i) implies

$$|u(L)-u(-L)| = \left|\int_{-L}^{L} u'(s)\ ds\right| \le \int_{-L}^{L} |u'(s)|\ ds \le \int_{-L}^{L} q\ ds = 2qL,$$

and so $-2qL \le u(L)-u(-L) \le 2qL$. Moreover, $u(L)-u(-L)=2qL$ if and only if $u'(s)\equiv q$ and $u(L)-u(-L)=-2qL$ if and only if $u'(s)\equiv -q$.

Since $v''=p(q^2-(v')^2)$, v is of the form

$$v(s)=\frac{1}{p}\log(\cosh(pqs)+\tau\sinh(pqs))+C,$$

where $\tau, C\in\mathbb{R}$.

We next show that $|\tau|\le 1$. To that end, note that

$$\begin{aligned}
v(s) &= \frac{1}{p}\log[\cosh(pqs)+\tau\sinh(pqs)]+\log C\\
&= \frac{1}{p}\log\left[\frac{1}{2}(\exp(pqs)+\exp(-pqs))+\frac{\tau}{2}(\exp(pqs)-\exp(-pqs))\right]+\log C\\
&= \frac{1}{p}\log\big(\exp(pqs)+\exp(-pqs)+\tau(\exp(pqs)-\exp(-pqs))\big)-\frac{1}{p}\log 2+\log C\\
&= \frac{1}{p}\log\Big(\exp(-pqs)\big(\exp(2pqs)+1+\tau\exp(2pqs)-\tau\big)\Big)-\frac{1}{p}\log 2+\log C\\
&= \frac{1}{p}\log\big(\exp(2pqs)+1+\tau\exp(2pqs)-\tau\big)-qs-\frac{1}{p}\log 2+\log C
\end{aligned}$$

Therefore,

$$\begin{aligned}
v'(s) &= \frac{1}{p}\left(\frac{2pq\exp(2pqs)(\tau+1)}{\exp(2pqs)(\tau+1)-\tau+1}\right)-q\\
&= \frac{2q\big(\exp(2pqs)(\tau+1)-\tau+1\big)-2q(-\tau+1)}{\exp(2pqs)(\tau+1)-\tau+1}-q\\
&= q+\frac{2q(\tau-1)}{\exp(2pqs)(\tau+1)-\tau+1}
\end{aligned}$$

By hypothesis, we have $|v'(s)|\le q$ for all s. Hence, $\left|q+\dfrac{2q(\tau-1)}{\exp(2pqs)(\tau+1)-\tau+1}\right|\le q$, and so we must have $\dfrac{2q(\tau-1)}{\exp(2pqs)(\tau+1)-\tau+1}\le 0$. Now, $2q(\tau-1)\le 0$ if and only if $\tau\le 1$. Moreover, we have

$$\begin{aligned}
\exp(2pqs)(\tau+1)-\tau+1<0 &\Rightarrow \tau(\exp(2pqs)-1)+\exp(2pqs)+1<0\\
&\Rightarrow \tau<\frac{1+\exp(2pqs)}{1-\exp(2pqs)}=\frac{-1}{\tanh(pqs)}\quad\text{for all } s\\
&\Rightarrow \tau<-1.
\end{aligned}$$

Therefore, $\dfrac{2q(\tau-1)}{\exp(2pqs)(\tau+1)-\tau+1} \leq 0$ only for $|\tau| \leq 1$. Hence, the condition $|v'| \leq q$ implies $|\tau| \leq 1$.

Now, if $|\tau| = 1$, we obtain $v'(s) = q$ or $v'(s) = -q$. If $v'(s) = q$, then we have $v(L) - v(-L) = \int_{-L}^{L} v'(s)\ dx = \int_{-L}^{L} q\ ds = 2qL$. Then also (by the boundary conditions) $u(L) - u(-L) = 2qL$, and so $u'(s) = q$ (by condition (i)). Similarly, if $v'(s) = -q$, then $u'(s) = -q$. Hence, for $|\tau| = 1$, we have $u'(s) = v'(s)$, $u(L) = v(L)$, and $u(-L) = v(-L)$. Thus, $u \equiv v$ and the desired inequality is trivial.

Therefore, we may assume that $|\tau| < 1$. Because $|\tau| < 1$, we have $2q(\tau-1) < 0$ and $\exp(2pqs)(\tau+1)-\tau+1 > 0$. Hence, $\dfrac{2q(\tau-1)}{\exp(2pqs)(\tau+1)-\tau+1} < 0$ and $|v'(s)| < q$ on $[-L, L]$.

Applying Rolle's Theorem to $u-v$, we see that there is an $s_0 \in (-L, L)$ with $u'(s_0) = v'(s_0)$.

We will prove that $u'(s) \leq v'(s)$ for all $s \in [s_0, L]$. Suppose this is not the case. Let $s_1 = \inf\{s \geq s_0 : u'(s) > v'(s)\}$. Then we have $u'(s_1) = v'(s_1) \in (-q, q)$. Now, there is a $\delta > 0$ with $|u'(s)| < q$ on $[s_1, s_1+\delta)$. Therefore, for $t \in (s_1, s_1+\delta)$, we have

$$\frac{u''(t)}{p(q^2-(u')^2(t))} \leq 1 = \frac{v''(t)}{p(q^2-(v')^2(t))}.$$

For $s \in [s_1, s_1+\delta)$, integrating from s_1 to s results in

$$\operatorname{arctanh}\frac{u'}{q}\bigg|_{s_1}^{s} \leq \operatorname{arctanh}\frac{v'}{q}\bigg|_{s_1}^{s}.$$

Since $u'(s_1) = v'(s_1)$, we must have $u'(s) \leq v'(s)$ on $[s_1, s_1+\delta)$. But this contradicts the definition of s_1, so $u'(s) \leq v'(s)$ for all $s \in [s_0, L]$.

Now, $u'(s) - v'(s) \leq 0$ for all $s \in [s_0, L]$, so $u - v$ is monotonically decreasing on this interval. Since $u(L) - v(L) = 0$, we must then have $u(s) - v(s) \geq 0$ on $[s_0, L]$.

A similar argument can be used to show that $u(s) \geq v(s)$ on $[-L, s_0]$. □

Note in particular that the solution v of the differential equation in the lemma can be expressed as

$$v(s) = \frac{1}{p}\log[\cosh(pqs) + \tau\sinh(pqs)] + \log C,$$

where $\tau \in [-1, 1]$ and $C \in \mathbb{R}$. In fact, it can be shown that $C = \left(\dfrac{\exp(pu(L)) + \exp(pu(-L))}{2\cosh(pqL)}\right)^{1/p}$.

Lemma 4.2. *For* $p > 1, q > 0$ *and* $\tau \in [-1, 1]$ *let* $B(\tau) = \int_{-L}^{L} (\cosh(pqs) + \tau \sinh(pqs))^{1/p}\, ds$*. Then for* $\tau \in (-1, 1)$*,* $B(\tau) > B(\pm 1) = \frac{2}{q} \sinh(qL)$*.*

Proof. Since $B(\tau) = \int_{-L}^{L} (\cosh(pqs) + \tau \sinh(pqs))^{1/p}\, ds$, we have

$$B'(\tau) = \frac{1}{p} \int_{-L}^{L} \sinh(pqs)\big(\cosh(pqs) + \tau \sinh(pqs)\big)^{(1-p)/p}\, ds$$

and

$$B''(\tau) = \frac{1-p}{p^2} \int_{-L}^{L} \sinh^2(pqs)\big(\cosh(pqs) + \tau \sinh(pqs)\big)^{(1-2p)/p}\, ds.$$

Since $p > 1$, we have $B''(\tau) < 0$. Thus, B is strictly concave on $[-1, 1]$ and the minimum value of B is either $B(1)$ or $B(-1)$. Since $B(1) = B(-1) = \frac{2}{q} \sinh(qL)$, the result follows. □

Theorem 4.3. *If* $g(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in S$*, then*

$$\left| a_3 - \left(\frac{3-p}{3}\right) a_2^2 \right| + \frac{p}{3}|a_2|^2 \le \begin{cases} 1 + 2\exp\left(\dfrac{2p-3}{p}\right), & 0 < p < \dfrac{3}{2}; \\ \dfrac{8p-3}{3}, & \dfrac{3}{2} \le p. \end{cases}$$

Proof. It suffices to find an upper bound for the functional

$$L_p(g) = \operatorname{Re}\left\{ a_3 - \left(\frac{3-p}{3}\right) a_2^2 \right\} + \frac{p}{3}|a_2|^2$$

over the family of normalized univalent functions.

Notice that

$$\begin{aligned} L_p(g) &= \operatorname{Re}\{a_3\} - \left(\frac{3-p}{3}\right) \operatorname{Re}\{a_2^2\} + \frac{p}{3}|a_2|^2 \\ &= \operatorname{Re}\{a_3\} - \left(\frac{3-p}{3}\right) [(\operatorname{Re}\{a_2\})^2 - (\operatorname{Im}\{a_2\})^2] + \frac{p}{3}[(\operatorname{Re}\{a_2\})^2 + (\operatorname{Im}\{a_2\})^2] \\ &= \operatorname{Re}\{a_3\} - \frac{3-2p}{3}(\operatorname{Re}\{a_2\})^2 + (\operatorname{Im}\{a_2\})^2 \end{aligned}$$

Since replacing $g(z)$ by $-g(-z)$ does not change the value of the functional, we may assume that $\operatorname{Re}\{a_2\} \ge 0$. Moreover, since $0 \le \operatorname{Re}\{a_2\} \le 2$, by Theorem 2.2, there is a unique $\lambda \in [0, 2]$ with $\operatorname{Re}\{a_2\} = \lambda \left(1 + \log \frac{2}{\lambda}\right)$.

In 1960, Jenkins (see [10], p.120) proved that

$$\operatorname{Re}\{a_3\} \le (\operatorname{Re}\{a_2\})^2 - (\operatorname{Im}\{a_2\})^2 - 2\lambda \operatorname{Re}\{a_2\} + \lambda^2 \log \frac{2}{\lambda} + \frac{3}{2}\lambda^2 + 1.$$

Therefore:

$$\begin{aligned}
L_p(g) &\leq \frac{2p}{3}(\mathrm{Re}\{a_2\})^2 - 2\lambda\mathrm{Re}\{a_2\} + \lambda^2 \log\frac{2}{\lambda} + \frac{3}{2}\lambda^2 + 1 \\
&= \frac{2p}{3}\lambda^2\left(1+\log\frac{2}{\lambda}\right)^2 - 2\lambda^2\left(1+\log\frac{2}{\lambda}\right) + \lambda^2\log\frac{2}{\lambda} + \frac{3}{2}\lambda^2 + 1 \\
&= \frac{2p}{3}\lambda^2\left(1+2\log\frac{2}{\lambda}+\left(\log\frac{2}{\lambda}\right)^2\right) - 2\lambda^2 - 2\lambda^2\log\frac{2}{\lambda} + \lambda^2\log\frac{2}{\lambda} + \frac{3}{2}\lambda^2 + 1 \\
&= \lambda^2\left(\frac{2p}{3} - 2 + \frac{3}{2}\right) + \lambda^2\log\frac{2}{\lambda}\left(\frac{4p}{3} - 2 + 1\right) + \lambda^2\left(\log\frac{2}{\lambda}\right)^2\left(\frac{2p}{3}\right) + 1 \\
&= \left(\frac{4p-3}{6}\right)\lambda^2 + \left(\frac{4p-3}{3}\right)\lambda^2\log\frac{2}{\lambda} + \frac{2p}{3}\lambda^2\left(\log\frac{2}{\lambda}\right)^2 + 1 \\
&= H(\lambda)
\end{aligned}$$

Note that $H(0) = 1$ and $H(2) = \left(\frac{4p-3}{6}\right)(4) + 1 = \frac{8p-3}{3}$.

Also,

$$\begin{aligned}
H'(\lambda) = 2\lambda\left(\frac{4p-3}{6}\right) + 2\lambda\left(\frac{4p-3}{3}\right)\log\frac{2}{\lambda} + \left(\frac{4p-3}{3}\right)\lambda^2\left(\frac{-2/\lambda^2}{2/\lambda}\right) \\
+ 2\lambda\left(\frac{2p}{3}\right)\left(\log\frac{2}{\lambda}\right)^2 + \frac{2p}{3}\lambda^2\left(2\log\frac{2}{\lambda}\right)\left(\frac{-2/\lambda^2}{2/\lambda}\right) \\
= \left(\frac{4p-3}{3}\right)\lambda + 2\lambda\left(\frac{4p-3}{3}\right)\log\frac{2}{\lambda} - \left(\frac{4p-3}{3}\right)\lambda + 2\lambda\left(\frac{2p}{3}\right)\left(\log\frac{2}{\lambda}\right)^2 - 2\lambda\left(\frac{2p}{3}\right)\log\frac{2}{\lambda} \\
= 2\lambda\left(\frac{2p-3}{3}\right)\log\frac{2}{\lambda} + 2\lambda\left(\frac{2p}{3}\right)\left(\log\frac{2}{\lambda}\right)^2 \\
= \frac{2\lambda}{3}\log\frac{2}{\lambda}\left[2p - 3 + 2p\log\frac{2}{\lambda}\right]
\end{aligned}$$

If $p \geq \frac{3}{2}$, then $H'(\lambda)$ has no zeros in $(0,2)$. Thus, $H(\lambda)$ is strictly increasing with maximum value $H(2) = \frac{8p-3}{3}$.

If $0 < p < \frac{3}{2}$, then $H'(\lambda)$ has a zero at $\lambda_0 = 2\exp\left(\frac{2p-3}{2p}\right) \in (0,2)$. Note that $H(\lambda)$ is strictly increasing on $(0,\lambda_0)$ and strictly decreasing on $(\lambda_0, 2)$. Therefore, H attains its maximum at $H(\lambda_0) = 1 + 2\exp\left(\frac{2p-3}{p}\right)$. □

Corollary 4.1. *If* $g(z) = z + a_2z^2 + a_3z^3 + \cdots \in S$, *then* $\left|a_3 - \frac{1}{2}a_2^2\right| + \frac{5}{6}|a_2|^2 \leq \frac{13}{3}$.

Proof. The result follows from Theorem 4.3 with $p = \frac{3}{2}$ and from Theorem 2.2. In fact, for $p = \frac{3}{2}$, we have:

$$\left|a_3 - \frac{1}{2}a_2^2\right| + \frac{5}{6}|a_2|^2 = \left|a_3 - \left(\frac{3-p}{3}\right)a_2^2\right| + \frac{p}{3}|a_2|^2 + \frac{1}{3}|a_2|^2 \leq \frac{8p-3}{3} + \frac{1}{3}(2)^2 = \frac{13}{3}.$$

□

The distortion theorems given by Kim and Minda in [6] and Ma and Minda in [8] use the following notation:

- $D_1f(z) = (1-|z|^2)f'(z)$,
- $D_2f(z) = (1-|z|^2)^2f''(z) - 2\overline{z}(1-|z|^2)f'(z)$,
- $D_3f(z) = (1-|z|^2)^3f'''(z) - 6\overline{z}(1-|z|^2)^2f''(z) + 6\overline{z}^2(1-|z|)^2f'(z)$,
- $R_f(z) = \dfrac{f'''(z)}{f'(z)} - \dfrac{3}{2}\left(\dfrac{f''(z)}{f'(z)}\right)^2$, and
- $Q_f(z) = \dfrac{D_2f(z)}{D_1f(z)} = (1-|z|^2)\dfrac{f''(z)}{f'(z)} - 2\overline{z}$.

We are finally ready to state and prove Kim and Minda's distortion theorem.

Theorem 4.4. *Suppose that* f *is univalent in* $\mathbb{D}$. *There is a constant* $P \in (1, 3/2]$ *such that for any* $p \geq P$ *and all* $a, b \in \mathbb{D}$,

$$|f(a) - f(b)| \geq \frac{\sinh(2d_{\mathbb{D}}(a,b))}{2[2\cosh(2pd_{\mathbb{D}}(a,b))]^{1/p}}\left(|D_1f(a)|^p + |D_1f(b)|^p\right)^{1/p}.$$

Conversely, if a nonconstant analytic function f *satisfies this inequality for all* $a, b \in \mathbb{D}$, *then* f *is univalent on* $\mathbb{D}$.

Proof. The proof that a function satisfying the inequality is univalent is the same as the proof of sufficiency in Theorem 4.1.

Now assume that f is univalent and let $a, b \in \mathbb{D}$. Suppose $p \geq 1$ is any number such that

$$\left|(1-|z|^2)^2R_f(z) + \frac{p}{2}(Q_f(z))^2\right| + \frac{p+1}{2}|Q_f(z)|^2 - 2 \leq 16p \tag{4.1}$$

for every univalent function f defined on $\mathbb{D}$ and all $z \in \mathbb{D}$.

Note that if inequality (4.1) holds for one value of $p \geq 1$, then it holds for all larger values of p. Let P be the minimum of all $p \geq 1$ such that (4.1) holds for all univalent functions f defined in $\mathbb{D}$. It suffices to establish (4.1) for $z = 0$ and normalized univalent functions. Now, Corollary 4.1 implies that the inequality

$$\left|a_3 - \left(\frac{3-p}{3}\right)a_2^2\right| + \left(\frac{p+1}{3}\right)|a_2|^2 - \frac{1}{3} \leq \frac{8p}{3}$$

is valid for $p = \dfrac{3}{2}$. Hence, we know that $P \leq \dfrac{3}{2}$. With $p = 1$, the inequality becomes

$$\left|a_3 - \frac{2}{3}a_2^2\right| \leq 3 - \frac{2}{3}|a_2|^2.$$

It has been shown that this inequality does not hold for all normalized univalent functions. Thus, $P > 1$. Hence, (4.1) holds for all $p \geq P$, where $1 < P \leq \dfrac{3}{2}$.

We first consider the case where $[f(a), f(b)]$ is contained in $f(\mathbb{D})$. In this case, there is a Jordan arc γ in $\mathbb{D}$ with hyperbolic arc length $2L$ joining a and b such that f maps γ injectively onto the segment $[f(a), f(b)]$. Let γ be parameterized by hyperbolic arc length as $\gamma : z = z(s)$, $s \in [-L, L]$. Then $z'(s) = (1 - |z(s)|^2)e^{i\theta(s)}$, where $\theta(s) = \arg z'(s)$. Let $u(s) = \log|D_1 f(z(s))|$. Then $u'(s) = \mathrm{Re}\{Q_f(z(s))e^{i\theta(s)}\}$, $(u')^2(s) = \dfrac{1}{2}\mathrm{Re}\{(Q_f(z(s)))^2 e^{2i\theta(s)}\} + \dfrac{1}{2}\big|Q_f(z(s))\big|^2$, and $u''(s) = \mathrm{Re}\{(1 - |z(s)|^2)^2 R_f(z(s))e^{2i\theta(s)}\} + \dfrac{1}{2}\big|Q_f(z(s))\big|^2 - 2$.

Therefore:

$$\begin{aligned} u''(s) + p(u')^2(s) &= \mathrm{Re}\{(1 - |z(s)|^2)^2 R_f(z(s))e^{2i\theta(s)}\} + \frac{1}{2}\big|Q_f(z(s))\big|^2 - 2 \\ &\qquad + p\left(\frac{1}{2}\mathrm{Re}\{(Q_f(z(s)))^2 e^{2i\theta(s)}\} + \frac{1}{2}\big|Q_f(z(s))\big|^2\right) \\ &= \mathrm{Re}\left\{\left[(1 - |z(s)|^2)^2 R_f(z(s)) + \frac{p}{2}(Q_f(z(s)))^2\right]e^{2i\theta(s)}\right\} + \frac{p+1}{2}\big|Q_f(z(s))\big|^2 - 2 \\ &\leq \left|(1 - |z(s)|^2)^2 R_f(z(s)) + \frac{p}{2}(Q_f(z(s)))^2\right| + \frac{p+1}{2}\big|Q_f(z(s))\big|^2 - 2. \end{aligned}$$

Since p satisfies inequality 4.1, we have $u''(s) + p(u')^2(s) \leq 16p$, so $u'' \leq p(4^2 - (u')^2)$. Apply Lemma 4.1 with $q = 4$, letting v be as in the statement of the lemma. Then $u(s) \geq v(s)$ for all $s \in [-L, L]$. Moreover, $v(s) = \dfrac{1}{p}\log[\cosh(4ps) + \tau\sinh(4ps)] + \log C$. Note that

$$C = \left(\frac{\exp(pu(L)) + \exp(pu(-L))}{2\cosh(4pL)}\right)^{1/p} = \left(\frac{|D_1 f(a)|^p + |D_1 f(b)|^p}{2\cosh(4pL)}\right)^{1/p},$$

since $u(L) = |D_1 f(z(L))| = |D_1 f(b)|$ and $u(-L) = |D_1 f(z(-L))| = |D_1 f(a)|$.

Therefore:

$$\begin{aligned}
|f(b) - f(a)| &= \int_{-L}^{L} |f'(z(s))||dz(s)| \\
&= \int_{-L}^{L} (1 - |z(s)|^2)|f'(z(s))| \frac{|dz(s)|}{1 - |z(s)|^2} \\
&= \int_{-L}^{L} |D_1 f(z(s))| \; ds \\
&= \int_{-L}^{L} \exp(u(s)) \; ds \\
&\geq \int_{-L}^{L} \exp(v(s)) \; ds \\
&\geq \frac{C}{2} \sinh(4L) \qquad \text{(by Lemma 4.2)} \\
&= \frac{\sinh(4L)}{2[2\cosh(4pL)]^{1/p}} \left(|D_1 f(a)|^p + |D_1 f(b)|^p\right)^{1/p}.
\end{aligned}$$

Note that the function $h(t) = \dfrac{\sinh t}{[2\cosh(pt)]^{1/p}}$ is increasing and $2\mathrm{d}_{\mathbb{D}}(a, b) \leq 4L$. Hence

$$|f(b) - f(a)| \geq \frac{\sinh(2\mathrm{d}_{\mathbb{D}}(a, b))}{2[2\cosh(2p\mathrm{d}_{\mathbb{D}}(a, b))]^{1/p}} \left(|D_1 f(a)|^p + |D_1 f(b)|^p\right)^{1/p}.$$

This establishes the inequality when $[f(a), f(b)]$ is contained in $f(\mathbb{D})$.

Next we find a limiting form for this inequality. Set $\Omega = f(\mathbb{D})$. If $\alpha \in \partial\Omega$ with $[f(a), \alpha) \subset \Omega$ and $b \in \mathbb{D}$ is such that $f(b) \in [f(a), \alpha)$, then we have

$$|f(a) - f(b)| \geq \frac{2\sinh(2\mathrm{d}_{\mathbb{D}}(a, b))}{2[2\cosh(2p\mathrm{d}_{\mathbb{D}}(a, b))]^{1/p}} |D_1 f(a)|.$$

Letting $f(b) \to \partial\Omega$ along the segment $[f(a), \alpha)$ results in $b \to \partial\mathbb{D}$ and $\mathrm{d}_{\mathbb{D}}(a, b) \to \infty$. Now, for the function $h(t) = \dfrac{\sinh t}{[2\cosh(pt)]^{1/p}}$, we have $\lim\limits_{t\to\infty} h(t) = \dfrac{1}{2}$. Hence

$$|f(a) - \alpha| \geq \frac{1}{4}|D_1 f(a)|.$$

Now, if $[f(a), f(b)]$ does not lie in Ω, then there are points $\alpha, \beta \in \partial\Omega$ such that $[f(a), \alpha)$ and $(\beta, f(b)]$ are disjoint and lie in Ω with $[f(a), \alpha) \cup (\beta, f(b)] \subseteq [f(a), f(b)]$. Then, by the limiting form, we have $|f(a)-\alpha| \geq \frac{1}{4}|D_1 f(a)|$ and $|f(b)-\beta| \geq \frac{1}{4}|D_1 f(b)|$. Hence:

$$\begin{aligned} |f(a)-f(b)| &\geq |f(a)-\alpha| + |f(b)-\beta| \\ &\geq \frac{1}{4}(|D_1 f(a)| + |D_1 f(b)|) \\ &\geq \frac{1}{4}(|D_1 f(a)|^p + |D_1 f(b)|^p)^{1/p} \end{aligned}$$

Since $\lim_{t\to\infty} h(t) = \frac{1}{2}$ and h is strictly increasing, we obtain

$$|f(a)-f(b)| > \frac{\sinh(2\mathrm{d}_{\mathbb{D}}(a,b))}{2[2\cosh(2p\mathrm{d}_{\mathbb{D}}(a,b))]^{1/p}}(|D_1 f(a)|^p + |D_1 f(b)|^p)^{1/p}.$$

Hence, the inequality has been established in all cases. ☐

Remarks.

1. Actually, the occurrence of P in Kim and Minda's distortion theorem is a result of the method used, and is not inherent in the problem. Jenkins [5] later proved that the inequality holds for all $p \geq 1$ and fails for $0 < p < 1$.
2. Note that the limiting form of this inequality ($p = \infty$) is Theorem 4.1, which is the invariant form of the lower bound given in Theorem 2.5.

4.3 Upper Bounds for $|f(a)-f(b)|$

A theorem giving a family of upper bounds for $|f(a)-f(b)|$ is given by Ma and Minda in [8]. The proof relies on an integral inequality which was proved by Ma and Minda in [9].

Lemma 4.3. *Suppose* $w \in C^2[a,b]$, $k > 0$, *and* $w'' \leq k^2 w$. *If* $w(a) \geq 0$ *and* $w(b) \geq 0$, *then* $w \geq 0$ *on* $[a,b]$.

Proof. Let $m = \min\{w(s) : s \in [a,b]\}$. If $m \geq 0$, there is nothing to prove, so we will assume $m < 0$. First, find $s_0 \in (a,b)$ with $w(s_0) = m$. Since $w(s_0)$ is the minimum of w on $[a,b]$, we have $w'(s_0) = 0$ and $w''(s_0) \geq 0$. But also $w''(s_0) \leq k^2 w(s_0) = k^2 m < 0$. This is a contradiction, and so $m \geq 0$. ☐

Corollary 4.2. *Suppose $u, v \in C^2[a,b]$, $k > 0$, $v'' \leq k^2 v$, and $u'' \leq k^2 u$. If $v(a) \geq u(a)$ and $v(b) \geq u(b)$, then $v \geq u$ on $[a,b]$.*

Proof. The result follows immediately by applying Lemma 4.3 to $w = v - u$. □

Lemma 4.4. *Suppose $v \in C^2[-L,L]$, $v > 0$, $k > 0$, $|v'| \leq kv$ and $v'' \leq k^2 v$. Then*

$$\int_{-L}^{L} \frac{ds}{v(s)} \leq \frac{4\cosh(kL)\sinh(kL)}{k[v(L)+v(-L)]}.$$

Proof. Let $u \in C^2[-L,L]$ satisfy $u'' = k^2 u$ with the boundary conditions $u(L) = v(L)$ and $u(-L) = v(-L)$. The general solution of $u'' = k^2 u$ is $u(s) = A\cosh(ks) + B\sinh(ks)$, where $A, B \in \mathbb{R}$. Now, the boundary conditions imply $A = \dfrac{v(L)+v(-L)}{2\cosh(kL)}$ and $B = \dfrac{v(L)-v(-L)}{2\sinh(kL)}$. Let $\tau = \dfrac{B}{A} = \dfrac{v(L)-v(-L)}{v(L)+v(-L)} \cdot \dfrac{\cosh(kL)}{\sinh(kL)}$. Then $u(s) = A[\cosh(ks) + \tau\sinh(ks)]$. We will next show that $\tau \in [-1,1]$. To that end, notice that $|v'| \leq kv$ implies $-k \leq \dfrac{v'}{v} \leq k$. Integrating over $[-L,L]$ and exponentiating results in

$$\exp(-2kL) \leq \frac{v(L)}{v(-L)} \leq \exp(2kL).$$

Now, the function $h(t) = \dfrac{t-1}{t+1}$ is strictly increasing for $t > -1$. Therefore:

$$\begin{aligned}
&\frac{\exp(-2kL)-1}{\exp(-2kL)+1} \leq \frac{\dfrac{v(L)}{v(-L)}-1}{\dfrac{v(L)}{v(-L)}+1} \leq \frac{\exp(2kL)-1}{\exp(2kL)+1} \\
&\Rightarrow \frac{\exp(-kL)-\exp(kL)}{\exp(-kL)+\exp(kL)} \leq \frac{v(L)-v(-L)}{v(L)+v(-L)} \leq \frac{\exp(kL)-\exp(-kL)}{\exp(kL)+\exp(-kL)} \\
&\Rightarrow -\frac{\sinh(kL)}{\cosh(kL)} \leq \frac{v(L)-v(-L)}{v(L)+v(-L)} \leq \frac{\sinh(kL)}{\cosh(kL)} \\
&\Rightarrow -1 \leq \frac{v(L)-v(-L)}{v(L)+v(-L)} \cdot \frac{\cosh(kL)}{\sinh(kL)} \leq 1 \\
&\Rightarrow -1 \leq \tau \leq 1
\end{aligned}$$

Since $A > 0$ and $\tau \in [-1,1]$, we have $u > 0$ and $|u'| \leq ku$. Hence, Corollary 4.2 implies $v \geq u$ on $[-L,L]$. Then we must have

$$\int_{-L}^{L} \frac{ds}{v(s)} \leq \int_{-L}^{L} \frac{ds}{u(s)} = \frac{1}{A}\int_{-L}^{L} \frac{ds}{\cosh(ks) + \tau\sinh(ks)}.$$

Let $I(\tau) = \displaystyle\int_{-L}^{L} \frac{ds}{\cosh(ks) + \tau\sinh(ks)}$. Then $I''(\tau) = 2\displaystyle\int_{-L}^{L} \frac{\sinh^2(ks)}{[\cosh(ks) + \tau\sinh(ks)]^2} > 0$. Thus, $I(\tau)$ is strictly concave up on $[-L, L]$, so $I(\tau) \leq I(\pm 1) = \dfrac{2\sinh(kL)}{k}$ for all $\tau \in [-1, 1]$. Hence,

$$\int_{-L}^{L} \frac{ds}{v(s)} \leq \frac{2\sinh(kL)}{Ak} = \frac{4\cosh(kL)\sinh(kL)}{k[v(L) + v(-L)]}.$$

□

The result that we will need for Ma and Minda's distortion theorem is the following corollary.

Corollary 4.3. *Suppose* $v \in C^2[-L, L]$, $v > 0$, $k > 0$, $|v'| \leq kv$ *and* $v'' \leq k^2 v$. *Then for any* $p \geq 1$

$$\int_{-L}^{L} \frac{ds}{v(s)} \leq \frac{2[2\cosh(pkL)]^{1/p}\sinh(kL)}{k[v(L)^p + v(-L)^p]^{1/p}}.$$

Proof. First note that $\dfrac{2[2\cosh(pkL)]^{1/p}\sinh(kL)}{k[v(L)^p + v(-L)^p]^{1/p}} = \dfrac{2\exp(kL)\sinh(kL)}{kv(L)}\left[\dfrac{1 + (\exp(-2kL))^p}{1 + \left(\dfrac{v(-L)}{v(L)}\right)^p}\right]^{1/p}$.

Now, in the proof of Lemma 4.4, we saw that $\dfrac{v(-L)}{v(L)} \geq \exp(-2kL)$. Moreover, for $0 < s < t \leq 1$, the function $h(p) = \left(\dfrac{1 + s^p}{1 + t^p}\right)^{1/p}$ is strictly increasing. Thus, if $\dfrac{v(-L)}{v(L)} \leq 1$, the corollary follows immediately from Lemma 4.4. If $\dfrac{v(-L)}{v(L)} > 1$, we factor out $v(-L)$ instead. □

We will also make use of the inequalities $(1 - |z|^2)^2|R_f(z)| \leq 6$ (which is due to Kraus and Nehari; see [3], p.263) and $|Q_f(z)| \leq 4$ (as shown by Blatter [2]). These inequalities hold whenever f is a univalent function.

Theorem 4.5. *Suppose* f *is univalent in* $\mathbb{D}$. *Then for* $a, b \in \mathbb{D}$ *and* $p \geq 1$

$$|f(a) - f(b)| \leq \frac{[2\cosh(2pd_{\mathbb{D}}(a,b))]^{1/p}\sinh(2d_{\mathbb{D}}(a,b))}{2\left[1/|D_1 f(a)|^p + 1/|D_1 f(b)|^p\right]^{1/p}}.$$

Proof. Let $a, b \in \mathbb{D}$. Let $\gamma : z = z(s)$, $s \in [-L, L]$, be a Jordan arc joining a and b parameterized by hyperbolic arc length such that $2L = \mathrm{d}_{\mathbb{D}}(a, b)$. We first assume only that f is locally univalent on $\mathbb{D}$ so that $f'(z) \neq 0$ for $z \in \mathbb{D}$. Let $v(s) = |D_1 f(z(s))|^{-1}$.

Since $\frac{d}{ds}|D_1f(z(s))| = |D_1f(z(s))|\mathrm{Re}\big\{e^{i\theta(s)}Q_f(z(s))\big\}$, we have

$$v'(s) = -v(s)\mathrm{Re}\big\{e^{i\theta(s)}Q_f(z(s))\big\}.$$

Then

$$\begin{aligned} v''(s) &= -v'(s)\mathrm{Re}\big\{e^{i\theta(s)}Q_f(z(s))\big\} - v(s)\frac{d}{ds}\mathrm{Re}\big\{e^{i\theta(s)}Q_f(z(s))\big\} \\ &= v(s)\left[\Big(\mathrm{Re}\big\{e^{i\theta(s)}Q_f(z(s))\big\}\Big)^2 - \frac{d}{ds}\mathrm{Re}\big\{e^{i\theta(s)}Q_f(z(s))\big\}\right]. \end{aligned}$$

A calculation yields

$$\frac{d}{ds}\big[e^{i\theta(s)}Q_f(z(s))\big] = e^{2i\theta(s)}(1-|z(s)|^2)^2R_f(z(s)) + \frac{1}{2}\big[e^{i\theta(s)}Q_f(z(s))\big]^2 - 2.$$

Therefore:

$$\begin{aligned} v''(s) &= v(s)\left[\Big(\mathrm{Re}\big\{e^{i\theta(s)}Q_f(z(s))\big\}\Big)^2 - \frac{1}{2}\mathrm{Re}\big\{[e^{i\theta(s)}Q_f(z(s))]^2\big\} - \mathrm{Re}\big\{e^{2i\theta(s)}(1-|z(s)|^2)^2R_f(z(s))\big\} + 2\right] \\ &= v(s)\left[\frac{1}{2}\big|Q_f(z(s))\big|^2 - \mathrm{Re}\big\{e^{2i\theta(s)}(1-|z(s)|^2)^2R_f(z(s))\big\} + 2\right] \\ &\le v(s)\left[\frac{1}{2}\big|Q_f(z(s))\big|^2 + (1-|z(s)|^2)^2\big|R_f(z(s))\big| + 2\right] \end{aligned}$$

Now assume f is univalent on $\mathbb{D}$. Then $|Q_f(z)| \le 4$ and $(1-|z|^2)^2|R_f(z)| \le 6$, so that $|v'(s)| \le 4v(s)$ and $v''(s) \le 16v(s)$.

Since $f \circ \gamma$ is a path connecting $f(a)$ to $f(b)$, we have

$$\begin{aligned} |f(a)-f(b)| &\le \int_{f\circ\gamma} |dw| \\ &= \int_\gamma |f'(z)||dz| \\ &= \int_{-L}^{L} |f'(z(s))|(1-|z(s)|^2)\ ds \\ &= \int_{-L}^{L} |D_1f(z(s))|\ ds \\ &= \int_{-L}^{L} \frac{ds}{v(s)}. \end{aligned}$$

Applying Corollay 4.3 with $k=4$ establishes the desired result. □

Remark. The case $p = \infty$ gives

$$|f(a) - f(b)| \leq \frac{1}{2} \exp(2\mathrm{d}_{\mathbb{D}}(a,b)) \sinh(2\mathrm{d}_{\mathbb{D}}(a,b)) \min\left\{|D_1 f(a)|, |D_1 f(b)|\right\},$$

which is an invariant form of the upper bound in Theorem 2.5. To see this, apply the bound in Theorem 4.5 to $g \in S$ with $a = 0$ and $b = z$.
We get $|g(z)| \leq \dfrac{|z|}{(1-|z|)^2} \min\{1, |D_1 g(z)|\} \leq \dfrac{|z|}{(1-|z|)^2}$.

4.4 Comparisons Between Hyperbolic and Euclidean Geometries on Simply Connected Regions

The two-point distortion theorems discussed in the previous two sections yield comparisons between hyperbolic and Euclidean geometries on simply connected regions. Before we state the theorems, we will introduce some more terminology and notation.

We say that a region Ω in the complex plane is hyperbolic if $\mathbb{C}\backslash\Omega$ contains at lease two points. Saying that a region is hyperbolic is equivalent to saying that there is a holomorphic universal covering projection $f : \mathbb{D} \to \Omega$. This f is a conformal mapping if Ω is simply connected.

If Ω is a hyperbolic region, then the density of the hyperbolic metric on Ω is obtained from

$$\lambda_\Omega(f(z))|f'(z)| = \lambda_{\mathbb{D}}(z),$$

where $f : \mathbb{D} \to \Omega$ is any holomorphic universal covering projection of $\mathbb{D}$ onto Ω. The density is independent of the choice of f. Notice that $\lambda_\Omega(f(z)) = \dfrac{\lambda_{\mathbb{D}}(z)}{|f'(z)|} = \dfrac{1}{(1-|z|^2)|f'(z)|} = \dfrac{1}{|D_1 f(z)|}$.

The hyperbolic distance function on Ω is given by

$$\mathrm{d}_\Omega(P,Q) = \inf\left\{\int_\gamma \lambda_\Omega(w)|dw| : \gamma \text{ is a path in } \Omega \text{ joining } P \text{ and } Q\right\}.$$

If f is a conformal mapping, then $\mathrm{d}_\Omega(f(a), f(b)) = \mathrm{d}_{\mathbb{D}}(a,b)$.

Theorem 4.6. *Let Ω be a simply connected hyperbolic region in $\mathbb{C}$. Then for any $p \geq 1$ and all $A, B \in \Omega$,*

$$|A - B| \geq \frac{\sinh(2d_\Omega(A,B))}{2[2\cosh(2p d_\Omega(A,B))]^{1/p}} \left(\frac{1}{\lambda_\Omega^p(A)} + \frac{1}{\lambda_\Omega^p(B)}\right)^{1/p}.$$

Proof. Let $A, B \in \Omega$ be given and let f be a conformal mapping of $\mathbb{D}$ onto Ω. Find $a, b \in \mathbb{D}$ with $f(a) = A$ and $f(b) = B$. Then the remark following Theorem 4.4 implies

$$\begin{aligned}
|A - B| &= |f(a) - f(b)| \\
&\geq \frac{\sinh(2\mathrm{d}_{\mathbb{D}}(a,b))}{2[2\cosh(2p\mathrm{d}_{\mathbb{D}}(a,b))]^{1/p}} \left(|D_1 f(a)|^p + |D_1 f(b)|^p\right)^{1/p} \\
&= \frac{\sinh(2\mathrm{d}_{\Omega}(f(a),f(b)))}{2[2\cosh(2p\mathrm{d}_{\Omega}(f(a),f(b)))]^{1/p}} \left(\frac{1}{\lambda_{\Omega}^p(f(a))} + \frac{1}{\lambda_{\Omega}^p(f(b))}\right)^{1/p} \\
&= \frac{\sinh(2\mathrm{d}_{\Omega}(A,B))}{2[2\cosh(2p\mathrm{d}_{\Omega}(A,B))]^{1/p}} \left(\frac{1}{\lambda_{\Omega}^p(A)} + \frac{1}{\lambda_{\Omega}^p(B)}\right)^{1/p}
\end{aligned}$$

□

Theorem 4.7. *Let Ω be a simply connected hyperbolic region in $\mathbb{C}$. Then for any $p \geq 1$ and all $A, B \in \Omega$,*

$$|A - B| \leq \frac{[2\cosh(2p d_{\Omega}(A,B))]^{1/p} \sinh(2 d_{\Omega}(A,B))}{2[\lambda_{\Omega}^p(A) + \lambda_{\Omega}^p(B)]^{1/p}}.$$

Proof. The proof is similar to that of Theorem 4.6. It follows from applying Theorem 4.5 to a conformal mapping $f : \mathbb{D} \to \Omega$. □

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