The Jordan Canonical Forms of complex orthogonal and skew-symmetric matrices: characterization and examples

by

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TABLE OF CONTENTS

CHAPTER 1. INTRODUCTION AND MAIN RESULTS

1.1 Introduction

The main goal of this paper is to characterize the Jordan Canonical Form of a matrix that is similar to a complex orthogonal or complex skew-symmetric matrix. Our work is based on a paper from Horn and Merino [5] that extends standard results for unitary, Hermitian and skew-Hermitian matrices. Furthermore, several new families of examples are provided.

The set of all $n \times m$ matrices over the complex numbers $\mathbb{C}$ will be denoted by $\mathbb{C}^{n\times m}$. In particular, $\mathbb{C}^{n\times n}$ represents all square matrices over the field $\mathbb{C}$, $I_n \in \mathbb{C}^{n\times n}$ is the identity matrix of dimension n and $0_n \in \mathbb{C}^{n\times n}$ denotes the zero matrix of dimension n. If $A = [a_{ij}] \in \mathbb{C}^{n\times m}$, then $A^T = [a_{ji}] \in \mathbb{C}^{m\times n}$ represents the transpose of A. The Hermitian adjoint A^* of A is defined by $\overline{A}^T$, where $\overline{A} = [\overline{a}_{ij}] \in \mathbb{C}^{n\times m}$ is the component-wise conjugate of A. Using this notation we can introduce an important class of square matrices, the normal matrices.

Definition 1.1.1. *A matrix $A \in \mathbb{C}^{n\times n}$ is normal if $AA^* = A^*A$.*

The class of normal matrices includes the Hermitian matrices defined by $A^* = A$, the unitary matrices with the property $U^* = U^{-1}$ and the skew-Hermitian matrices where $A^* = -A$. However, there are normal matrices which are not Hermitian, unitary or skew-symmetric. For example $M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ is normal, yet does not satisfy $M^* = M$, $M^* = M^{-1}$ nor $M^* = -M$.

One central property of a normal matrix is that it is diagonalizable by a unitary matrix. In other words, if $A \in \mathbb{C}^{n\times n}$ is normal then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that $U^*AU = \text{diag}(\lambda_1, \ldots, \lambda_n)$, where $\text{diag}(\lambda_1, \ldots, \lambda_n)$ denotes a diagonal matrix in $\mathbb{C}^{n\times n}$ with the diagonal entries $\lambda_1, \ldots, \lambda_n$. For a proof of this assertion see Horn and Johnson [3], Theorem 2.5.4 (b) or Zhang [6], Theorem 8.1.2. Hence, the Jordan Canonical Form for this type of matrix is always a diagonal matrix and there is no need for further analysis.

In this paper we will examine the Jordan Canonical Forms of subclasses of the class of matrices which satisfy a similar property as the normal matrices, namely $AA^T = A^TA$. We will denote this class by

$$\mathbb{A}_n = \{A \in \mathbb{C}^{n\times n} \text{ such that } AA^T = A^TA\}.$$

We will show that while the definition of $\mathbb{A}_n$ differs only slightly from the definition of the class of normal matrices, the Jordan Canonical Forms of matrices in $\mathbb{A}_n$ have a completely different characterization. Now we will introduce our notation.

Definition 1.1.2. *A Jordan block $J_k(\lambda) \in \mathbb{C}^{k\times k}$ is an upper triangular matrix of the form*

$$J_k(\lambda) = \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}.$$

Definition 1.1.3. *A Jordan Canonical Form $J \in \mathbb{C}^{n\times n}$ is a direct sum of Jordan blocks.*

$$J = J_{k_1}(\lambda_1) \oplus \cdots \oplus J_{k_r}(\lambda_r) = \begin{bmatrix} J_{k_1}(\lambda_1) & & & \\ & J_{k_2}(\lambda_2) & & 0 \\ & & \ddots & \\ 0 & & & J_{k_r}(\lambda_r) \end{bmatrix},$$

where $k_1 + \cdots + k_r = n$.

In order to determine the Jordan Canonical Form of a matrix it is often necessary to identify its characteristic and minimal polynomials. The set of all polynomials in x with coefficients from $\mathbb{C}$ is represented by $\mathbb{C}[x]$.

Definition 1.1.4. *The characteristic polynomial $p_A(x)$ of a matrix $A \in \mathbb{C}^{n\times n}$ is a polynomial in $\mathbb{C}[x]$ of degree n and is defined by $p_A(x) = \det(xI_n - A)$. The minimal polynomial of A is the unique monic polynomial $m_A(x) \in \mathbb{C}[x]$ of minimum degree such that $m_A(A) = 0$.*

Note. If $p_A(x) = (x - \lambda_1) \cdots (x - \lambda_n)$, then $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A \in \mathbb{C}^{n \times n}$.

The next theorem is a fundamental result in Linear Algebra and it will be used several times throughout the paper. We will state it without proof, which can be found in [3], Theorem 2.4.2 or [6], Theorem 3.8.

Theorem 1.1.5. (Cayley-Hamilton) *Let $A \in \mathbb{C}^{n \times n}$ and let $p_A(x) \in \mathbb{C}[x]$ be the characteristic polynomial of A. Then $p_A(A) = 0$.*

Note. An important property of the minimal polynomial is that it divides every polynomial $f(x) \in \mathbb{C}[x]$ which is satisfied by A. That is, if $f(A) = 0$ then $f(x) = q(x)m_A(x)$ for some polynomial $q(x) \in \mathbb{C}[x]$. Therefore, from the Cayley-Hamilton Theorem, the minimal polynomial divides the characteristic polynomial.

For several proofs we will use the backward identity matrix

$$B_n = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix} \in \mathbb{C}^{n \times n}.$$

Note that this matrix is symmetric and $B_n^2 = I_n$, hence we have $B_n = B_n^T = B_n^{-1}$.

1.2 Statement of Main Results

We will now examine subclasses of $\mathbb{A}_n$ which correspond to the subclasses of normal matrices presented in the previous section. These are the symmetric, orthogonal and skew-symmetric matrices. Note that there exist matrices in $\mathbb{A}_n$ which do not belong to these three classes (an example is the matrix M from the previous section).

The class of matrices which is related to the Hermitian matrices are the symmetric matrices.

Definition 1.2.1. *A complex matrix $S \in \mathbb{C}^{n \times n}$ is symmetric if $S^T = S$.*

Note that the definition of a symmetric matrix $S = [s_{ij}]_{i,j=1,\ldots,n}$ gives us that $s_{ij} = s_{ji}$ for all $i, j = 1, \ldots, n$. One example of a symmetric matrix is

$$S = \begin{bmatrix} i & 1+i \\ 1+i & -1 \end{bmatrix}.$$

We will show that every matrix is similar to a complex symmetric matrix (Theorem 2.1.4). Thus the Jordan Canonical Form of a complex symmetric matrix can adopt any form.

The next class is the class of complex orthogonal matrices, which correspond to the unitary matrices.

Definition 1.2.2. *A nonsingular complex matrix* $Q \in \mathbb{C}^{n\times n}$ *is orthogonal if* $Q^T = Q^{-1}$.

Frequently used representatives for real orthogonal matrices are the Givens matrices G which perform a rotation by the angle θ,

$$G = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

More examples of orthogonal matrices can be found in Chapter 3.

If Q is an orthogonal matrix, then since any matrix is similar to its transpose (Theorem 2.1.5) and the inverse of an orthogonal matrix is its transpose, it is immediate that Q must be similar to its inverse. It is natural to consider whether a nonsingular matrix is always similar to its inverse. This is the case if and only if the Jordan Canonical Form of the matrix has a special form (see Proposition 4.1.1); thus the Jordan Canonical Form of an orthogonal matrix must satisfy these conditions. This argument led Horn and Merino to investigate the Jordan Canonical Form of orthogonal matrices in more detail in [5]. Their main result is outlined in the following theorem.

Theorem 1.2.3. ([5], Theorem 1) *An* $n \times n$ *complex matrix is similar to a complex orthogonal matrix if and only if its Jordan Canonical Form can be expressed as a direct sum of matrices of only the following three types:*

(a) $J_k(\lambda) \oplus J_k(\lambda^{-1})$ *for* $\lambda \in \mathbb{C} \setminus \{0\}$ *and any* k*,*

(b) $J_k(1)$ *for any odd* k *and*

(c) $J_k(-1)$ *for any odd* k*.*

To prove that the Jordan blocks of types (a), (b) and (c) are similar to an orthogonal matrix is rather technical, but it is the easier part of the proof. The main difficulty is to show that even-dimensional Jordan blocks with eigenvalue 1 or -1 must occur in pairs. The detailed proof of this theorem will be presented in Chapter 4.

The last category to consider is the skew-symmetric matrices, the definition of which is similar to skew-Hermitian matrices.

Definition 1.2.4. *A complex matrix* $A \in \mathbb{C}^{n\times n}$ *is skew-symmetric if* $A^T = -A$.

Note that if $A = [a_{ij}]_{i,j=1,\ldots,n}$ is skew-symmetric, then we have the condition $a_{ij} = -a_{ji}$ for all $i, j = 1, \ldots, n$ and therefore all diagonal entries must be equal to zero. An example of a complex skew-symmetric matrix is

$$A = \begin{bmatrix} 0 & i & 1-i \\ -i & 0 & -1 \\ -1+i & 1 & 0 \end{bmatrix}.$$

It follows from the definition and Theorem 2.1.5 that if A is skew-symmetric then it is similar to its additive inverse $-A$. Since this is not true in general, it is of interest to study the skew-symmetric case more closely. In order for a matrix to be similar to its additive inverse, it has to possess a special type of Jordan Canonical Form (see Proposition 5.1.1). In particular this must be true for skew-symmetric matrices, so Horn and Merino analyzed this case in [5] and found the following interesting result.

Theorem 1.2.5. ([5], Theorem 2) *An* $n \times n$ *complex matrix is similar to a complex skew-symmetric matrix if and only if its Jordan Canonical Form can be expressed as a direct sum of matrices of only the following two types:*

(a) $J_k(\lambda) \oplus J_k(-\lambda)$ *for* $\lambda \in \mathbb{C}$ *and any* k*,*

(b) $J_k(0)$ *for any odd* k*.*

We will give the proof of this theorem in Chapter 5. As in the orthogonal case, showing the sufficiency of the conditions (a) and (b) will be done by utilizing technical arguments. That the Jordan blocks of even dimension corresponding to the eigenvalue 0 have to occur in pairs is a direct consequence of the constraints on the even-dimensional Jordan blocks with eigenvalue

1 of an orthogonal matrix. We will show this by using the matrix exponential which will be introduced in Section 2.3.

Before we can prove these main results, we will present various results about Jordan Canonical Forms, as well as some general technical properties for matrices in the next chapter. We will provide several examples in Chapter 3 which will illustrate Theorem 1.2.3 and Theorem 1.2.5.

CHAPTER 2. IMPORTANT RESULTS AND USEFUL TOOLS

2.1 Results for Jordan Canonical Forms of Complex Matrices

In this section we will provide useful results for complex square matrices and their Jordan Canonical Forms. We first give two theorems which are essential for the study of Jordan Canonical Forms. We will state these fundamental results without proof; however the proofs can be found in Horn and Johnson [3], Theorem 3.1.11 and Friedberg, Insel and Spence [2], Theorem 7.11.

Theorem 2.1.1. *Let* $A \in \mathbb{C}^{n\times n}$ *be a square complex matrix. Then there exists an invertible matrix* $P \in \mathbb{C}^{n\times n}$ *such that*

$$P^{-1}AP = J(A) = J_{k_1}(\lambda_1) \oplus \cdots \oplus J_{k_r}(\lambda_r)$$

where the $J_{k_i}(\lambda_i)$ *are the Jordan blocks of* A *for the eigenvalues of* A *and* $k_1 + \cdots + k_r = n$. *The Jordan Canonical Form* $J(A)$ *is unique up to permutation of the Jordan blocks.*

Theorem 2.1.2. *Let* $A, B \in \mathbb{C}^{n\times n}$ *be square complex matrices. Then* A *and* B *are similar if and only if their Jordan Canonical Forms are the same up to permutation of the Jordan blocks.*

Our next goal will be to show that every complex matrix is similar to a complex symmetric matrix. In order to do this, we will first show that this assertion is true for single Jordan blocks (cf. [3], pp. 208-209).

Proposition 2.1.3. *Every Jordan block* $J_k(\lambda) \in \mathbb{C}^{k\times k}$ *is similar to a complex symmetric matrix in* $\mathbb{C}^{k\times k}$.

Proof. Define the following symmetric matrix $S_k = \frac{1}{\sqrt{2}}(I_k + iB_k) \in \mathbb{C}^{k\times k}$, where B_k is the backward identity matrix. For convenience we will omit the subscripts k which specify the dimension. Since

$$S\bar{S} = \frac{1}{2}(I + iB)(I - iB) = \frac{1}{2}(I - iB + iB - (iB)^2) = \frac{1}{2}(I + I) = I$$

we have $\bar{S} = S^{-1}$. For $k = 1$ all matrices have dimension 1×1, thus everything commutes trivially. Therefore $S_1 J_1(\lambda) S_1^{-1} = S_1 S_1^{-1} J_1(\lambda) = J_1(\lambda)$ which is symmetric. Next, note that for $k \geq 2$, $J_k(\lambda) = \lambda I_k + N_k$ where N_k stands for the nilpotent matrix $J_k(0)$. Again we will omit the subscripts. Then direct computation gives us

$$\begin{aligned} SJ_k(\lambda)S^{-1} &= S(\lambda I + N)\bar{S} = \lambda I + \frac{1}{2}(I + iB)N(I - iB) \\ &= \lambda I + \frac{1}{2}(N + BNB) + \frac{i}{2}(BN - NB) \\ &= \frac{1}{2}\begin{bmatrix} 2\lambda & 1 & & \\ 1 & 2\lambda & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 2\lambda \end{bmatrix} + \frac{i}{2}\begin{bmatrix} & & -1 & 0 \\ & \iddots & 0 & 1 \\ -1 & \iddots & \iddots & \\ 0 & 1 & & \end{bmatrix} \end{aligned}$$

which is evidently a symmetric matrix, thereby proving the lemma. □

Utilizing this proposition we can extend the result to all complex square matrices.

Theorem 2.1.4. *Every matrix $A \in \mathbb{C}^{n \times n}$ is similar to a complex symmetric matrix.*

Proof. By Theorem 2.1.1 we have that A is similar to its Jordan Canonical Form $J(A) = J_{k_1}(\lambda_1) \oplus \cdots \oplus J_{k_r}(\lambda_r)$. For all k_j, where $j = 1, \ldots, r$, define $S_{k_j} = \frac{1}{\sqrt{2}}(I_{k_j} + iB_{k_j})$. Then by Proposition 2.1.3 we know that $S_{k_j} J_{k_j}(\lambda_j) S_{k_j}^{-1}$ is symmetric for all $j = 1, \ldots, r$. If we define $S = S_{k_1} \oplus \cdots \oplus S_{k_r}$ then $SJ(A)S^{-1} = S_{k_1} J_{k_1}(\lambda_1) S_{k_1}^{-1} \oplus \cdots \oplus S_{k_r} J_{k_r}(\lambda_r) S_{k_r}^{-1}$ is symmetric which proves the theorem. □

Thus, for every possible Jordan Canonical Form J, there exists a corresponding complex symmetric matrix S such that S is similar to J.

Note. The result of Theorem 2.1.4 does not hold for matrices over the real numbers, that is, there exists a real matrix $A \in \mathbb{R}^{n \times n}$ which is not similar to any real symmetric matrix. To see this recall Theorem 6.20 in [2]: A real matrix $A \in \mathbb{R}^{n \times n}$ is symmetric if and only if A is orthogonally equivalent to a real diagonal matrix. Thus the Jordan Canonical Form of a real symmetric matrix is a diagonal matrix, which is a direct sum of 1×1 blocks. Therefore, we have that a matrix is similar to a real symmetric matrix if and only if it is diagonalizable.

One application of the Jordan Canonical Form is to use it to prove that every square matrix is similar to its transpose (see for example [6], Theorem 3.13.1).

Theorem 2.1.5. *Any complex square matrix $A \in \mathbb{C}^{n \times n}$ is similar to its transpose matrix A^T.*

Proof. First notice that any Jordan block $J_k(\lambda)$ is similar to its transpose. To prove this we will use the backward identity matrix B_k. Since $B_k^{-1} = B_k$, then direct computation shows that $B_k^{-1} J_k(\lambda) B_k = J_k(\lambda)^T$. Using $B = B_{k_1} \oplus \cdots \oplus B_{k_r}$ we can prove more generally that the Jordan Canonical Form of A, $J(A) = J_{k_1}(\lambda_1) \oplus \cdots \oplus J_{k_r}(\lambda_r)$, is similar to its transpose $J(A)^T = J_{k_1}(\lambda_1)^T \oplus \cdots \oplus J_{k_r}(\lambda_r)^T$. Next, by Theorem 2.1.1 we know that A is similar to its Jordan Canonical Form $J(A)$ which is similar to $J(A)^T$ as shown above. Since $J(A)^T$ is similar to A^T, this proves the theorem. □

Since the rank of a matrix is similarity invariant, we can deduce that the row rank of a complex matrix is the same as the column rank. Furthermore, it also follows that A and A^T must have the same Jordan Canonical Form and hence the same set of eigenvalues.

The last objective of this section is to present similarity results for single Jordan blocks.

Lemma 2.1.6. *Any Jordan block of the form $J_k(-\lambda)$ is similar to $-J_k(\lambda)$.*

Proof. Let $D = \operatorname{diag}(1, -1, 1, \ldots, (-1)^{k+1}) \in \mathbb{C}^{k \times k}$. Note that this is a nonsingular matrix with $D^{-1} = D$. Then $DJ_k(-\lambda)D^{-1} = -J_k(\lambda)$ and therefore we have that $J_k(-\lambda)$ is similar to $-J_k(\lambda)$. □

Lemma 2.1.7. *If $\lambda \neq 0$, then the Jordan block $J_k(\lambda^{-1})$ is similar to $J_k(\lambda)^{-1}$.*

Proof. For convenience let J denote $J_k(\lambda^{-1})$. Notice that the characteristic polynomial $p_J(x)$ and the minimal polynomial $m_J(x)$ of the Jordan block J are the same and of the following form

$$p_J(x) = m_J(x) = (x - \lambda^{-1})^k,$$

because $(J-\lambda^{-1}I_k)^k=0$, but $(J-\lambda^{-1}I_k)^l\neq 0$ for all $l=0,1,\ldots,k-1$. Since $\lambda\neq 0$, J^{-1} exists and we can compute that

$$\begin{aligned} &(\lambda J^{-1})^k(J-\lambda^{-1}I_k)^k=(\lambda I_k-J^{-1})^k=0,\\ \text{but}\qquad &(\lambda J^{-1})^l(J-\lambda^{-1}I_k)^l=(\lambda I_k-J^{-1})^l\neq 0\quad \text{for all } l=0,1,\ldots,k-1. \end{aligned}$$

Hence, J^{-1} satisfies the polynomial $q(x)=(x-\lambda)^k$ and therefore the minimal polynomial $m_{J^{-1}}(x)$ of J^{-1} divides $q(x)$. But since J^{-1} does not satisfy $(x-\lambda)^l$ for all $l=0,1,\ldots,k-1$, it follows that the minimal polynomial is equal to $q(x)$. Note that $q(x)$ is also the characteristic polynomial $p_{J^{-1}}(x)$ because its degree is k. Therefore the Jordan Canonical Form of J^{-1} is $J_k(\lambda)$. Hence, $J=J_k(\lambda^{-1})$ is similar to $J_k(\lambda)^{-1}$. □

2.2 Useful Results for Complex Matrices

This section provides useful tools which we will apply throughout Chapters 4 and 5. The first result is the fact that we can decompose every nonsingular matrix into the product of a symmetric matrix and an orthogonal matrix.

Theorem 2.2.1. *Let $A\in\mathbb{C}^{n\times n}$. If A is nonsingular, then $A=SQ$, where $Q\in\mathbb{C}^{n\times n}$ is complex orthogonal, $S\in\mathbb{C}^{n\times n}$ is complex symmetric, and S is a polynomial in AA^T.*

Proof. We will give a short outline of the proof. More details can be found in Horn and Johnson [4], Theorem 6.4.16.

Since $AA^T\in\mathbb{C}^{n\times n}$ is symmetric and nonsingular there exists a complex square root $S\in\mathbb{C}^{n\times n}$ of A (that is, $AA^T=S^2$) such that S is a polynomial in AA^T (compare [4], Theorem 6.4.12(a)). Since S is a polynomial in A^TA, S is symmetric. Define $Q=S^{-1}A$. Then

$$QQ^T=S^{-1}A(S^{-1}A)^T=S^{-1}AA^T(S^T)^{-1}=S^{-1}S^2S^{-1}=I_n.$$

Thus Q is orthonormal and $SQ=SS^{-1}A=A$. □

Proving the main results about Jordan Canonical Forms for orthogonal and skew-symmetric matrices will require that we use symmetric matrices. If S is symmetric, then obviously S^2

and S^{-1} are both symmetric matrices. The next proposition states an important property of symmetric matrices which can be also found in [3], Corollary 4.4.6.

Proposition 2.2.2. *Let $S \in \mathbb{C}^{n\times n}$. Then S is a nonsingular complex symmetric matrix if and only if there exists a nonsingular matrix $X \in \mathbb{C}^{n\times n}$ such that $S = X^T X$.*

Proof. First assume that S is symmetric, so that $S^* = \bar{S}^T = \bar{S}$. Moreover, we have that S and therefore S^* is nonsingular, hence SS^* has rank n and so all eigenvalues must be nonzero. Let μ be an eigenvalue of SS^* and $x \neq 0$ the corresponding eigenvector, then

$$\mu(x^*x) = x^*(\mu x) = x^*(SS^*x) = (S^*x)^*(S^*x) > 0, \text{ since } S^*x \neq 0.$$

Therefore we have

$$\mu = \frac{(S^*x)^*(S^*x)}{x^*x} > 0,$$

so the eigenvalues $\mu_1, \ldots, \mu_n$ of $SS^* = S\bar{S}$ are all positive and real. Hence, by a more general form of the Schur unitary-triangular decomposition (which can be found in Horn and Johnson [3], Theorem 4.4.3) there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ and an upper triangular matrix $\Delta \in \mathbb{C}^{n\times n}$ with $\mu_1, \ldots, \mu_n$ on the diagonal such that $S = U\Delta U^T$. Since S is symmetric we have $U\Delta U^T = S = S^T = U\Delta^T U^T$ and therefore $\Delta = \Delta^T = \Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$. Since all the entries of Σ are positive, we can define $X = (UD)^T$ with $D = \operatorname{diag}(\sqrt{\sigma_1}, \ldots, \sqrt{\sigma_n}) = D^T$. Then

$$X^T X = (UD)(UD)^T = UDD^T U^T = U\Sigma U^T = S.$$

Conversely, assume that there exists a nonsingular matrix X such that $S = X^T X$. Then we have $S^T = (X^T X)^T = X^T X$ and therefore S is symmetric. □

The following lemma is an expanded form of Lemma 4 in [5] and will be useful in the rather technical proofs that we will give in Chapter 4 and Chapter 5.

Lemma 2.2.3. *Let $X = [x_{i,j}] \in \mathbb{C}^{n\times m}$ that satisfies $J_n^T(1)XJ_m(1) = X$. Then all entries of X above its counter-diagonal, which starts in the upper right corner if $n \leq m$ or in the lower left corner otherwise, are equal to zero, that is*

$$x_{i,j} = 0 \quad \text{for all } i + j \leq \max\{m, n\}. \tag{2.1}$$

More generally, we have that X *satisfies* $J_n^T(1)XJ_m(1) = X$ *if and only if*

$$x_{i-1,j-1} + x_{i-1,j} + x_{i,j-1} = 0 \quad \text{for all } i = 1, \ldots, n \text{ and } j = 1, \ldots, m \tag{2.2}$$

where $x_{k,l} = 0$ *if* $k = 0$ *or* $l = 0$*. Furthermore, the first column of* X *is zero if* $n = m$ *is even and* X *is symmetric.*

Proof. First, direct computation reveals that $J_n^T(1)XJ_m(1) = X$ holds if and only if

$$\begin{bmatrix} x_{1,1} & x_{1,1}+x_{1,2} & \cdots & x_{1,m-1}+x_{1,m} \\ x_{1,1}+x_{2,1} & x_{1,1}+x_{1,2}+x_{2,1}+x_{2,2} & \cdots & x_{1,m-1}+x_{1,m}+x_{2,m-1}+x_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n-1,1}+x_{n,1} & x_{n-1,1}+x_{n-1,2}+x_{n,1}+x_{n,2} & \ldots & x_{n-1,m-1}+x_{n-1,m}+x_{n,m-1}+x_{n,m} \end{bmatrix}$$
$$= \begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,m} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,1} & x_{n,2} & \ldots & x_{n,m} \end{bmatrix}.$$

This gives us the equations $x_{i,j} = x_{i-1,j-1} + x_{i-1,j} + x_{i,j-1} + x_{i,j}$ for all $i = 1, \cdots, n$ and $j = 1, \cdots, m$, where $x_{k,l} = 0$ if $k = 0$ or $l = 0$. Canceling $x_{i,j}$ in each of these equations gives us exactly the constraints (2.2) which we are going to use in order to prove the first assertion of the lemma.

- If $i = 1$ and $j = 2, \ldots, m$ then $0 = x_{0,j-1} + x_{0,j} + x_{1,j-1} = x_{1,j-1}$, which proves that the first $m-1$ entries of the first row are zero, that is $x_{1,1} = \cdots = x_{1,m-1} = 0$.

- Next if $i = 2$ and $j = 2, \ldots, m-1$ then $0 = x_{1,j-1} + x_{1,j} + x_{2,j-1} = x_{2,j-1}$ and thus all entries, except the last two, of the second row are forced to be zero, that is $x_{2,1} = \cdots = x_{2,m-2} = 0$. Note that for $i = 2$ and $j = m$ we have $0 = x_{1,m-1} + x_{1,m} + x_{2,m-1} = x_{1,m} + x_{2,m-1}$, therefore $x_{1,m}$ and $x_{2,m-1}$ do not have to be 0.

- If $i = 3, \ldots, \min\{m,n\}-1$ and $j = 2, \ldots, m-i+1$, then similarly to the above, $x_{i,j-1} = 0$. Now note that $i + (j-1) \in \{i+1, \ldots, m\}$ and thus $x_{k,l} = 0$ for all $k + l \leq m$.

If $m \geq n$ then there is nothing else to show. For $n > m$ consider

- If $j = 1$ and $i = m+1, \ldots, n$ then $0 = x_{i-1,0} + x_{i-1,1} + x_{i,0} = x_{i-1,1}$ and therefore $x_{m,1} = \cdots = x_{n-1,1} = 0$.
- If $j = 2, \ldots, m$ and $i = m+2-j, \ldots, n+1-j$ then $0 = x_{i-1,j-1} + x_{i-1,j} + x_{i,j-1} = x_{i-1,j}$. Since $(i-1)+j \in \{m+1, \ldots, n\}$ this proves that $x_{k,l} = 0$ for all $k + l \leq n$.

Next, we want to prove the last part of the lemma. Assume that $n = m$ is even and X is symmetric. First, we have that the first $n-1$ entries of the first column are zero by (2.1) for $i = 1, \ldots, n-1$ and $j = 1$, that is $x_{1,1} = \cdots = x_{n-1,1} = 0$. Hence, it remains to show that $x_{n,1} = 0$. Note that for all $i = 2, \ldots, n$ and $j = n-i+2$ we have $x_{i-1,n-i+1} + x_{i-1,n-i+2} + x_{i,n-i+1} = 0$. Since $(i-1)+(n-i+1) = n = m$, it follows that $x_{i-1,n-i+1} = 0$, so we have $x_{i-1,n-i+2} = -x_{i,n-i+1}$. Hence

$$x_{1,n} = -x_{2,n-1}, \quad x_{2,n-1} = -x_{3,n-2}, \quad \ldots \quad x_{n-1,2} = -x_{n,1}.$$

Using this result and the facts that X is symmetric and n is even gives us

$$x_{n,1} = x_{1,n} = (-1)^{n-1} x_{n,1} = -x_{n,1}, \text{ which implies } x_{n,1} = 0.$$

Therefore the first column of X is zero. □

We end this section with another technical result which will be key in proving that the even-dimensional Jordan blocks for the eigenvalues 1 and -1 have to appear in pairs in the Jordan Canonical Form of an orthogonal matrix.

Lemma 2.2.4. *Let $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{m\times m}$ and suppose A and B have no eigenvalues in common. If a matrix $X \in \mathbb{C}^{n\times m}$ satisfies $AX = XB$, then $X = 0$.*

Proof. First, we want to prove by induction on k that $A^k X = X B^k$ for all $k \in \mathbb{Z}_+$. The case for $k = 1$ follows by assumption. Suppose the assertion is true for k, so we have

$$A^{k+1}X = A(A^k X) = A(XB^k) = (AX)B^k = XBB^k = XB^{k+1}.$$

Next, let $p(x) = a_n x^n + \cdots + a_1 x + a_0$ be any polynomial in $\mathbb{C}[x]$. Then

$$p(A)X = a_n A^n X + \cdots + a_1 AX + a_0 = a_n XB^n + \cdots + a_1 BX + a_0 = Xp(B).$$

Now choose $p(x)$ to be the characteristic polynomial of A, that is $p_A(x) = (x - \lambda_1)\cdots(x - \lambda_n)$. Then by the Cayley-Hamilton Theorem (1.1.5) we have that $0 = p_A(A)X = Xp_A(B)$. Since the eigenvalues of A and B are distinct, we know that $(B - \lambda_i I_n)$ is nonsingular for all $i = 1, \ldots, n$. Hence $p_A(B) = (B - \lambda_1 I_n)\cdots(B - \lambda_n I_n)$ is a nonsingular matrix and therefore $Xp_A(B) = 0$ only has the solution $X = 0$. $\square$

2.3 Matrix Exponential

Matrix functions in general are an interesting area in matrix analysis. We can define primary matrix functions such as the exponential, logarithmic and trigonometric functions using their common real power series applied to matrices. Showing the convergence of these functions requires the theory of matrix norms and power series which are studied in detail in Chapter 5 of [4]. More information about matrix functions can be found in Chapter 11 of Handbook of Linear Algebra [1]. After we define the matrix exponential, we will then study its properties.

Definition 2.3.1. *For $A \in \mathbb{C}^{n\times n}$, the exponential of A, denoted by e^A or $\exp(A)$, is given by the power series*

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

Note. Using matrix norms it can be shown that the matrix exponential is well-defined and converges for all matrices $A \in \mathbb{C}^{n\times n}$.

Proposition 2.3.2. *The matrix exponential e^A for a matrix $A \in \mathbb{C}^{n\times n}$ has the following properties*

(i) *$e^{(0_n)} = I_n$, where 0_n is the zero matrix in $\mathbb{C}^{n\times n}$;*

(ii) *If $A, B \in \mathbb{C}^{n\times n}$ such that $AB = BA$, then $e^{A+B} = e^A e^B$;*

(iii) *If* $C \in \mathbb{C}^{n\times n}$ *is nonsingular, then* $\exp(C^{-1}AC) = C^{-1}\exp(A)C$*;*

(iv) *If* $A = A_1 \oplus \cdots \oplus A_r$*, then* $\exp(A) = \exp(A_1) \oplus \cdots \oplus \exp(A_r)$*;*

(v) $(e^A)^T = e^{A^T}$*;*

(vi) $(e^A)^{-1} = e^{-A}$*.*

Proof. In order to prove **(i)** notice that $(0_n)^0 = I_n$ and $(0_n)^k = 0_n$ for all $k \in \mathbb{Z}_+$. Hence $e^{(0_n)} = \frac{1}{0!}I_n + \sum_{k=1}^{\infty}\frac{0_n}{k!} = I_n$.

Now, let $A, B \in \mathbb{C}^{n\times n}$. To prove **(ii)** assume $AB = BA$. Since A and B commute we can apply the binomial formula for matrices

$$(A+B)^k = \sum_{l=0}^{k}\frac{k!}{l!(k-l)!}A^lB^{k-l}.$$

So we have

$$\begin{aligned} e^{A+B} &= \sum_{k=0}^{\infty}\frac{(A+B)^k}{k!} = \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{l=0}^{k}\frac{k!}{l!(k-l)!}A^lB^{k-l} \\ &= \sum_{l=0}^{\infty}\sum_{k=l}^{\infty}\frac{1}{l!(k-l)!}A^lB^{k-l} = \sum_{l=0}^{\infty}\frac{1}{l!}A^l\sum_{k=0}^{\infty}\frac{1}{k!}B^k = e^Ae^B. \end{aligned}$$

Property **(iii)** follows by direct computation, since

$$\exp(C^{-1}AC) = \sum_{k=0}^{\infty}\frac{(C^{-1}AC)^k}{k!} = \sum_{k=0}^{\infty}\frac{C^{-1}(A)^kC}{k!} = C^{-1}\exp(A)C.$$

To prove **(iv)**, notice that $(A_1 \oplus \cdots \oplus A_r)^k = A_1^k \oplus \cdots \oplus A_r^k$ for all $k \in \mathbb{N} \cup \{0\}$, and apply the definition of matrix exponential. Next, property **(v)** follows from $e^{A^T} = \sum_{k=0}^{\infty}\frac{(A^T)^k}{k!} = \sum_{k=0}^{\infty}\frac{(A^k)^T}{k!} = (\sum_{k=0}^{\infty}\frac{A^k}{k!})^T = (e^A)^T$. Finally, to show **(vi)**, note that since A commutes with $-A$ we have

$$I_n \overset{(i)}{=} e^{0_n} = e^{A-A} \overset{(ii)}{=} e^Ae^{-A},$$

by properties **(i)** and **(ii)**. This proves that e^{-A} is the inverse of e^A. □

Corollary 2.3.3. *Let* $J_k(\lambda) \in \mathbb{C}^{k\times k}$ *be the Jordan block for the eigenvalue* $\lambda \in \mathbb{C}$. *Then* $\exp(J_k(\lambda)) = e^{\lambda}M$ *where*

$$M = \begin{bmatrix} \frac{1}{0!} & \frac{1}{1!} & \frac{1}{2!} & \cdots & \frac{1}{(k-1)!} \\ & \frac{1}{0!} & \frac{1}{1!} & \cdots & \frac{1}{(k-2)!} \\ & & \ddots & \ddots & \vdots \\ & \text{\huge0} & & \frac{1}{0!} & \frac{1}{1!} \\ & & & & \frac{1}{0!} \end{bmatrix} = \begin{bmatrix} 1 & 1 & \frac{1}{2} & \cdots & \frac{1}{(k-1)!} \\ & 1 & 1 & \cdots & \frac{1}{(k-2)!} \\ & & \ddots & \ddots & \vdots \\ & \text{\huge0} & & 1 & 1 \\ & & & & 1 \end{bmatrix}.$$

If $A \in \mathbb{C}^{n\times n}$ *has the Jordan Canonical Form* $J(A) = J_1(\lambda_1) \oplus \cdots \oplus J_r(\lambda_r)$, *then*

$$e^A = P\left[\exp(J_1(\lambda_1)) \oplus \cdots \oplus \exp(J_r(\lambda_r))\right]P^{-1}$$

where P *is the transition matrix such that* $A = PJ(A)P^{-1}$.

Proof. Note that $J_k(\lambda) = L+N$ where $L = \operatorname{diag}(\lambda, \ldots, \lambda)$ and $N = J_k(0)$. Since L and N commute we have by Proposition 2.3.2 (ii) that $\exp(J_k(\lambda)) = e^L e^N$. Now, $e^L = \operatorname{diag}(e^{\lambda}, \ldots, e^{\lambda}) = e^{\lambda}I_k$, since it is a special case of property (iv) in Proposition 2.3.2, where all blocks have size one by one. In order to determine e^N, note that

$$N^2 = \begin{bmatrix} 0 & 0 & 1 & & 0 \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & \text{\huge0} & & \ddots & 0 \\ & & & & 0 \end{bmatrix}, \ldots, \; N^{k-1} = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ & \ddots & & 0 \\ & & \ddots & \vdots \\ \text{\huge0} & & & 0 \end{bmatrix}$$

and $N^l = 0_k$ for all $l \geq k$. Thus,

$$e^N = \frac{1}{0!}N^0 + \frac{1}{1!}N + \cdots + \frac{1}{(k-1)!}N^{k-1} = I_k + N + \cdots + \frac{1}{(k-1)!}N^{k-1} = M.$$

This proves the first part of the corollary. The second part follows directly from Proposition 2.3.2 (iii) and (iv). □

The matrix exponential is our central tool in connecting the orthogonal and skew-symmetric matrices. For instance, we will use the following property to generate some examples in Chapter 3.

Corollary 2.3.4. *If $A \in \mathbb{C}^{n \times n}$ is skew-symmetric, then e^A is orthogonal.*

Proof. Since A is skew-symmetric we have $A^T = -A$ and hence by properties (v) and (vi) of Proposition 2.3.2 we can compute that

$$(e^A)^T = e^{A^T} = e^{-A} = (e^A)^{-1}.$$

Therefore, e^A is an orthogonal matrix. □

We will complete this section by showing that the matrix exponential of a Jordan block for the eigenvalue 0 corresponds to the Jordan block of the same size for the eigenvalue 1.

Proposition 2.3.5. *The Jordan Canonical Form of* $\exp\left(J_k(0)\right)$ *is* $J_k(1)$.

Proof. According to the result in Corollary 2.3.3, we know that

$$\exp(J_k(0)) = e^0 M = M = \begin{bmatrix} 1 & 1 & \frac{1}{2} & \cdots & \frac{1}{(k-1)!} \\ & 1 & 1 & \cdots & \frac{1}{(k-2)!} \\ & & \ddots & \ddots & \vdots \\ & \text{\Large 0} & & 1 & 1 \\ & & & & 1 \end{bmatrix}.$$

Thus, all k eigenvalues of $\exp(J_k(0)) = M$ are equal to 1 and $p_M(x) = (x-1)^k$. Moreover the minimal polynomial of M must be of the form $m_M(x) = (x-1)^l$, where $l \in \{1, \ldots, k\}$ is the minimal number such that $(M - I_k)^l = 0_k$. But since $M - I_k$ is a nilpotent matrix of degree $k-1$, we can deduce that $l = k$ and $m_M(x) = p_M(x) = (x-1)^k$. This proves that the Jordan Canonical Form of $M = \exp(J_k(0))$ is $J_k(1)$. □

CHAPTER 3. EXAMPLES

Before we prove the main results we will illustrate them with examples. Our goal is to exhibit complex orthogonal matrices and complex skew-symmetric matrices similar to the Jordan blocks of the types (a), (b) and (c), described in Theorem 1.2.3 and the Jordan blocks of the types (a) and (b), described in Theorem 1.2.5. Additional examples demonstrating how single Jordan blocks of size 2 for the eigenvalue 0 or 1 fail, are given in Sections 4.3 and 5.3.

3.1 Odd-dimensional Jordan Blocks for the Eigenvalues 0 and 1

First, we want to present some examples which show that odd-dimensional Jordan blocks for the eigenvalue 0 are similar to a family of complex skew-symmetric matrices and that Jordan blocks of odd size for the eigenvalue 1 are similar to a family of complex orthogonal matrices.

Example 3.1.1. *There are infinitely many complex skew-symmetric matrices which are similar to* $J_3(0)$.

If $J_3(0)$ is similar to a skew-symmetric matrix $A \in \mathbb{C}^{3\times 3}$, then A must be of the following form

$$A = \begin{bmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{bmatrix}$$

The characteristic polynomial of this matrix is $p_A(x) = x^3 + (a_1^2 + a_2^2 + a_3^2)x$. Since all eigenvalues of $J_3(0)$ are 0, it follows that $p_A(x) = x^3$ and therefore we have $a_1^2 + a_2^2 + a_3^2 = 0$. To achieve this, one possibility is to set $a_1 = i$ which gives us the equation $a_2^2 + a_3^2 = 1$. Therefore $a_2 = \cos(\theta)$ and $a_3 = \sin(\theta)$ satisfy this equation for all $\theta \in [0, 2\pi)$. Using these values for a_1, a_2 and a_3 we can define a matrix A_θ for each $\theta \in [0, 2\pi)$

$$A_\theta = \begin{bmatrix} 0 & i & \cos(\theta) \\ -i & 0 & \sin(\theta) \\ -\cos(\theta) & -\sin(\theta) & 0 \end{bmatrix}.$$

Since A_θ, A_θ^2 are not zero, we have $m_{A_\theta}(x) = p_{A_\theta}(x) = x^3$. Thus, $J_3(0)$ is the Jordan Canonical Form of A_θ. Hence, there is an infinite family of skew-symmetric matrices which are similar to $J_3(0)$. For completeness we provide a similarity matrix. If

$$\begin{aligned} P_\theta &= \begin{bmatrix} 0 & 0 & 1 \\ -\cos(\theta) & -\sin(\theta) & 0 \\ i\sin(\theta) & -i\cos(\theta) & -1 \end{bmatrix} \\ \text{then } P_\theta^{-1} &= \begin{bmatrix} -i\sin(\theta) & -\cos(\theta) & -i\sin(\theta) \\ i\cos(\theta) & -\sin(\theta) & i\cos(\theta) \\ 1 & 0 & 0 \end{bmatrix}, \end{aligned}$$

and $P_\theta^{-1} J_3(0) P_\theta = A_\theta$ for all $\theta \in [0, 2\pi)$.

Note. The matrices A_θ for $\theta \in [0, 2\pi)$ are not the only possible matrices which are similar to $J_3(0)$. Clearly, there are other matrices that meet the requirements $a_1^2 + a_2^2 + a_3^2 = 0$ and $m_A(x) = p_A(x)$.

The next example uses the preceding result in order to prove a similar result for the orthogonal case.

Example 3.1.2. *There are infinitely many complex orthogonal matrices which are similar to* $J_3(1)$.

Using the notation from Example 3.1.1, recall that for all $\theta \in [0, 2\pi)$ there exists a nonsingular complex matrix P_θ such that $P_\theta^{-1} J_3(0) P_\theta = A_\theta$. By Corollary 2.3.4 we have that $Q_\theta = \exp(A_\theta)$ is orthogonal. Moreover, Proposition 2.3.2 (iii) gives us that

$$Q_\theta = P_\theta^{-1} \exp(J_3(0)) P_\theta.$$

Next, Proposition 2.3.5 shows that the Jordan block $J_3(1)$ is the Jordan Canonical Form of $\exp(J_3(0))$. One similarity matrix for $T^{-1}J_3(1)T = \exp(J_3(0))$ is

$$T = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, we obtain $Q_\theta = P_\theta^{-1}\exp(J_3(0))P_\theta = (TP_\theta)^{-1}J_3(1)TP_\theta$. So, for all $\theta \in [0, 2\pi)$, $J_3(1)$ is similar to the orthogonal matrix

$$Q_\theta = \frac{1}{4}\begin{bmatrix} 5-\cos(2\theta) & 4i - 2\cos(\theta)\sin(\theta) & 4\cos(\theta) + 2i\sin(\theta) \\ -2\cos(\theta)\sin(\theta) - 4i & \cos(2\theta) + 5 & 4\sin(\theta) - 2i\cos(\theta) \\ 2i\sin(\theta) - 4\cos(\theta) & -2i\cos(\theta) - 4\sin(\theta) & 2 \end{bmatrix}$$

via the similarity matrix

$$T_\theta = TP_\theta = \begin{bmatrix} \frac{\cos(\theta)}{2} & \frac{\sin(\theta)}{2} & 1 \\ -\cos(\theta) & -\sin(\theta) & 0 \\ i\sin(\theta) & -i\cos(\theta) & -1 \end{bmatrix}.$$

The next two examples demonstrate that we can also achieve the previous results also for dimension 5.

Example 3.1.3. *There are infinitely many complex skew-symmetric matrices which are similar to $J_5(0)$.*

The construction of a skew-symmetric matrix C which is similar to $J_5(0)$ can be done similarly to the matrix A in Example 3.1.1. Obviously, we get more difficult constraints from the characteristic polynomial and there are many ways to construct a skew-symmetric matrix which has the desired properties. The following skew-symmetric matrix

$$C_\theta = \begin{bmatrix} 0 & 1 & \cos(\theta) & i & i \\ -1 & 0 & 0 & 0 & 0 \\ -\cos(\theta) & 0 & 0 & 0 & \sin(\theta) \\ -i & 0 & 0 & 0 & 0 \\ -i & 0 & -\sin(\theta) & 0 & 0 \end{bmatrix}$$

satisfies $p_{C_\theta}(x) = m_{C_\theta}(x) = x^5$ for all $\theta \in [0, 2\pi)$ and therefore is similar to $J_5(0)$. The complex matrix

$$R_\theta = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & -i & -i\cos(\theta) & 1 & 1 \\ -i\sin^2(\theta) & 0 & -\sin(\theta) & 0 & -i\cos(\theta)\sin(\theta) \\ 0 & -i\sin^2(\theta) & 0 & \sin^2(\theta) & 0 \end{bmatrix}$$

is a corresponding similarity matrix, hence $R_\theta^{-1} J_5(0) R_\theta = C_\theta$ for all $\theta \in [0, 2\pi)$.

Example 3.1.4. *There are infinitely many complex orthogonal matrices which are similar to* $J_5(1)$.

Following the argument from Example 3.1.2 and using the matrices found in Example 3.1.3, we know that $\widetilde{Q}_\theta = \exp(C_\theta) = R_\theta^{-1} \exp(J_5(0)) R_\theta$ is complex orthogonal. Since $J_5(1)$ is the Jordan Canonical Form of $\exp(J_5(0))$ there exists $\widetilde{T}$ such that $\widetilde{T}^{-1} J_5(1) \widetilde{T} = \exp(J_5(0))$:

$$\widetilde{T} = \begin{bmatrix} 2 & -3 & \frac{5}{6} & 0 & 0 \\ 0 & 2 & -2 & -\frac{1}{3} & 0 \\ 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

Hence $\widetilde{Q}_\theta = \widetilde{T}_\theta^{-1} J_5(1) \widetilde{T}_\theta$, where $\widetilde{T}_\theta = \widetilde{T} R_\theta$.

3.2 Paired Jordan Blocks

Similar to the examples in the previous section, we want to explicitly exhibit a family of skew-symmetric matrices A_x and nonsingular matrices P_x, such that $J_2(0) \oplus J_2(0)$ is similar to A_x via the similarity matrix P_x for all $x \in \mathbb{C} \setminus \{0\}$.

Example 3.2.1. $J_2(0) \oplus J_2(0)$ *is similar to infinitely many complex skew-symmetric matrices.*

If there exists a skew-symmetric matrix

$$A = \begin{bmatrix} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & a_4 & a_5 \\ -a_2 & -a_4 & 0 & a_6 \\ -a_3 & -a_5 & -a_6 & 0 \end{bmatrix}$$

which is similar to $J_2(0) \oplus J_2(0)$, then its characteristic polynomial must be $p_A(x) = x^4$ and its minimal polynomial must be $m_A(x) = x^2$. Thus

$$A^2 = \begin{bmatrix} -a_1^2 - a_2^2 - a_3^2 & -a_2a_4 - a_3a_5 & a_1a_4 - a_3a_6 & a_1a_5 + a_2a_6 \\ -a_2a_4 - a_3a_5 & -a_1^2 - a_4^2 - a_5^2 & -a_1a_2 - a_5a_6 & a_4a_6 - a_1a_3 \\ a_1a_4 - a_3a_6 & -a_1a_2 - a_5a_6 & -a_2^2 - a_4^2 - a_6^2 & -a_2a_3 - a_4a_5 \\ a_1a_5 + a_2a_6 & a_4a_6 - a_1a_3 & -a_2a_3 - a_4a_5 & -a_3^2 - a_5^2 - a_6^2 \end{bmatrix} = 0_4.$$

This give us 10 different equations in 6 unknowns. One possible solution is to set $a_3 = a_4 = 0$ and $a_2 = a_5 = x$, $a_1 = ia_2$, $a_6 = -ia_2$ for some $x \in \mathbb{C} \setminus \{0\}$ which gives us the matrix

$$A_x = \begin{bmatrix} 0 & ix & x & 0 \\ -ix & 0 & 0 & x \\ -x & 0 & 0 & -ix \\ 0 & -x & ix & 0 \end{bmatrix}.$$

The corresponding nonsingular similarity matrix for all complex numbers $x \neq 0$ which satisfies $P_x^{-1}(J_2(0) \oplus J_2(0))P_x = A_x$ is

$$P_x = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -x & ix & 0 \\ 0 & 0 & 1 & 0 \\ -x & 0 & 0 & -ix \end{bmatrix}.$$

Example 3.2.2. *There are infinitely many complex orthogonal matrices which are similar to* $J_2(1) \oplus J_2(1)$.

Using the same argument and notation as in Example 3.2.1, we can show that

$$Q_x = \exp(A_x) = \begin{bmatrix} 1 & ix & x & 0 \\ -ix & 1 & 0 & x \\ -x & 0 & 1 & -ix \\ 0 & -x & ix & 1 \end{bmatrix} = P_x^{-1} \exp(J_2(0) \oplus J_2(0)) P_x$$

is orthogonal for all complex $x \neq 0$. Since $\exp(J_2(0) \oplus J_2(0)) = J_2(1) \oplus J_2(1)$, this establishes our claim.

CHAPTER 4. ORTHOGONAL CASE

4.1 General Properties

We have shown that every matrix is similar to its transpose (Theorem 2.1.5). Now we are interested in determining which properties a matrix has to possess in order to be similar to its inverse.

Proposition 4.1.1. *A nonsingular complex square matrix* $A \in \mathbb{C}^{n\times n}$ *is similar to its inverse matrix* A^{-1} *if and only if its Jordan Canonical Form contains only Jordan blocks with eigenvalues* ± 1 *and pairs of blocks of the form* $J_k(\lambda) \oplus J_k(\lambda^{-1})$ *for* $\lambda \in \mathbb{C} \setminus \{-1, 0, 1\}$.

Proof. First, we have by Theorem 2.1.1 that A is similar to its Jordan Canonical Form $J(A) = J_{k_1}(\lambda_1) \oplus \cdots \oplus J_{k_r}(\lambda_r)$. Therefore A^{-1} is similar to the $J(A)^{-1} = J_{k_1}(\lambda_1)^{-1} \oplus \cdots \oplus J_{k_r}(\lambda_r)^{-1}$. But, according to Lemma 2.1.7, $J(A)^{-1}$ is similar to $J_{k_1}(\lambda_1^{-1}) \oplus \cdots \oplus J_{k_r}(\lambda_r^{-1})$, hence the Jordan Canonical Form of A^{-1} is

$$J(A^{-1}) = J_{k_1}(\lambda_1^{-1}) \oplus \cdots \oplus J_{k_r}(\lambda_r^{-1}).$$

Next, by Theorem 2.1.2 we have that A and A^{-1} are similar if and only if they have the same Jordan Canonical Form. Therefore, for every $\lambda \in \mathbb{C} \setminus \{-1, 0, 1\}$ the Jordan Canonical Form $J(A)$ has to contain both Jordan blocks $J_{k_i}(\lambda_i)$ and $J_{k_i}(\lambda_i^{-1})$. Since $(\pm 1)^{-1} = \pm 1$ the Jordan blocks for these eigenvalues do not have to appear in pairs. $\square$

For an orthogonal matrix Q, we have by definition that $Q^T = Q^{-1}$, so we can deduce that it must be similar to its inverse. Thus its Jordan Canonical Form must fulfill the properties described in the proposition above. However, it is not obvious that the Jordan Canonical Form of a complex orthogonal matrix has an additional constraint on its Jordan structure: its even-dimensional Jordan blocks with eigenvalue 1 or -1 must occur in pairs. We will show this in the next section, but first we need the following key observation.

Lemma 4.1.2. ([5], Lemma 1) *Let $A \in \mathbb{C}^{n\times n}$ be nonsingular. The following statements are equivalent:*

(i) *A is similar to a complex orthogonal matrix;*

(ii) *A is similar to a complex orthogonal matrix via a complex symmetric similarity;*

(iii) *there exists a nonsingular complex symmetric matrix $S \in \mathbb{C}^{n\times n}$ such that $A^T = SA^{-1}S^{-1}$;*

(iv) *there exists a nonsingular complex symmetric matrix $S \in \mathbb{C}^{n\times n}$ such that $A^TSA = S$.*

Proof.

(i) ⇒ (ii) Assume there exists a nonsingular matrix R such that $RAR^{-1} = Q$ is orthogonal. Since A is a nonsingular matrix, Theorem 2.2.1 ensures the existence of a nonsingular complex symmetric matrix S and a complex orthogonal matrix P such that $R^T = SP^T$ and thus $R = PS$. Therefore

$$Q = RAR^{-1} = (PS)A(PS)^{-1} = P(SAS^{-1})P^T.$$

Since P is orthogonal this gives us $P^TQP = SAS^{-1}$. This proves (ii) since P^TQP is orthogonal as it is a product of orthogonal matrices.

(ii) ⇒ (iii) Assume there exists a nonsingular symmetric complex matrix S and an orthogonal complex matrix Q such that $A = SQS^{-1}$ and therefore $A^{-1} = SQ^TS^{-1}$. Using $S = S^T$ we have

$$A^T = (SQS^{-1})^T = S^{-1}Q^TS = S^{-2}(SQ^TS^{-1})S^2 = S^{-2}A^{-1}(S^{-2})^{-1}.$$

This proves (iii) since S^{-2} is symmetric.

(iii) ⇒ (i) Assume that $A^T = SA^{-1}S^{-1}$ where S is nonsingular and symmetric. According to Proposition 2.2.2 there exists a nonsingular complex matrix $X^T \in \mathbb{C}^{n\times n}$ such that

$S = X^T X$. So we have

$$A^T = SA^{-1}S^{-1} = (X^T X)A^{-1}(X^T X)^{-1} = X^T(XA^{-1}X^{-1})X^{-T}$$

which is equivalent to

$$X^{-T}A^T X^T = XA^{-1}X^{-1} \text{ and therefore } (XAX^{-1})^T = (XAX^{-1})^{-1}.$$

Hence XAX^{-1} is an orthogonal matrix and A is similar to an orthogonal matrix via the nonsingular matrix X.

(iii) $\Leftrightarrow$ **(iv)** This equivalence is trivial. □

4.2 Proof of the Main Result for Orthogonal Matrices

The first goal is to show that any pair of Jordan blocks for eigenvalues $\lambda \neq 0$ and λ^{-1} is similar to a complex orthogonal matrix.

Lemma 4.2.1. ([5], Lemma 2) *For any positive integer k and any $\lambda \neq 0$, $J_k(\lambda) \oplus J_k(\lambda^{-1})$ is similar to a complex orthogonal matrix.*

Proof. According to Proposition 2.1.3 the Jordan block $J_k(\lambda)$ is similar to a symmetric matrix S. Furthermore, we know by Lemma 2.1.7 that $J_k(\lambda) \oplus J_k(\lambda^{-1})$ is similar to $J_k(\lambda) \oplus J_k(\lambda)^{-1}$, which in turn is similar to $S \oplus S^{-1}$. Hence, it is enough to show that $S \oplus S^{-1}$ is similar to an orthogonal matrix. Define the symmetric matrix

$$H = \begin{bmatrix} 0 & I_k \\ I_k & 0 \end{bmatrix} \in M_{2k} \text{ where } H^2 = I_{2k} \text{ and hence } H^{-1} = H.$$

Therefore we have

$$H(S \oplus S^{-1})^{-1}H^{-1} = H(S^{-1} \oplus S)H = S \oplus S^{-1} = (S \oplus S^{-1})^T$$

According to Lemma 4.1.2 (iii), this proves the assertion. □

Our next step is to show that odd-dimensional Jordan blocks for the eigenvalues 1 and -1 which are not paired are similar to complex orthogonal matrices.

Lemma 4.2.2. ([5], Lemma 3) *For any odd positive integer k, each of $J_k(1)$ and $J_k(-1)$ is similar to a complex orthogonal matrix.*

Proof. By Lemma 2.1.6 we know that the Jordan block $J_k(-1)$ is similar to $-J_k(1)$, hence it is enough to prove that $J_k(1)$ is similar to a complex orthogonal matrix whenever k is an odd positive integer. We are going to establish the identity $J_k(1)^T S_k J_k(1) = S_k$ (for S_k defined below) by induction on $n \in \mathbb{N} \cup \{0\}$, where $k = 2n+1$, and obtain the desired result by Lemma 4.1.2 (iv).

If $n = 0$ then $k = 1$. Let $S_1 = [1]$, then $J_1(1)^T S_1 J_1(1) = [1] = S_1$. For $n = 1$, we have $k = 3$ and define

$$S_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -S_1 & x_1 \\ 1 & x_1 & 0 \end{bmatrix} \in \mathbb{C}^{3\times 3},$$

which is symmetric and satisfies the conditions (2.1) from Lemma 2.2.3. The constraints (2.2) of Lemma 2.2.3 give us $2x_1 - S_1 = 0$, so $x_1 = \frac{1}{2}$. With this choice of x_1, $J_3(1)^T S_3 J_3(1) = S_3$. For $n \geq 2$ we use the same idea to construct S_{2n+1}. Let

$$S_{2n+1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -S_{2n-1} & x \\ 1 & x^T & 0 \end{bmatrix}, \text{ where } x^T = [x_1, \ldots, x_{2n-1}] \in \mathbb{C}^{2n-1}.$$

By the induction hypothesis we know that S_{2n-1} is symmetric and satisfies the constraints (2.1) and (2.2). Hence S_{2n+1} is symmetric and satisfies (2.1), so it remains to find values for x such that (2.2) holds. First notice that by definition of $S_{2n-1} = [s_{ij}]_{i,j=1,\ldots,2n-1}$ we have that $s_{2n-1,2n-1} = 0$ for all $n \geq 2$. So $2x_{2n-1} - s_{2n-1,2n-1} = 0$ gives us $x_{2n-1} = 0$. Moreover, by (2.2) applied to $i = 2n+1$ and $j = 3, \ldots, 2n-1$, we have the following system of $2n-2$ equations for the unknowns $x_1, \ldots, x_{2n-2}$:

$$\begin{aligned} x_1 &= s_{2n-1,1} + s_{2n-1,2}, \\ x_2 &= s_{2n-1,2} + s_{2n-1,3}, \\ &\vdots \\ x_{2n-2} &= s_{2n-1,2n-2} + s_{2n-1,2n-1}. \end{aligned}$$

This gives us a unique solution for the vector x such that S_{2n+1} satisfies all necessary conditions for $J_{n+1}(1)^T S_{n+1} J_{n+1}(1) = S_{n+1}$. $\square$

We have now shown that all the Jordan Canonical Forms in Theorem 1.2.3 are similar to orthogonal matrices. The next Lemma will be used to exclude unpaired Jordan block for the eigenvalues 1 and -1 whenever its dimension is even.

Lemma 4.2.3. ([5], Lemma 5) *Let $r, k_1, \dots, k_r$ be positive integers with k_1 is even. If $r > 1$ suppose that $k_1 > k_2 \geq \cdots \geq k_r$. Then neither $J_{k_1}(1) \oplus \cdots \oplus J_{k_r}(1)$ nor $J_{k_1}(-1) \oplus \cdots \oplus J_{k_r}(-1)$ is similar to a complex orthogonal matrix.*

Proof. Since $J_{k_i}(-1)$ is similar to $-J_{k_i}(1)$ we only need to consider the eigenvalue 1. Assume to the contrary that $J = J_{k_1}(1) \oplus \cdots \oplus J_{k_r}(1)$ is similar to a complex orthogonal matrix. Then by Lemma 4.1.2 (iv) there exists a nonsingular symmetric matrix S such that $J^T S J = S$. Partition S conformally so that $S = [S_{k_i,k_j}]_{i,j=1,\dots,r}$, where each $S_{k_i,k_i} \in \mathbb{C}^{k_i \times k_i}$ is symmetric for all $i = 1, \dots, r$ and $S_{k_i,k_j} = S^T_{k_j,k_i} \in \mathbb{C}^{k_i \times k_j}$ for all $i, j = 1, \dots, r$. Then

$$\begin{aligned} J^T S J &= \begin{bmatrix} J^T_{k_1} & & \\ & \ddots & \\ & & J^T_{k_r} \end{bmatrix} \begin{bmatrix} S_{k_1,k_1} & \cdots & S_{k_1,k_r} \\ \vdots & \ddots & \vdots \\ S_{k_r,k_1} & \cdots & S_{k_r,k_r} \end{bmatrix} \begin{bmatrix} J_{k_1} & & \\ & \ddots & \\ & & J_{k_r} \end{bmatrix} \\ &= \begin{bmatrix} J^T_{k_1} S_{k_1,k_1} J_{k_1} & \cdots & J^T_{k_1} S_{k_1,k_r} J_{k_r} \\ \vdots & \ddots & \vdots \\ J^T_{k_r} S_{k_r,k_1} J_{k_1} & \cdots & J^T_{k_r} S_{k_r,k_r} J_{k_r} \end{bmatrix} = \begin{bmatrix} S_{k_1,k_1} & \cdots & S_{k_1,k_r} \\ \vdots & \ddots & \vdots \\ S_{k_r,k_1} & \cdots & S_{k_r,k_r} \end{bmatrix} \end{aligned}$$

so that $J^T_{k_i} S_{k_i,k_j} J_{k_j} = S_{k_i,k_j}$ for all $i, j = 1, \dots r$. Then in particular $S_{k_1,k_1} = J^T_{k_1} S_{k_1,k_1} J_{k_1}$. Since k_1 is even by assumption, Lemma 2.2.3 ensures that the first column of S_{k_1,k_1} is zero. Moreover, we know that $k_1 > k_i$ for all $i = 2, \dots, r$, and since $J^T_{k_i} S_{k_i,k_1} J_{k_1} = S_{k_i,k_1}$ the constraints (2.1) from Lemma 2.2.3 give us that the first column of each of $S_{k_2,k_1}, \dots, S_{k_r,k_1}$ must be zero. Hence, the first column of S is zero which contradicts the fact that S is nonsingular. $\square$

Although a Jordan block of even dimension for the eigenvalue 1 or -1 can not appear by itself, we did not exclude the possibility that we can group even-dimensional Jordan blocks for the eigenvalue 1 and -1 together.

Theorem 4.2.4. ([5], Theorem 3) *Let* $r, k_1, \ldots, k_r$ *and* $p, l_1, \ldots, l_p$ *be positive integers where* k_1 *and* l_1 *are even. If* $r > 1$ *suppose that* $k_1 > k_2 \geq \cdots \geq k_r$ *and if* $p > 1$ *assume* $l_1 > l_2 \geq \cdots \geq l_r$*. Then* $J_{k_1}(1) \oplus \cdots \oplus J_{k_r}(1) \oplus J_{l_1}(-1) \oplus \cdots \oplus J_{l_p}(-1)$ *is not similar to a complex orthogonal matrix.*

Proof. For convenience let $J_+ = J_{k_1}(1) \oplus \cdots \oplus J_{k_r}(1)$ and $J_- = J_{l_1}(-1) \oplus \cdots \oplus J_{l_p}(-1)$. Now assume the converse, that is $J_+ \oplus J_-$ is similar to a complex orthogonal matrix. Thus by Lemma 4.1.2 there exist a nonsingular, symmetric matrix S such that $(J_+^T \oplus J_-^T)S = S(J_+^{-1} \oplus J_-^{-1})$. Now we partition $S = [S_{i,j}]_{i,j=1,2}$ conformally to $J_+ \oplus J_-$, then

$$(J_+^T \oplus J_-^T)S = \begin{bmatrix} J_+^T & 0 \\ 0 & J_-^T \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} = \begin{bmatrix} J_+^T S_{11} & J_+^T S_{12} \\ J_-^T S_{12}^T & J_-^T S_{22} \end{bmatrix} \quad \text{and}$$

$$S(J_+^{-1} \oplus J_-^{-1}) = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} \begin{bmatrix} J_+^{-1} & 0 \\ 0 & J_-^{-1} \end{bmatrix} = \begin{bmatrix} S_{11}J_+^{-1} & S_{12}J_-^{-1} \\ S_{12}^T J_+^{-1} & S_{22}J_-^{-1} \end{bmatrix}.$$

Comparing the $(1,2)$-entries of the two matrices above gives us $J_+^T S_{12} = S_{12}J_-^{-1}$. Since J_+ and $J-$ have distinct eigenvalues, we can deduce from Lemma 2.2.4 that $S_{12} = 0$. Hence $S = S_{11} \oplus S_{22}$ and therefore both J_+ and J_- are similar to complex orthogonal matrices, which contradicts Lemma 4.2.3. Thus $J_+ \oplus J_-$ is not similar to a complex orthogonal matrix. ☐

The final tool needed to prove the necessity of the Jordan blocks of types (a), (b) and (c) in Theorem 1.2.3 is the following cancellation lemma.

Lemma 4.2.5. ([5], Lemma 6) *Let* $C \in \mathbb{C}^{k \times k}$ *be similar to a complex orthogonal matrix. If* $B \oplus C$ *is similar to a complex orthogonal matrix for some* $B \in \mathbb{C}^{n \times n}$*, then* B *is similar to a complex orthogonal matrix.*

Proof. We know that RCR^{-1} is complex orthogonal for some nonsingular matrix R. Then $(I_n \oplus R)(B \oplus C)(I_n \oplus R)^{-1} = B \oplus RCR^{-1}$ is still similar to a complex orthogonal matrix, so we can assume without loss of generality that C is complex orthogonal. Furthermore, there exists a nonsingular matrix X such that $A = X(B \oplus C)X^{-1}$ is complex orthogonal, hence

$$\begin{aligned} X^{-T}(B^T \oplus C^T)X^T &= A^T = A^{-1} = X(B^{-1} \oplus C^T)X^{-1}, \quad \text{so} \\ (B^T \oplus C^T)X^T X &= X^T X(B^{-1} \oplus C^T). \end{aligned}$$

Thus, there exists a nonsingular and symmetric matrix $S = X^T X$ such that

$$(B^T \oplus C^T)S = S(B^{-1} \oplus C^T). \tag{4.1}$$

Now partition $S = [S_{ij}]_{i,j=1,2}$ conformally to $B \oplus C$ where $S_{21} = S_{12}^T \in \mathbb{C}^{n\times k}$ and both $S_{11} \in \mathbb{C}^{n\times n}$ and $S_{22} \in \mathbb{C}^{k\times k}$ are symmetric. The next step will be to prove that we can adjust S_{22} if necessary, so that S_{22} is nonsingular. For this purpose we add $0_n \oplus tC^T$ to both sides of (4.1), where $t \in \mathbb{C}$

$$\begin{bmatrix} B^T & 0 \\ 0 & C^T \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22}+tI_k \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22}+tI_k \end{bmatrix} \begin{bmatrix} B^{-1} & 0 \\ 0 & C^T \end{bmatrix}$$

Thus if we let $S(t) = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22}+tI_k \end{bmatrix}$, then $S(t)$ satisfies (4.1) for all $t \in \mathbb{C}$. Now define $p(t) = \det S(t)$. Since $S = S(0)$ is nonsingular we have that $p(0) \neq 0$, which proves that $p(t)$ is not the zero polynomial. Moreover, there are only finitely many values $t \in \mathbb{C}$ such that $p(t) = 0$ and $S_{22} + tI_k$ is singular, so we may assume that S_{22} is nonsingular and satisfies (4.1). The identity

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -S_{22}^{-1}S_{12}^T & I_k \end{bmatrix} = \begin{bmatrix} S_{11} - S_{12}S_{22}^{-1}S_{12}^T & S_{12} \\ 0 & S_{22} \end{bmatrix}$$

shows that $Y \equiv S_{11} - S_{12}S_{22}^{-1}S_{12}^T$, the Schur complement of S_{22} in S, is nonsingular and symmetric since S_{11} and S_{22} are both symmetric. The block multiplication of both sides in (4.1) gives us

$$\begin{bmatrix} B^T S_{11} & B^T S_{12} \\ C^T S_{12}^T & C^T S_{22} \end{bmatrix} = \begin{bmatrix} S_{11}B^{-1} & S_{12}C^T \\ S_{12}^T B^{-1} & S_{22}C^T \end{bmatrix} \tag{4.2}$$

Now, comparing the $(1,2)$-entries and taking its transpose gives us (a) $S_{12}^T B = CS_{12}^T$, and comparing the entries $(2,2)$ and taking its inverse gives us (b) $S_{22}^{-1}C = CS_{22}^{-1}$. Thus (a) and (b) give us

$$S_{22}^{-1}S_{12}^T B \overset{(a)}{=} S_{22}^{-1}CS_{12}^T \overset{(b)}{=} CS_{22}^{-1}S_{12}^T. \tag{4.3}$$

Next, comparing the $(1,1)$-entries yields (c) $B^T S_{11} = S_{11} B^{-1}$. Then, the transpose of (a), equations (c) and (4.3), and the orthogonality of C give us finally

$$\begin{aligned} B^T Y &= B^T S_{11} - B^T S_{12} S_{22}^{-1} S_{12}^T \overset{(c)}{=} S_{11} B^{-1} - (B^T S_{12})(S_{22}^{-1} S_{12}^T B) B^{-1} \\ &\overset{(a)^T,(4.3)}{=} S_{11} B^{-1} - (S_{12} C^T)(C S_{22}^{-1} S_{12}^T) B^{-1} = S_{11} B^{-1} - (S_{12} S_{22}^{-1} S_{12}^T) B^{-1} \\ &= Y B^{-1}. \end{aligned}$$

Since Y is nonsingular and symmetric, we have by Lemma 4.1.2 that B is similar to a complex orthogonal matrix. □

Theorem 4.2.6. ([5], Theorem 4) *Let $A \in \mathbb{C}^{n \times n}$ be a complex orthogonal matrix. Then the even-sized Jordan blocks of A corresponding to each of the eigenvalues 1 and -1 appear in pairs.*

Proof. Let $J(A)$ be the Jordan Canonical Form of A. Now let C_1 be the direct sum of all the pairs of Jordan blocks in $J(A)$ of the form $J_k(\lambda) \oplus J_k(\lambda^{-1})$. Notice that by Proposition 4.1.1 the Jordan blocks for the eigenvalues $\lambda \in \mathbb{C} \setminus \{-1, 0, 1\}$ have to appear in pairs, so that they are all incorporated in C_1. Furthermore, C_1 contains all paired blocks of A with the eigenvalues ± 1. Therefore, the remaining blocks of $J(A)$ are all unpaired blocks for the eigenvalues ± 1. Let C_2 and C_3 respectively denote the direct sum of all the remaining odd-sized Jordan blocks and even-sized Jordan blocks in $J(A)$. According to Lemmas 4.2.1 and 4.2.2 we know that $C_1 \oplus C_2$ is similar to a complex orthogonal matrix. Moreover, the Jordan Canonical Form $J(A) = (C_1 \oplus C_2) \oplus C_3$ is similar to A which is orthogonal. Hence according to Lemma 4.2.5, C_3 must be similar to a complex orthogonal matrix. But, by Theorem 4.2.4, if C_3 is nonempty, it is not similar to a complex orthogonal matrix, and therefore the direct sum forming C_3 must be empty. □

Now we have all required tools to show our main result for orthogonal matrices. Recall Theorem 1.2.3 before we give its proof:

Theorem 1.2.3. *An $n \times n$ complex matrix is similar to a complex orthogonal matrix if and only if its Jordan Canonical Form can be expressed as a direct sum of matrices of only the* *following three types:*

(a) $J_k(\lambda) \oplus J_k(\lambda^{-1})$ *for* $\lambda \in \mathbb{C} \setminus \{0\}$ *and any* k*,*

(b) $J_k(1)$ *for any odd* k *and*

(c) $J_k(-1)$ *for any odd* k*.*

Proof. As a first step, Lemma 4.2.1 proves that the Jordan blocks of type (a) are similar to a complex orthogonal matrix. The next step is to prove the same result for the Jordan blocks of types (b) and (c), which is accomplished by Lemma 4.2.2. Since a direct sum of matrices that are individually similar to complex orthogonal matrices must be similar to a complex matrix, these two lemmas establish the sufficiency of the conditions in Theorem 1.2.3.

To prove their necessity we only need to show that it is not possible to have an odd number of Jordan blocks of a given even dimension associated with either of the eigenvalues 1 or -1. This is done by Theorem 4.2.6 whose proof in turn is based on Theorem 4.2.4 and Lemma 4.2.5. □

4.3 Examples

Next, we want to present examples of matrices which are similar to their inverses, but are not similar to any orthogonal matrix. We showed in Example 3.2.2 that $J_2(1) \oplus J_2(1)$ is similar to infinitely many orthogonal matrices. Now we want to prove that this Jordan block can not appear by itself.

Example 4.3.1. $J_2(1)$ *is not similar to a complex orthogonal matrix.*

To see this, assume to the contrary that $J_2(1)$ is similar to a complex orthogonal matrix. Then according to Lemma 4.1.2 (iv), there exists a nonsingular symmetric matrix S such that

$J_2(1)^T S J_2(1) = S$. That is,

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} s_{1,1} & s_{1,2} \\ s_{1,2} & s_{2,2} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} s_{1,1} & s_{1,1}+s_{1,2} \\ s_{1,1}+s_{1,2} & s_{1,1}+2s_{1,2}+s_{2,2} \end{bmatrix}$$

$$= \begin{bmatrix} s_{1,1} & s_{1,2} \\ s_{1,2} & s_{2,2} \end{bmatrix}$$

Therefore, $s_{1,1} = s_{1,2} = 0$ which contradicts the fact that S is nonsingular.

Example 4.3.2. *The nonsingular complex matrix*

$$A = \begin{bmatrix} i & -2 & -2i & 2 \\ 0 & -i & 1-i & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

is similar to its inverse, however A is not similar to any orthogonal matrix.

Since A is upper-triangular its eigenvalues are the diagonal entries $i, -i, 1$ and 1. Direct computation shows that $p_A(x) = m_A(x) = (x-i)(x+i)(x-1)^2$ and therefore $J(A) = J_1(i) \oplus J_1(-i) \oplus J_2(1)$, where $i^{-1} = -i$. Hence, according to Proposition 4.1.1 we have that A is similar to its inverse.

To prove the second part of our claim, we assume to the contrary that A is similar to an orthogonal matrix. So, by Lemma 4.1.2 (iv), there exists a nonsingular symmetric matrix S such that $A^T S A = S$. Comparing the entries of $A^T S A$ and S forces

$$S = \begin{bmatrix} 0 & s_{12} & is_{12} & -s_{12} \\ s_{12} & 2is_{12} & -2s_{12} & -2is_{12} \\ is_{12} & -2s_{12} & -2is_{12} & 2s_{12} \\ -s_{12} & -2is_{12} & 2s_{12} & s_{44} \end{bmatrix}.$$

The fact that the third column of S is i times the second column contradicts the assumption that S is nonsingular and hence proves the assertion.

These examples emphasize the importance of the characterization of the Jordan Canonical Forms of orthogonal matrices which we showed throughout this chapter.

CHAPTER 5. SKEW-SYMMETRIC CASE

5.1 General Properties

First, we want to analyze the properties of a matrix which is similar to its additive inverse. Similarly to the orthogonal case, we can exhibit the Jordan Canonical Form of such a matrix.

Proposition 5.1.1. *A complex square matrix* $A \in \mathbb{C}^{n \times n}$ *is similar to* $-A$ *if and only if its Jordan Canonical Form contains only Jordan blocks with eigenvalue* 0 *and pairs of blocks of the form* $J_k(\lambda) \oplus J_k(-\lambda)$ *for* $\lambda \in \mathbb{C} \setminus \{0\}$.

Proof. The matrix A is similar to its Jordan Canonical Form $J(A) = J_{k_1}(\lambda_1) \oplus \cdots \oplus J_{k_r}(\lambda_r)$, hence $-A$ is similar to $-J(A) = [-J_{k_1}(\lambda_1)] \oplus \cdots \oplus [-J_{k_r}(\lambda_r)]$. But according to Lemma 2.1.6, we know that $-J(A)$ is similar to $J_{k_1}(-\lambda_1) \oplus \cdots \oplus J_{k_r}(-\lambda_r)$. Therefore, the Jordan Canonical Form of $-A$ is of the following form

$$J(-A) = J_{k_1}(-\lambda_1) \oplus \cdots \oplus J_{k_r}(-\lambda_r).$$

Furthermore, we have that A and $-A$ are similar to each other if and only if they have the same Jordan Canonical Form. Therefore, for every $\lambda \in \mathbb{C} \setminus \{0\}$ the Jordan Canonical Form $J(A)$ has to contain both Jordan blocks $J_{k_i}(\lambda_i)$ and $J_{k_i}(-\lambda_i)$. Since $0 = -0$, the Jordan blocks for this eigenvalue do not have to appear in pairs. $\square$

A useful tool which will be needed to prove that a given matrix is similar to a complex skew-symmetric matrix is provided in the following Lemma.

Lemma 5.1.2. ([5], Lemma 7) $A \in \mathbb{C}^{n \times n}$ *is similar to a complex skew-symmetric matrix if and only if there is a nonsingular symmetric* S *such that* $A^T = -SAS^{-1}$.

Proof. First assume that A is similar to a complex skew-symmetric matrix M. Then there exists a nonsingular complex matrix R such that $RAR^{-1} = M$, or equivalently, $A = R^{-1}MR$. Thus,

$$\begin{aligned} A^T &= (R^{-1}MR)^T = R^TM^TR^{-T} = -R^TMR^{-T} = -R^TRAR^{-1}R^{-T} \\ &= -(R^TR)A(R^TR)^{-1} = -SAS^{-1} \end{aligned}$$

Since $S = R^TR$ is symmetric this establishes the necessity.

Now assume there exists a nonsingular symmetric S such that $A^T = -SAS^{-1}$. According to Proposition 2.2.2 there exists a nonsingular complex matrix $X^T \in \mathbb{C}^{n\times n}$ such that $S = X^TX$. So we have

$$A^T = -SAS^{-1} = -(X^TX)A(X^TX)^{-1} = -X^T(XAX^{-1})X^{-T}$$

which is equivalent to

$$X^{-T}A^TX^T = -XAX^{-1} \text{ and therefore } (XAX^{-1})^T = -(XAX^{-1}).$$

Hence XAX^{-1} is a skew-symmetric matrix and A is similar to a skew-symmetric matrix via the nonsingular matrix X.

□

5.2 Proof of the Main Result for Skew-symmetric Matrices

From the definition of a skew-symmetric matrix and Proposition 5.1.1, we know that a Jordan block for an eigenvalue $\lambda \neq 0$ has to appear paired with the Jordan block for $-\lambda$. Now, we show that the direct sum given by this pair is indeed similar to a skew-symmetric matrix for any eigenvalue $\lambda \in \mathbb{C}$.

Lemma 5.2.1. ([5], Lemma 8) *For any positive integer k and any $\lambda \in \mathbb{C}$, $J_k(\lambda) \oplus J_k(-\lambda)$ is similar to a skew-symmetric matrix.*

Proof. By Proposition 2.1.3 there exists a symmetric matrix S which is similar to $J_k(\lambda)$. Using the same matrix H as in the proof of Lemma 4.2.1, we see that

$$(S \oplus -S)^T = H(-S \oplus S)H = -H(S \oplus -S)H^{-1}.$$

Hence $S \oplus -S$ is similar to a complex skew-symmetric matrix by Lemma 5.1.2, and $J_k(\lambda) \oplus J_k(-\lambda)$ is similar to $S \oplus -S$. □

Next, we prove that a single odd-dimensional Jordan block for the eigenvalue 0 is similar to a skew-symmetric matrix.

Lemma 5.2.2. ([5], Lemma 9) *For any odd positive integer k, $J_k(0)$ is similar to a complex skew-symmetric matrix.*

Proof. The proof will be by induction on $n \in \mathbb{N} \cup \{0\}$ where $k = 2n+1$ and will establish the identity $J_k(0)^T = -S_k J_k(0) S_k^{-1}$, for S_k defined below. Then using Lemma 5.1.2, this will prove the assertion.

For $n = 0$, we have $k = 1$. Define $S_1 = [1]$, then $-S_1 J_1(0) S_1^{-1} = -[1][0][1]^{-1} = [0] = J_1(0)^T$. Now, assume for $n \geq 1$ there exits a matrix S_{2n-1} such that $J_{2n-1}(0)^T = -S_{2n-1} J_{2n-1}(0) S_{2n-1}^{-1}$ and define

$$S_{2n+1} = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & -1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & -1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix} \in M_{2n+1},$$

where 1 and -1 alternate on the counter-diagonal. This matrix is nonsingular and symmetric since $2n+1$ is odd. Moreover, we have $S_{2n+1}^2 = I_{2n+1}$ and therefore $S^{-1} = S$. Let $e_j \in \mathbb{C}^{2n-1}$ denote the j-th unit vector, then by the induction hypothesis we have

$$
\begin{aligned}
-S_{2n+1}J_{2n+1}(0)S_{2n+1} &= \\
&= -\begin{bmatrix} 0 & 0 & 1 \\ 0 & -S_{2n-1} & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & e_1^T & 0 \\ 0 & J_{2n-1}(0) & e_{2n-1} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & -S_{2n-1} & 0 \\ 1 & 0 & 0 \end{bmatrix} \\
&= -\begin{bmatrix} 0 & 0 & 0 \\ -Se_{2n-1} & S_{2n-1}J_{2n-1}(0)S_{2n-1} & 0 \\ 0 & -e_1^T S_{2n-1} & 0 \end{bmatrix} \\
&= -\begin{bmatrix} 0 & 0 & 0 \\ -e_1 & -J_{2n-1}(0)^T & 0 \\ 0 & -e_{2n-1}^T & 0 \end{bmatrix} \\
&= J_{2n+1}(0)^T.
\end{aligned}
$$

□

Proposition 5.1.1 does not give any evidence that Jordan blocks for the eigenvalue 0 appear in pairs. Nevertheless, the even-dimensional Jordan blocks for the eigenvalue 0 cannot appear by themselves. Using the matrix exponential we will see that this result can be shown from on the fact that the Jordan Canonical Form of an orthogonal matrix does not contain single, even-dimensional Jordan blocks for the eigenvalue 1.

Lemma 5.2.3. ([5], Lemma 10) *Let $r, k_1, \ldots, k_r$ be positive integers where k_1 is even. If $r > 1$ suppose that $k_1 > k_2 \geq \cdots \geq k_r$. Then $J_{k_1}(0) \oplus \cdots \oplus J_{k_r}(0)$ is not similar to a complex skew-symmetric matrix.*

Proof. Suppose to the contrary that $J = J_{k_1}(0) \oplus \cdots \oplus J_{k_r}(0)$ is similar to a skew-symmetric matrix. Then by Corollary 2.3.4, e^J is similar to a complex orthogonal matrix, whose Jordan Canonical Form is $J_{k_1}(1) \oplus \cdots \oplus J_{k_r}(1)$ according to Proposition 2.3.5. But this contradicts the result of Lemma 4.2.3, thereby proving the claim. □

The following cancellation lemma is similar to Lemma 4.2.5, and will be used for the analogous purpose.

Lemma 5.2.4. ([5], Lemma 11) *Let $C \in \mathbb{C}^{k\times k}$ be similar to a complex skew-symmetric matrix. If $B \oplus C$ is similar to a complex skew-symmetric matrix for some $B \in \mathbb{C}^{n\times n}$, then B is also similar to a complex skew-symmetric matrix.*

Proof. Using the argument and notation of Lemma 4.2.5, we can assume without loss of generality that C is skew-symmetric. Next, we have that $B \oplus C$ is similar to a skew-symmetric matrix, hence by Lemma 5.1.2 there is a nonsingular symmetric matrix S such that

$$\begin{aligned} (B\oplus C)^T &= -S(B\oplus C)S^{-1} \\ (B^T \oplus -C)S &= S(-B\oplus -C). \end{aligned} \tag{5.1}$$

After partitioning S conformally to $B \oplus C$ (compare Lemma 4.2.5), the block multiplication in (5.1) gives us (a) $B^TS_{11} = -S_{11}B$, (b) $B^TS_{12} = -S_{12}C$, (c) $CS_{12}^T = S_{12}^TB$ and (d) $CS_{22} = S_{22}C$. Then,

$$B^TS_{12}S_{22}^{-1}S_{12}^T \overset{(b)}{=} -S_{12}CS_{22}^{-1}S_{12}^T \overset{(d)}{=} -S_{12}S_{22}^{-1}CS_{12}^T \overset{(c)}{=} -S_{12}S_{22}^{-1}S_{12}^TB. \tag{5.2}$$

Now, define $Y = S_{11} - S_{12}S_{22}^{-1}S_{12}^T$, the Schur compliment which is nonsingular by the same argument as in Lemma 4.2.5, and use (a) and (5.2) to show

$$B^TY = B^TS_{11} - B^TS_{12}S_{22}^{-1}S_{12}^T \overset{(a),(5.2)}{=} -S_{11}B + S_{12}S_{22}^{-1}S_{12}^TB = -YB.$$

Since S_{11} and S_{22} are both symmetric, we have $Y^T = Y$ and thus Lemma 5.1.2 proves that B must be similar to a complex skew-symmetric matrix. □

Finally, the next theorem will complete the proof of the main result for skew-symmetric matrices.

Theorem 5.2.5. ([5], Theorem 5) *Let A be a complex skew-symmetric matrix. Then the even-sized singular Jordan blocks of A appear in pairs.*

Proof. This proof is entirely parallel to that of Theorem 4.2.6. If $J(A)$ denotes the Jordan Canonical Form of A we can decompose $J(A)$ into C_1, C_2 and C_3, where C_1 is the direct sum of all paired Jordan blocks, and C_2 and C_3 respectively denote the direct sum of all the remaining odd-sized Jordan blocks and even-sized Jordan blocks for the eigenvalue 0. Lemmas 5.2.1 and 5.2.2 show that $C_1 \oplus C_2$ is similar to a complex skew-symmetric matrix. Since $J(A) = (C_1 \oplus C_2) \oplus C_3$ is similar to a complex skew-symmetric matrix, we know that C_3 must be similar to a complex skew-symmetric matrix by Lemma 5.2.4. If C_3 is nontrivial, this is impossible by Lemma 5.2.3; thus the direct sum forming C_3 must be empty. □

We now summarize the results from this section to prove Theorem 1.2.5:

Theorem 1.2.5. *An $n \times n$ complex matrix is similar to a complex skew-symmetric matrix if and only if its Jordan Canonical Form can be expressed as a direct sum of matrices of only the following two types:*

(a) $J_k(\lambda) \oplus J_k(-\lambda)$ *for* $\lambda \in \mathbb{C}$ *and any* k*,*

(b) $J_k(0)$ *for any odd* k*.*

Proof. First, Lemma 5.2.1 and Lemma 5.2.2 prove that the blocks (a) and (b) described in Theorem 1.2.5 are similar to a complex skew-symmetric matrix. This establishes the sufficiency of the conditions in Theorem 1.2.5. Finally, Theorem 5.2.5 states the necessity of the paired appearance of even-dimensional Jordan blocks for the eigenvalue 0. □

5.3 Examples

Example 5.3.1. $J_2(0)$ *is not similar to a complex skew-symmetric matrix.*

Assume the converse, then by Lemma 5.1.2 there exists a nonsingular symmetric matrix S such that $J_2(0)^T = -SJ_2(0)S^{-1}$. Thus

$$\begin{aligned}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} &= J_2(0)^T = -SJ_2(0)S^{-1} = -\frac{1}{s_{1,1}s_{2,2} - s_{1,2}^2}\begin{bmatrix} s_{1,1} & s_{1,2} \\ s_{1,2} & s_{2,2} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} s_{2,2} & -s_{1,2} \\ -s_{1,2} & s_{1,1} \end{bmatrix} \\ &= \frac{1}{s_{1,1}s_{2,2} - s_{1,2}^2}\begin{bmatrix} s_{1,1}s_{1,2} & -s_{1,1}^2 \\ s_{1,2}^2 & -s_{1,1}s_{1,2} \end{bmatrix}.\end{aligned}$$

This forces $s_{1,1} = 0$ and hence $-SJ_2(0)S^{-1} = -J_2(0)^T \neq J_2(0)^T$, a contradiction. Therefore $J_2(0)$ is not similar to a skew-symmetric matrix.

Example 5.3.2. *The complex matrix*

$$A = \begin{bmatrix} i & -2 & -2i & 2 \\ 0 & -i & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

is similar to its additive inverse, however A is not similar to any skew-symmetric matrix.

The eigenvalues of A are its diagonal entries $i, -i, 0$ and 0 since it is upper-triangular. Direct computation shows that $p_A(x) = m_A(x) = x^2(x-i)(x+i)$ and therefore $J(A) = J_2(0) \oplus J_1(i) \oplus J_1(-i)$. Hence, according to Proposition 5.1.1 we have that A is similar to its additive inverse.

To prove the second part of our claim, we assume to the contrary that A is similar to a skew-symmetric matrix. By Lemma 5.1.2, there exists a nonsingular symmetric matrix S such that $A^T = -SAS^{-1}$ which is equivalent to $A^TS + SA = 0$. Solving this system forces S to have the same form as in Example 4.3.2 which is singular. However, since S must be nonsingular this contradicts the assumption and proves the claim.

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