Journey in the Learning and Teaching of Mathematics by Incorporating Technology

Lana Lyddon Hatten

Iowa State University
Introduction

I focused on two components in my program of study for the Masters of School Mathematics, the mathematical content I learned in and beyond the required courses and my participation in the National Science Foundation funded research project, Discourse analysis: Catalyst for reflective inquiry in the mathematics classroom. This creative component reflects the mathematics I learned with the aid of technology. It also reflects how I have used technology to support my students’ understanding in mathematics. Both components challenged and invigorated me more than other professional development I have experienced. The leadership in both components models high expectations for themselves and me. Together the components provided an integrated and enriched way for me to advance both in the mathematics I know and the way I wish to have my students learn. The mathematics was rigorous, intense, and compacted. My participation
in reflective inquiry through the lens of discourse analysis enabled me to examine what happens in my classroom to make positive changes. The discourse analysis project allowed me to participate in a project which supported and held me accountable to consider the mathematics education aspects of teaching I knew I needed to consider. I appreciate and value how the new mathematics I learned places the mathematics I teach in a larger context.

Researchers Beth Herbel-Eisenmann and Michelle Cirillo lead eight classroom teachers from central Iowa in the discourse project. Its purpose is to study discourse in mathematics classrooms and its use as a focus for engaging in action-research projects. My participation came as I wanted to closely examine my instructional practices. I have completed four of the five phases of the project.

During Phase I, in the spring and summer of 2005, I completed the QUASAR Beliefs Inventory and the Nature of Mathematics Survey for Teachers and was interviewed about my views on teaching and learning math and the norms of my classroom. I participated in an activity in which I reflected on those things which were important to me in mathematics and teaching. I wrote those things which I professed to be closest to my heart in my teaching and in mathematics on post-it notes. I arranged the notes so the most important were toward the center of a paper. This paper of post-it notes became my beliefs map. This phase was important because, as Tim Rowland believes (1995), pedagogy rests on a teacher’s philosophy of mathematics and her or his goals and beliefs about the nature of mathematics. This was the first example of coursework and project integration; it deepened my learning in both. During this phase, I read and discussed the nature and philosophies of Mathematics with other secondary and
During Phase II, in the fall and spring of 2005-2006, researchers developed a case study of classroom discourse. Researchers collected data for one class period of eighth graders taking Integrated Geometry by observing, video-taping and writing field notes for four weeks. I wrote pre- and post-observation plans and interpretations. During the first observation week in September, students studied surface area and volumes of spheres and direct variation. In November, students solved quadratic equations by factoring, extracting square roots and using the quadratic formula. During the January observation week, students talked about and worked problems with probability, odds, combinations, and compound events. In the February observation week, students wrote proofs for angles formed by transversals intersecting parallel lines, and similar and congruent triangles. Researchers looked for discourse patterns and norms embedded in and carried by the patterns.

The case study combines quantitative information and qualitative analysis. The quantitative information included a list for each day of the four week observation of the activity structures (e.g., going over homework, setting up, partner work, board work, whole class, etc.) of the class in the order they occurred. Circle graphs displayed the percentages of time the students and I spent on each structure for each observation week. The Turn Taking Analysis compared my turns speaking to the students’ turns speaking in numbers of words spoken per turn. The last quantitative report analyzed the Teacher Communication Behavior Questionnaire (TCBQ), which my students took both in the fall and in the spring. The TCBQ assessed student perceptions of five teacher communication
behaviors: Challenging, Encouragement and Praise, Non-Verbal Support, Understanding and Friendly, and Controlling.

In order to examine how my students and I use language to talk about content, create interpersonal relationships, and create continuity across time, researchers performed a linguistic analysis of the Activity Structures of Going Over Group work, Going Over Seatwork, Review, Setting Up, and Summary. The results indicated the use of pronouns and vague references, the word *if*, and repetition. The analytic memos described the processes evident whether they be describing/defining or thinking (material, relational, mental).

During Phase III, in the summer, fall, and winter of 2006-2007, I read literature related to classroom discourse and teacher or action-research on classroom discourse. The discourse of the classroom include the oral and written communication, interaction patterns, representations used, reasoning processes, and styles of argumentation. I learned of the use of pronouns (Rowland, 1992), hedges (Rowland, 1992), politeness strategies (Bills, 2000) and ways to ask better questions of my students (Manouchehri & Lapp, 2003) leading them to better understand mathematics. Discussions and readings focused on features such as wait time, revoicing (O’Connor & Michaels, 1996), focusing and funneling (Wood, 1998), the benefits of setting up routines and norms, and explicitly commenting on mathematical conversations (Rittenhouse, 1998). Readings suggested seating arrangements, gaze changes and listening responsibilities to improve discourse. The research/readings presented an array of possibilities rather than a mandate.

Through reading *Classroom Discourse: The language of teaching and learning* by Courtney Cazden, I expanded my understanding of the stakes of the discourse in the
classroom. Analyzing the discourse challenged me not only in the words I used to teach mathematics but also in what the ideas entail. Examining interaction patterns reveals possible inequities. As a teacher, I acknowledge who does and does not participate, why or why not, and the types of responses received. Those students whose informal talk differs most from the discourse of school should not be denied access to educational and therefore economic success. Focusing on discourse involves explicitly helping students have access to and become confident in the writing and speaking that is required for success in school. I expanded my belief, that a mathematical community of learners is a socially positive way to do mathematics, to include the possibility that students will cognitively benefit by making connections in community (Cazden, 1988; NCTM, 1991).

Phase IV began in the spring of 2007 and continued through spring 2008. I generated an action research question. To choose a question, I reviewed video of a class session from November 21, 2005 to compare what actually happens in my class with my perceptions during the class. I read the case study of classroom discourse generated from the observation weeks. I reviewed what was most important to me about teaching mathematics and looked for a performance gap, a disconnect between what I wish to occur and what actually occurs (Hopkins, 2002). I considered the ideas from the discourse readings which were most salient to me and my students to help me bridge this disconnect. The action research question I chose became “How can I enhance the mathematical discourse of my students when technology is present?”

During this phase, I collected and analyzed data, video and audio recordings, my journal reflections, and student writings, and read research literature to improve the quality of talk while using technology to learn mathematics. I wrote a chapter which will
be published in a book with accompanying chapters written by the other seven project members. In it I recount the awareness and growth I gained through my experience in the project.

The project culminates in Phase V during the 2008-2009 academic year when participants help develop aspects of the pilot for professional development for other mathematics teachers. As a participant in the project I presented at the National Council of Teachers of Mathematics national and regional conferences. A highlight was the invitation and participation in an NSF funded conference in May, *Investigating equitable discourse practices in mathematics classrooms*. Thirty scholars and practitioners in discourse and equity attended the conference and addressed the need for equitable participation and culturally-relevant strategies in mathematics classes. I was one of five classroom teachers invited to attend and participate in a plenary panel which focused attention on the realities and complexities of teachers’ contexts. During the conference, participants developed research and action agendas.

I use technology as a tool for both engaging students and managing the classroom. I decided to use technology in the classroom to motivate and demand students think hard. I assumed from previous teaching experience student enjoyment would be high and intimidation low. I sought to examine how the student talk when using technology to learn mathematics could be improved. In response to NAEP achievement from 1996, 2000, 2003, and 2005, The National Center for Educational Studies in *The Nation’s Report Card, Mathematics 2000* reports:

Eighth-graders whose teachers reported that calculators were used almost every day scored highest...teachers who permitted unrestricted use of calculators and those who permitted calculator use on tests had eighth-graders with higher
average scores than did teachers who did not indicate such use of calculators in their classrooms. (Braswell et al, 2001)

The assessment measures mathematical complexity which includes conceptual understanding, procedural knowledge, and problem solving, and mathematical power, reasoning, connections, and communication.

I paid closer attention to how the calculator was used and what the talk was like during its use. I noticed that what I was actually doing and what the students were watching was not very meaningful. I noticed and was disappointed to learn the talk amounted to me telling students which buttons to push. I expected the use of technology to be a vehicle for students to make important and meaningful connections between multiple representations, algebraic, geometric, numeric, and verbal.

Integrating, not simply adding, the technology with the mathematics being studied can help students develop essential understandings about the nature, use, and limits of the tool and promote deeper understanding of the mathematical concepts involved (Burrill et al, 2002).

The calculator was not granting access to mind expanding mathematical concepts that increase students’ mathematical power. When the TI 84+ calculator was present, the talk sounded like me saying,

First push stat, choose the calc menu, select four linear regression type, comma—the comma is above the seven key, to type L one, push second and the one key—type L two, to paste the regression equation in Y two, go to vars, y-vars, number one function number two Y two and hit enter.

To find the point of intersection on the graph you have, do not use trace, that is not accurate enough. Go to the calc menu which is second trace, select number 5 intersect, hit enter, enter, enter.

I do not perceive myself as a teacher who tells students what to do, yet that is what I was doing.
I thought about this *performance gap*, as David Hopkins calls it. Once I realized the students’ experiences around calculators were frequently reduced to pushing the buttons at my direction, I quickly generalized that I frequently tell the kids what to do in other contexts and leave it at that. I appeared to be a procedural teacher but did not feel like one. I realized I did not wish to underestimate or patronize my students for two specific reasons.

The first reason is I do not wish to patronize my students by stating what I think might be obvious to them. For example I might show students how to find the intersection point of two lines on a graphing calculator, but not discuss the point of intersection as the solution to a system of equations. The students at my magnet school are identified as gifted (though not always in mathematics). If anything, I erred on the side of assuming too much understanding and independent sense making. I believed students cannot help but make sense of the mathematics as I believed with my key stroke pedagogy. I assumed students grasped conceptual understanding; I only needed to share the particular syntaxes as to how a calculator could deliver a result. I believed if they could make the calculator perform, then students would internalize the mathematical concept. I believed sense making on a student’s part was a given, so it did not demand explanation, exploration, or communication.

The second reason is I do not wish to patronize my students by giving them more time to process than they need. Because these were gifted kids, I believed they would learn and understand quickly. I thought if we spent time in conversation, it would be unnecessary busy work. For example, a conversation about why every point on the perpendicular bisector of a line segment must be equidistant from the endpoints would
not be necessary. Why should they listen to classmates talk about something I assumed they already understood? I thought they processed fast enough, so if I showed them why a concept made sense or how a procedure worked, they would hear and understand at the same time. I did this out of respect for their intelligence and time. By experimenting with my assumptions about students’ understanding, I did not believe I would do any lasting damage to anyone’s mathematical experience. Possibly and hopefully more students could increase their mathematical power as a result.

The National Research Council (2001), in *Adding It Up*, provides support for me to challenge my assumption. “When teachers learn to see and hear students’ work during a lesson and to use that information to shape their instruction, instruction becomes clearer, more focused and more effective.”

My discourse interest focused on justification, argumentation, reasoning and proof. The NCTM discourse standard views “mathematical reasoning and evidence as the basis for discourse. In order for students to develop the ability to formulate problems, to explore, conjecture, and reason logically, to evaluate whether something makes sense, classroom discourse must be founded on mathematical evidence” (NCTM, 1991, p. 34). I wanted to be more purposeful to reach students who did not engage with mathematics with interest. Through *Agreeing to Disagree: Developing Sociable Mathematical Discourse* (1996), written by Magdalene Lampert, Peggy Rittenhouse, and Carol Crumbaugh, I learned of social costs and discomforts and other nonmathematical reasons students may not like to make conjectures, disagree, or be wrong in front of their peers. Students’ previous experiences in school may only include the textbook and teacher as authority.
The National Research Council (1999) suggests, in *How People Learn: Brain, mind, experience, and school*, “students learn when they are actively involved in choosing and evaluating strategies, considering assumptions, and receiving feedback, by building on or transferring knowledge from previous experiences” (p. 57). This too supports my discourse interest in encouraging students to justify and reason.

*Handheld Technology at the Secondary Level: Research Findings and Implications for the Classroom* is a synthesis of forty-three studies examined by educators led by Gail Burrill. Evidence indicates handheld technology helps students develop a better understanding of mathematical concepts, score higher on performance measures, and achieve a higher level of problem solving skills. In general, teachers using graphing technology created classrooms with more conjecturing, multiple approaches, and higher levels of discourse. How and why teachers use technology in their classrooms affects the gains in learning with graphing calculators implying teachers should increase cognitive demand, choose tasks wisely, and improve questions in their use of technology (Burrill et al, 2002).

After reading *Technology and Mathematics Education*, by James Kaput (1993), the use of technology, much like discourse, suggests larger questions about knowing and learning.

New technologies frequently reenergize age-old questions. These include questions regarding educations goals, appropriate pedagogical strategies, and underlying beliefs about the nature of the subject matter, the nature of learners and learning, and the relation between knowledge and knower. Implementation of new technologies also forces reconsideration of traditional questions about control and the social structure of classrooms and organizational structure of schools. (p. 516)

As a student learning new mathematics, dynamic geometric software, java applets, a graphing calculator and spreadsheets have allowed me, according to the van
Hiele theory of geometric thought, to visualize and analyze first in order to deduce both informally and formally much of the mathematics I studied: calculus, algorithms, discrete mathematics, statistics, and geometry. In geometry I experienced level 4: Rigor when comparing Euclidean geometry with geometries on the hyperbolic plane, sphere, and other surfaces (Van de Walle, 2001). Visualizing a graphical representation of a calculus concept like Newton’s Method helped me attend to the analytic representation. An appreciation for the analytic, in turn allowed me to think algorithmically and then write an algorithmic program.

Mathematics I Learned

Ten years after I finished my undergraduate math degree, I came across Ivars Peterson’s book, *The Mathematical Tourist, snapshots of modern mathematics*. The book contains discussions about several branches of contemporary mathematics. One of the topics discussed was fractals. I was introduced to and fascinated by the compelling aesthetic of fractals enough to lure me to become a teacher. Reflecting as a member of the Discourse Project, I concluded that I desire the heart of my teaching to be consistently introducing students to mind expanding mathematics. The use of algorithms and technology to develop an understanding of fractals is one such example. Previously I knew iterating a simple equation using complex numbers produces unique images, Julia and Mandelbrot sets being among them, but I did not know how. I explored a path beginning with a linear recurrence relation to gain an understanding.

*Linear Recurrence Relation*

I began with the Linear recurrence relation $P_{t+1} = a \cdot P_t + b$ and explored iterating by hand with various values for $a$, $b$, and $P_0$. I experimented numerically and graphically
unsystematically using a graphing calculator and then web applets. I noticed upon iteration, the sequence \( \{P_t\} \) either is constant, tends to infinity, approaches a fixed point, or oscillates between two values. I then derived the explicit equation in order to justify the various behaviors in general.

For each of the six cases for \( a \) I tried, I considered three sub cases for \( P_0 \). The cases for \( a \) are \( a > 1, a = 1, 0 < a < 1, -1 < a < 0, a = -1, \) and \( a < -1 \). I considered the following three sub cases for \( P_0 \): \( P_0 = \frac{b}{1-a}, P_0 > \frac{b}{1-a}, \) and \( P_0 < \frac{b}{1-a} \).

The fixed point is the value \( P_t \) such that \( P_{t+1} = P_t \). To find the fixed point, solve \( P_t = a P_t + b \) for \( P_t \) and obtain \( P_t = \frac{b}{1-a} \).

I first demonstrate what I determined numerically and graphically. I justify algebraically in general for simpler cases. I then use the explicit equation and derivative of the linear recurrence relation to justify what I observed graphically.

**The sequence \( \{P_t\} \) is constant.**

i) \( \{P_t\} \) is constant for \( P_0 = \frac{b}{1-a} \) for all \( b \) and \( a \neq 1 \).

\[
\begin{align*}
P_t &= aP_0 + b \\
P_t &= a \frac{b}{1-a} + b \\
P_t &= \frac{ab + b - ab}{1-a} \\
P_t &= \frac{b}{1-a} \\
P_t &= \frac{b}{1-a} \\
P_t &= \frac{b}{1-a} \\
P_t &= \frac{b}{1-a}
\end{align*}
\]
ii) \{P_t\} is constant for \(a=1\) and \(b=0\).
\[ P_{t+1} = 1 \cdot (P_t) + 0 \]
\[ P_{t+1} = P_t. \]

iii) \{P_t\} is constant for \(a=-1\) and \(P_0 = \frac{b}{2}\).
\[ P_1 = a \cdot P_0 + b \]
\[ P_1 = -1 \cdot \frac{b}{2} + b \]
\[ P_1 = \frac{b}{2} \]
\[ P_2 = -1 \cdot \frac{b}{2} + b \]
\[ P_2 = \frac{b}{2} \]
\[ P_{t+1} = \frac{b}{2} \]

The sequence \{P_t\} attracts to the fixed point.

i) If \(0 < a < 1\), \{P_t\} approaches the fixed point. If \(P_0 > \frac{b}{1-a}\), \{P_t\} approaches the fixed point, \(\frac{b}{1-a}\), from above. In this example, \(a=0.5\), \(b=3\), \(P_0=10\), \(\frac{b}{1-a}=6\).

ii) Similarly if \(0 < a < 1\) and \(P_0 < \frac{b}{1-a}\), \{P_t\} approaches the fixed point, \(\frac{b}{1-a}\), from below. In this example \(a=0.5\), \(b=3\), \(P_0=2\), \(\frac{b}{1-a}=6\).
iii) When $-1 < a < 0$, $\{P_t\}$ approaches the fixed point. $\{P_t\}$ approaches the fixed point by oscillating above and below it. In this example, $a=-0.5$, $b=3$, $P_0=10$, $\frac{b}{1-a}=2$.

The sequence $\{P_t\}$ oscillates between two values.

i) If $a=-1$ and $P_0=\frac{b}{1-a}$, $\{P_t\}$ oscillates between $P_0$ and $-P_0+b$. 
The sequence \( \{P_t\} \) tends to infinity.

i) When \( a=1 \), \( \frac{b}{1-a} \) is undefined and \( \{P_t\} \) increases or decreases linearly by \( b \) each iteration for \( b>0 \) or \( b<0 \) respectively.

\[
\begin{align*}
P_{t+1} &= aP_t + b \\
P_1 &= aP_0 + b \\
P_2 &= 1P_0 + b \\
P_3 &= 1(P_0 + b) + b \\
P_4 &= 1(P_0 + 2b) + b \\
P_n &= P_0 + nb
\end{align*}
\]

ii) If \( a>1 \) and \( P_0 \neq \frac{b}{1-a} \), \( \{P_t\} \) tends to infinity. The relation increases for \( P_0 > \frac{b}{1-a} \).

In this example, \( a=2, b=3, P_0=2, \frac{b}{1-a}=-3 \).

<table>
<thead>
<tr>
<th>Plot1</th>
<th>Plot2</th>
<th>Plot3</th>
<th>( n \mid u(n) )</th>
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<td>( \text{Min}=1 ) ( w(n)\text{=} )</td>
<td>( \text{Min}=1 ) ( w(n)\text{=} )</td>
<td>( \text{Min}=1 ) ( w(n)\text{=} )</td>
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</tbody>
</table>
iii) Similarly if $a > 1$ and $P_0 \neq \frac{b}{1-a}$, the relation decreases for $P_0 < \frac{b}{1-a}$. In this example, $a=2$, $b=3$, $P_0=-6$, $P_0 < \frac{b}{1-a} = -3$.

iv) If $a < -1$, the absolute value of $\{P_i\}$ tends to infinity. Successive terms have opposite signs.
Resources exist online with which users move sliders to change the values for $a$, $b$, and $P_0$. The applet displays the web and time series of the iterations simultaneously.

This example, $a=-1.4$, $b=1.8$, $P_0=0$, $\frac{b}{1-a} = 0.75$ and demonstrates oscillating behavior.

This image displays a case when $-1 < a < 0$ and $\{P_t\}$ attracts by oscillating toward the fixed point. In this example, $a=-0.8$, $b=5$, $P_0=11$, $\frac{b}{1-a} = \frac{25}{9} \approx 2.778$. 
I observed $P_{t+1} - P_t$ for different values for $|a| > 1$, $b$, and $P_0$ and concluded exponential increase or decrease since the difference between successive terms increased or decreased by a factor of $a$. In the example below, list $L_1=t$, list $L_2=P_{t+1}=2P_t+3$, and list $L_3=L_{t+1} - L_t$ with $a=2$, $b=3$, and $P_0=2$. Since the difference in list $L_3$ doubles each time, I determined the explicit form of $\{P_{t+1}\}$ is an exponential function with a base of two.

I iterated $P_{t+1} = aP_t + b$ in general to obtain the explicit equation.

\[
P_1 = aP_0 + b
\]

\[
P_2 = a(aP_0 + b) + b
\]

\[
P_3 = a^2P_0 + ab + b
\]
By induction,

\[ P_t = a^t P_0 + a^{t-1}b + a^{t-2}b + \ldots + ab + b \]

\[ P_t = a^t P_0 + b(a^{t-1} + a^{t-2} + \ldots + a + 1) \]

\[ P_t = a^t P_0 + b \sum_{i=0}^{t-1} a^i, \text{ if } a=1, P_t = P_0 + bt \]

\[ P_t = a^t P_0 + b \left( \frac{1 - a^t}{1 - a} \right) \]

\[ P_t = a^t P_0 + \frac{b}{1 - a} - a^t \frac{b}{1 - a} \]

\[ P_t = a^t (P_0 - \frac{b}{1 - a}) + \frac{b}{1 - a} \]

By using the equation above, it is clear to see for \( a < 0 \), behavior oscillates. For \( |a| > 1 \), \( \{P_t\} \) exponentially repels from the fixed point. For \( a = 1 \) the fixed point is undefined and the increase or decrease linear. For \( |a| < 1 \), \( \{P_t\} \) exponentially attracts to the fixed point.

The derivative of a linear function is a constant equal to the slope of the line. \( a \) is the slope of \( f(x) = ax + b \) which represents \( P_{t+1} = a P_t + b \). I concluded the \( \{P_t\} \) attracts to the fixed point when the absolute value of derivative at the point, \( P_0 \), is less than one, i.e. \( |f'(P_0)| < 1 \). The sequence repels from the fixed point when the absolute value of the derivative is greater than one. In the case of the linear recurrence relation, the sequence attracts to the fixed point for \( |a| < 1 \). The sequence repels from the fixed point for \( |a| > 1 \).
Quadratic Recurrence Relation

I explored the quadratic recurrence relation, \( P_{t+1} = P_t^2 + c \), using real numbers for \( c \) and \( P_0 \). For example, I let \( c = -2 \) and \( P_0 = 0 \) and calculated \( P_1 = -2 \) and \( P_2 = 2 \) thereafter. To find the fixed points in the quadratic recurrence relation, solve \( P_t = P_t^2 + c \) for \( P_t \).

\[
P_{t+1} = P_t^2 + c
\]
\[
P_t = P_t^2 + c
\]
\[
P_t^2 - P_t + c = 0
\]
\[
P_t = \frac{1}{2} (1 \pm \sqrt{1 - 4c})
\]

If \( c = 0 \), the fixed points are 1 and 0. For real number fixed points, \( c \leq \frac{1}{4} \).

I used a calculator to iterate numerically from the home screen (e.g., \( c = -2 \), \( P_0 = 0.5 \)).

\[
\begin{array}{c|c}
\text{Ans} & -2 \\
\hline
& -1.788921055 \\
& 1.20023654 \\
& 1.75 \\
& -0.5594274476 \\
& 1.0625 \\
& -1.687040931 \\
& -0.87109375 \\
& -1.241195679 \\
& 0.4594332871 \\
& -1.284102771 \\
& -0.3510800734 \\
\end{array}
\]

-2 < \( P_t \) < 2 for the next fifty iterations, but I did not recognize a pattern in the behavior within the interval.

I used the fill down command of a spreadsheet (e.g., \( c = -2 \), \( P_0 = 2.1 \)). \( \{ P_t \} \) increases quickly.

**Iterate** \( x^2 + c \)

<table>
<thead>
<tr>
<th>( t )</th>
<th>( P_t )</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>2.1</td>
</tr>
<tr>
<td>1</td>
<td>2.41</td>
</tr>
<tr>
<td>2</td>
<td>3.8081</td>
</tr>
<tr>
<td>3</td>
<td>12.5016256</td>
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<tr>
<td>4</td>
<td>154.290643</td>
</tr>
<tr>
<td>5</td>
<td>23803.6025</td>
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</table>
An existing Excel quadratic iterator spreadsheet numerically iterates $P_{t+1} = P_t^2 + c$ and displays the behavior of the iterated function as a time series graph. (e.g., $c = -1$, $P_0 = 0.5$) \{P_t\} oscillates between -1 and 0.

<table>
<thead>
<tr>
<th>Iterate</th>
<th>$X^2+c$</th>
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<tbody>
<tr>
<td>0</td>
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</tr>
<tr>
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<tr>
<td>2</td>
<td>-0.4375</td>
</tr>
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<tr>
<td>4</td>
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<tr>
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<td>6</td>
<td>-0.2253147</td>
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<td>7</td>
<td>-0.9492333</td>
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<td>-1</td>
</tr>
<tr>
<td>26</td>
<td>0</td>
</tr>
<tr>
<td>27</td>
<td>-1</td>
</tr>
<tr>
<td>28</td>
<td>0</td>
</tr>
</tbody>
</table>
Another page in the same spreadsheet displays the time series for two seed values, $P_0$, close to each other. The numerical lists and graph display the chaotic, vastly different behavior, for initial conditions that differ by only 0.001 produce.

<table>
<thead>
<tr>
<th>iterate $x^2+c$</th>
<th>$c= -2$</th>
<th>Iterate $x^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>seed 1 seed 2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0.001</td>
</tr>
<tr>
<td>1</td>
<td>-2</td>
<td>-2</td>
</tr>
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<td>2</td>
<td>2</td>
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</tr>
<tr>
<td>3</td>
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<td>2</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>1.9999</td>
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<tr>
<td>5</td>
<td>2</td>
<td>1.9997</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>1.999</td>
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<td>2</td>
<td>1.9959</td>
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<tr>
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<td>2</td>
<td>1.9836</td>
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<tr>
<td>9</td>
<td>2</td>
<td>1.9348</td>
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<tr>
<td>10</td>
<td>2</td>
<td>1.7435</td>
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<tr>
<td>11</td>
<td>2</td>
<td>1.0399</td>
</tr>
<tr>
<td>12</td>
<td>2</td>
<td>-0.919</td>
</tr>
<tr>
<td>13</td>
<td>2</td>
<td>-1.156</td>
</tr>
<tr>
<td>14</td>
<td>2</td>
<td>-0.663</td>
</tr>
</tbody>
</table>

On the third page of this spreadsheet, users move a slider to change the value for $c$ to notice the behavior of the time series graph of $P_{t+1} = P_t^2 + c$ dynamically. For example, there is a six point orbit when $c= -1.48$. 

<p>| | | | | |</p>
<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
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</tr>
<tr>
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<td></td>
<td></td>
<td>-0.528256</td>
<td>-1.200945</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td>-0.03773</td>
<td>-1.156</td>
</tr>
</tbody>
</table>
There are two fixed points when \( c < \frac{1}{4} \). I used an online web graph to determine that when there are two fixed points for the quadratic recurrence relation, the sequence \( \{P_t\} \) attracts to one fixed point and repels from the other. I observed if

\[
- \frac{1}{2} (1 + \sqrt{1 - 4c}) < P_0 < \frac{1}{2} (1 + \sqrt{1 - 4c}), \ \{P_t\} \ \text{attracted to or orbited around the smaller fixed point, } \frac{1}{2} (1 - \sqrt{1 - 4c}). \end{eqnarray}

If \( P_0 > \frac{1}{2} (1 + \sqrt{1 - 4c}) \) or \( P_0 < -\frac{1}{2} (1 + \sqrt{1 - 4c}) \), \( \{P_t\} \) repels from the larger fixed point. In this image \( P_{t+1} = P_t^2 - 1, \ P_0 = -0.93 \). When \( c = -1 \), the fixed points are \( \frac{1}{2} (1 \pm \sqrt{5}) \) with \( -\frac{1}{2} (1 + \sqrt{5}) < -0.93 < \frac{1}{2} (1 - \sqrt{5}) \).

http://math.bu.edu/DYSYS/applets/nonlinear-web.html
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There is one fixed point when \( c = \frac{1}{4} \). \( \{ P_t \} \) attracts to the fixed point, \( \frac{1}{2} \), when

\[-\frac{1}{2} < P_0 < \frac{1}{2} \]

and repels when the seed value, \( P_0 \), is not in that interval. In order to attract, the absolute value of the derivative at the seed value must be less than one. In order to repel, the absolute value of the derivative evaluated at the seed value must be greater than one. In the function \( f(x) = x^2 + c \) which represents \( P_{t+1} = P_t^2 + c \), the derivative is \( 2x \). \( x \) represents the seed value, \( P_0 \). \( |2x| < 1 \) corresponds to \( \frac{1}{2} < P_0 < \frac{1}{2} \), the interval where \( P_0 \) attracts to the fixed point.

To determine for which \( c \) values \( \{ P_t \} \) attracts to its fixed points, we evaluate where the absolute value of the derivative of \( P_{t+1} = P_t^2 + c \), at the fixed points,

\[ \frac{1}{2} (1 \pm \sqrt{1 - 4c}) \], is less than one.

\[ P_{t+1} = P_t^2 + c, \]

\[ f(P_t) = P_t^2 + c \]
\[ f'(P_t) = 2P_t \]
\[ f'(P_t) < 1 \text{ when } |2P_t| < 1 \]

\[ \left| \frac{1}{2} (1 \pm \sqrt{1 - 4c}) \right| < 1 \]

\[ -1 < 1 \pm \sqrt{1 - 4c} < 1 \]
\[ -1 < 1 + \sqrt{1 - 4c} < 1 \quad -1 < 1 - \sqrt{1 - 4c} < 1 \]
\[ -2 < \sqrt{1 - 4c} < 0 \quad -2 < -\sqrt{1 - 4c} < 0 \]
\[ \text{Impossible} \quad 4 > 1 - 4c > 0 \]
\[ 3 > -4c > -1 \]
\[ -3 < c < \frac{1}{4} \]
\[ \frac{1}{4} < c < \frac{1}{4} \]
\[ -\frac{3}{4} < c < \frac{1}{4} \] is the interval for which the absolute value of the derivative at the smaller fixed point, \( \frac{1}{2}(1 - \sqrt{1-4c}) \), is less than one. \( \{P_t\} \) attracts to \( \frac{1}{2}(1 - \sqrt{1-4c}) \) if

\[-\frac{1}{2}(1 + \sqrt{1-4c}) < P_0 < \frac{1}{2}(1 + \sqrt{1-4c}) .\]

Since \( \sqrt{1-4c} < 0 \) has no real solutions, \( \{P_t\} \) does not attract to the larger fixed point, \( \frac{1}{2}(1 + \sqrt{1-4c}) \), unless \( P_0 = \frac{1}{2}(1 + \sqrt{1-4c}) \).

If \( c < -\frac{3}{4} \), the absolute value of the derivative at both fixed points is greater than one. If \( -\frac{1}{2}(1 + \sqrt{1-4c}) < P_0 < \frac{1}{2}(1 + \sqrt{1-4c}) \) in this case, \( \{P_t\} \) orbits or cycles around the smaller fixed point, \( \frac{1}{2}(1 - \sqrt{1-4c}) \). The previous six point orbit, if \( c = -1.48 \) and \( P_0 = 0 \), is one example.

If \( c > \frac{1}{4} \), there are no real number fixed points and \( \{P_t\} \) increases for all \( P_0 \).

I used complex numbers, \( z = a + bi \), as seed values, \( P_0 \), and constants, \( c \), in the quadratic recurrence equation \( P_{t+1} = P_t^2 + c \). I calculated the value of each iteration, plotted each \( (a, b) \) on the complex plane, connected the points of successive iterations, and observed the path of the iterations. I observed behaviors similar to my previous investigations. For some iterations, the modulus, \( \sqrt{a^2 + b^2} \), of successive terms gets larger so each point is farther from the origin than the preceding one. Other iterations stayed within a circle centered at the origin with radius two. This related to a previous quadratic case where behavior was unpredictable but remained within an interval. Corresponding to
the attraction to fixed points, other iterations seemed to stay stuck on a few points. I observed how quickly the numbers left the circle of radius less than or equal to 2.

I show three examples demonstrating different behaviors of the sequence \( \{P_i\} \) using an online applet which plots and connects successive iterations.

For \( P_0 = i, c=0 \), \( P_1 = 1 \) after 2 iterations of \( P_{i+1} = P_i^2 + c \).

For \( P_0 = 0.43 + 0.43i \), and \( c = -0.8 \), the orbit escapes and the iterate value increases.
For $P_0 = 0.6 + 0.4i$, and $c = -0.5 - 0.3i$, the iterations stay within a circular region, meaning the modulus never is larger than two, the radius of the circle.

Julia Sets

A Julia set is defined to be the boundary between the seed values whose orbits escape and seed values whose orbits do not escape for $P_{t+1} = P_t^2 + c$. Consider the Julia set for $P_{t+1} = P_t^2$. If $|P_0| < P_t$ then the orbit, the seed value and the iterates will escape, that is, increase without bound, because a larger number is squared each iteration. If $|P_0| = P_t$ then the orbit attracts to a fixed point, in this case, one. If $|P_0| > P_t$, then the orbits do not escape. This means the Julia set for $c = 0$ is a circle of radius one. Each value of $c$ generates its own Julia set.

When I first explored this using complex numbers, I plotted each value of the iteration. In a Julia set, each complex number, $z = a + bi$ in a particular window (e.g., $x: (-2, 2)$ and $y: (-2, 2)$) of the complex plane, is iterated for the same $c$ value. On the complex plane, the complex number, $z_0$, is colored black if the orbit of the iterations stays within a boundary. The other colors represent how fast iterations escape the boundary. Pixels
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colored the same escape at the same rate. Pixels colored differently escape at different rates.

To help me notice how fast a seed value escapes, using a TI-89 calculator, I wrote a program which accepts as inputs a seed value and a constant. It outputs the iterated value, the number of times it has iterated, and the modulus. The program iterates until the modulus is greater than or equal to two. I observed it this way since only the seed value is plotted in a Julia set for a particular constant; neither the path of the iteration nor the particular iterated values appear. To be in the Julia set, the seed value must escape within fifty iterations.

Julia program

```
() Prgm
DeMon a,b,t
Disp "ai=[a(i-1)]^2+c"
Disp "enter a(i-1)"
Input a
Disp "enter constant"
Input b
1→t
While abs(a)≤2
a^2+b→a
a→li
Disp a,abs(a),t
EndWhile
Pause
EndPrgm
```

Another online resource exists in which users input the constant value and the output is a graph of the associated Julia set.

[http://www.shodor.org/interactivate/activities/JuliaSets/](http://www.shodor.org/interactivate/activities/JuliaSets/) Graphing Julia sets demonstrates the power of algorithm. If I evaluated each pixel on a graphing calculator, I would iterate 92·64=5888 points for $P_0$ up to fifty times to generate a single Julia set.
Lana Lyddon Hatten

Journey in the Learning and Teaching image. Since each complex number for \( c \) generates a new image, I used existing software to display the images for different Julia sets. Two examples of Julia sets are displayed below.

\[
c = i \quad \quad \quad \quad c = -0.391 - 0.587i
\]

(Wolfram Mathworld)

The Dynamical Systems and Technology Project at Boston University produced an interactive online site which includes explorations designed to teach the mathematics of the Julia and Mandelbrot sets. In one exploration, the viewer observes where the iterated value plots next on the complex plane, on the Julia set graph of \( P_{t+1} = P_t^2 - 1 \) for eight points in the Julia set. The fixed points are real numbers, \( F = \frac{1 + \sqrt{5}}{2} + 0i \) and \( G = \frac{1 - \sqrt{5}}{2} + 0i \).
A Mandelbrot set is a table of contents, dictionary, or atlas for the collection of Julia sets. A Julia set is a visual display for a single constant over many complex number seed values. The Mandelbrot set is a visual description of the behavior of the quadratic recurrence relation for every constant value; the seed value is always zero. Points which do not escape are in the Mandelbrot set. As in Julia sets, points are colored according to how quickly the iterations escape. Because of the magnitude of complex numbers to iterate up to fifty times each, an existing graphical representation helped my understanding of the relationship between the Julia set and the Mandelbrot set. In a Julia set, the seed values are the plotted points for a particular $c$ value. Each $c$ value has its own Julia set. The Mandelbrot set always seeds with 0. The constant is plotted on the plane.
In the Mandelbrot Set, the image on the left, the white point, $c=-0.123+0.745i$ produces the Julia Set on the right. Since that constant lies in the set, the black region, the Julia set is connected in one piece. If the constant is not colored black, meaning it is outside of the Mandelbrot set, then the Julia set is not connected but has an infinite number of pieces.

**Algorithm for images by Scott Burns**

Because of my coursework in the MSM program, I gained a more mathematical understanding in several of the branches of contemporary mathematics. Mathematically understanding how to generate images of Julia and Mandelbrot sets from simple recurrence relations is one example. While exploring and programming algorithms in Math 546, *Algorithms in Analysis and Their Computer Implementation* and learning about fractals, I obtained a digital image by artist, Scott Burns, an engineering professor who designs his artwork by algorithm. Short computer programs, no longer than one-quarter page in length, generate the fractal art. Patrons purchase both the image and the algorithm. Coincidentally, Scott Burns generated one of the color plates that was so compelling to me in Peterson’s *The Mathematical Tourist*.

I analyzed three visually very different images by Scott Burns in which the
These algorithms do not always iterate with the same constant each time, as in the Julia set. The algorithm for image 51 uses \( c = -4 \) for the first iteration. After the first iteration, \( c = 1 \) is used for the subsequent iterations. The Julia set iterates until the number escapes a distance of two from the origin or after fifty iterations which ever occurs first. The algorithm for Scott Burn’s image 51 iterates a maximum of sixteen times. The first test is whether the absolute value of the real part of the iterated value \((a + bi)\) added
to one-half was less than one tenth. This means the iteration is within 0.1 of the vertical line \( x = -\frac{1}{2} \). It lies between the vertical lines, \( x = -0.6 \) and \( x = -0.4 \). If it is not within sixteen iterations, then the next test before iterating another point representing a complex number is to test if the absolute value of the imaginary part \((b \text{ in } a + bi)\) added to one-half is less than one tenth. This means the iteration is within 0.1 of the horizontal axis. It lies between the horizontal lines, \( y = -0.1 \) and \( y = 0.1 \). The point is colored according to how many iterations it takes for one of those two conditions to occur.

The window for image 51 is \( x: (0, 14) \) \( y: (0, 10) \). The step is the step is 0.004762. The for loops in the algorithm set up the order in which that the complex numbers used as seed values. It begins in the lower left corner with \( (0, 0) \). The next points have an \( x \) coordinates of 0 and the \( y \) values increase until \( (0, 10) \) is iterated and colored. The next column of complex numbers iterates vertically and so on until the point \( (14, 10) \) representing \( z_{0} = 14 + 10i \) is colored. 2940 real coordinates and 2100 imaginary coordinates fit in this window which means 6,174,000 complex numbers are used as seed values. Each seed value iterate up to 16 times. Image 51 is in Appendix J.

The difference in the algorithms between image 51 and 79 is slight. The only difference is the points that are iterated in the viewing window. Image 79 zooms to the lower left corner of image 51. The viewing window is \( x: (-0.049, 1.345) \) \( y: (-0.154, 1.798) \). For this image, the step is 0.000428. Image 79 is in Appendix J.

Image 53 has viewing dimensions \( x: (-2.8, 2.8) \), \( y: (-2, 2) \) and a scale of 0.0019. The algorithm uses \( c = -4 \) for the first three iterations before iterating for \( c = 1 \) henceforth. Additionally, the tests to end the iterations are when first the absolute value of the real part is less than one or then if the absolute value of the imaginary part is less than one but
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not both. If after the fourth iteration, the $x$ coordinate lies between the vertical lines

$x = -1$ and $x = 1$ or the $y$ coordinate lies between the horizontal lines $y = -1$ and $y = 1$

but not both, the iteration ends. Image 53 is in Appendix J.

Learning the mathematics to understand the algorithm of a Scott Burns digital print led to more fascinating related mathematics. This excursion has given me a foundation, and I look forward to following these mathematical paths too.

Using Technology in Teaching High School Mathematics

I have three examples from my classroom of students using technology to understand a mathematical concept.

1st classroom story

On October 5, 2007, the class was in a familiar large group structure. I asked the class how long it would take the investment to reach $1800 if the initial investment was $400 and the money doubles every nine and nine-tenths years. The students concluded that there was no integer number of doublings, but the number of times the money doubled was between two and three. They approximated a numerical solution only. They also determined that the number of times the money doubled would be the solution to the equation $1800 = 400 \cdot 2^x$ where $x$ is the number of times the money doubled. Students solved algebraically one step further, $\frac{9}{2} = 2^x$. They did not know logarithms, however, so they did not have the algebra skills to solve an equation with the variable in the exponent. I told the students that they would need to use their calculators. Before they picked up
their calculators, they made a quick sketch by hand of $y= \frac{9}{2}$ and $y=2^x$. I asked, “If this is a picture on our graphing calculator, how is that picture going to help us solve $\frac{9}{2} = 2^x$?”

Twenty seconds later, one student suggested the point of intersection but was not able to say if the point or one or both of the coordinates were the solution to the equation. Next another student suggests the $x$ coordinate of the intersection point is the solution.

When I ask the rest of the class if the $x$ value gives the solution to the equation, another ten seconds go by before someone responds. Two students propose reasoning that is not correct. For the first time that day, I allowed students to graph the equation on their own calculators instead of viewing my projected image.

I did not employ the push-button pedagogy I used in the past. The students were not yet familiar with the graphing capabilities of the calculator. Rather than walking students through the process and telling them what the entries in the standard viewing menu meant, I asked them what they thought. For example, one student correctly guessed the $x scl$ and $y scl$ was “how much each notch is”. Unfortunately, I felt in the moment that I had to tell the students,
If that’s the equation we want to solve, \( \frac{9}{2} = 2^x \), then if we graph all the points where this true, \( y = \frac{9}{2} \), and all the points where this is true, \( y = 2^x \). If we want the right sides to be equal, then this point of intersection is the place where these two [equations] are true at the same time. They both equal nine halves when \( x \) is 2.170.

Later I asked, “Now let’s go back and answer this question. How long will it take to have $1800? How many doubling periods? \( x \) represents the number of times to double.

Now looking at this solution, how many doubling periods will it take? A full twelve seconds later I say,

So you can graph two curves and find the intersection point, but you have to do it for a reason. To come back here and answer these: How many doubling periods, and how many years? Talk at your tables. Show me what you get.

Three minutes later several had answered the questions in their small groups. I tell the rest of the class the correct answers I had heard. Wow, they either had a bad day or my assumption about them automatically making connections desperately needed to be challenged.

In previous years, I just told the students how to calculate the point of intersection, used the \( x \)-value to get the number of doublings, multiplied that number by nine and nine-tenths to get the number of years and was finished with the problem. This year even with some careful changes, the students had not been able to articulate connections between multiple representations in this large group use of technology where I displayed the information to the students. After viewing a recording of this class I wrote in my notes.

How to [structure] a discussion/activity for understanding a graphical solution to an algebraic problem? Next time set up an activity so that each kid and small group must consider how the graphical representation solves the algebraic
I decided to shift from a large group structure to smaller groups so I could hear the thought process of more students. This way I can answer specific calculator questions of individuals and spare the entire room from unnecessary review.

2nd classroom story

During the next opportunity to use the calculator to make algebraic and graphical connections, I decided to be more purposeful in ensuring students articulate the connection between multiple representations. I made sure the activity included important mathematics that they would need to grasp and articulate. As a result of the reflection of the earlier lesson, I asked students to work individually or in small groups rather than have a discussion as a large group. Students used a graphing calculator to graph six quadratic functions (e.g., \( y = 5x^2 + 2x - 3 \)) and observe the number of x-intercepts. They algebraically solved the corresponding quadratic equations (e.g., \( 0 = 5x^2 + 2x - 3 \)) to determine the number of solutions. Students calculated the value of the discriminant, \( b^2 - 4ac \), the radicand in the quadratic formula, \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \), and noted whether this discriminant was positive, negative, or zero. I insisted every student write a one sentence conjecture about the relationship between the discriminant and the real-number solutions of a quadratic equation. The students worked for ten minutes, and I said very little. I listened for understanding at each table. I helped two students with specific but minor calculator questions and attended to two students about future absences. The rest of my energy and attention went toward observation. The next time I spoke I said, “Keep going;
you’re working great. [The conjecture] is really the main point of it. You get to write a sentence, and I want to read the sentence.”

After I read one student’s conjecture I said to him, “I am going to have you write that [conjecture] more mathematically. Twos, what are twos talking about? And what are positives talking about?” My conversation with another pair went like this.

T: C, what did you write?

C: If the discriminant is two there is,

T: What do you mean, if the discriminant is two?

M: If the answer is zero

T: But what do you mean if the answer is zero?

M: If b squared minus four a c is zero.

T: And what is that called?

C: The discriminant.

T: OK, if the discriminant is zero,

M: then there is one x-intercept.

C: I got it.

(Ninety seconds later)
C: If the amount of solutions is two, there is a positive discriminant.

Once each student wrote conjecture, the whole class recollected.

T: For the last couple minutes, let’s say some things out loud that I know was going on at the tables. We noticed that when y equals zero, there’s one solution to this equation \( 0 = -4x^2 + 4x - 1 \) and when we graph the parabola \( y = -4x^2 + 4x - 1 \), there was one x-intercept. How come those match up?

C: There is only one spot [on the graph] when y is zero.
T: When you saw one x-intercept [on the graph] or you saw there was one solution [to the equation], what was the discriminant like? The choices were positive, zero, or negative.

J: The discriminant equaled zero.

T: Why does the discriminant of zero force there to be one solution? We can use this [algebra written on whiteboard by two students]

\[ 0 = -4x^2 + 4x - 1 \]

\[
x = \frac{-4 \pm \sqrt{16 - 4 \cdot -4 \cdot -1}}{2 \cdot -4}
\]

\[
x = \frac{-4 \pm \sqrt{0}}{\text{-8}}
\]

\[
x = \frac{1}{2} + 0 \quad x = \frac{1}{2} - 0
\]

T: Why does a discriminant of zero mean only one solution? Algebraically, what is the significance of zero under the square root?

C: It takes away the plus or minus.

T: There is nothing to add or subtract from the line of symmetry to get to the x-intercepts. The vertex is the x-intercept.

T: So now we have one like this, which is really the exciting part of the day! So, how many x-intercepts? (A graph of \( y = 2x^2 + x + 5 \) is displayed on the overhead calculator.)

S: None. Zero.

T: Alright so then how many solutions would you expect to find if we let y equal zero?

Students observed \( y = 2x^2 + x + 5 \) has no x-intercepts because there are no places on the parabola where \( y=0 \).

T: So tell me about the discriminant in that case.

Students had calculated the discriminate to be negative.
T: Why does it make sense that there is no solution \([0 = 2x^2 + x + 5\), algebra displayed on white board)]?  

\[
0 = 2x^2 + x + 5
\]
\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
\[
x = \frac{-1 \pm \sqrt{1^2 - 4(2)(5)}}{2(2)}
\]
\[
x = \frac{-1 \pm \sqrt{-39}}{4}
\]

S: Oh because there is no negative square root.

T: In the real world there is no square root of a negative number. Why not? Why can’t you take the square root of a negative number?

S: It wasn’t difficult for them to articulate why a number squared is always positive. Because we had pressed for an articulated understanding, the transition to complex numbers was more dramatic and fun. “You are right; we cannot take the square root of a negative number, not in the real world.

T: So, close your eyes; let’s imagine that we can take the square root of a negative number. Now we can. In the real world, this is not possible so it doesn’t show up as an \(x\)-intercept. There are two solutions to this quadratic equation \([0 = 2x^2 + x + 5]\)... We are going to imagine that we can take the square root of a negative number. So we come back tomorrow and we can.

I believe this activity allowed students to experience the wonder of mathematics much beyond me saying something like, \(i = \sqrt{-1}\) and here is how the arithmetic goes with imaginary and complex numbers. I asked students what the most amazing thing they learned in the first semester was, and many students responded similarly to this:

“Learning about and working with imaginary numbers. I love the algebra involved, and thought it was a cool way to solve an impossible problem.”
In April, the same students were asked to find four centers of triangles, the circumcenter, incenter, centroid, and orthocenter, by constructing the points of concurrency of perpendicular bisectors, angle bisectors, medians, and altitudes, respectively. Students used squares of patty paper that can be folded, creased, written and traced on. First, students folded the perpendicular bisectors to locate the circumcenter. They used a compass to construct circles with the center at the circumcenter. This circumcircle intersects the triangle at the vertices. Next, several students justified for their classmates why the circumcenter must be equidistant from the vertices. The exchange was terrific but not surprising. The students were successful in the construction and had become accustomed to justifying why things were true. Most were willing and many even were eager to do so.

Next students folded the angle bisectors of their triangles to find that point of concurrency, the incenter. They did a nice job of reasoning why, unlike the circumcenter, the incenter will always be on the interior of the triangle. I told them perpendiculars dropped from the incenter to each side of the triangle are congruent. Then I told them that they could make a circle that intersected each side exactly once. This was difficult because these incircle constructions were not as precise as the circumcircles had been, and students were not very successful. To their credit the students were not convinced by their constructions and were not willing to take my word about inscribing circles in triangles. I was pleased with their ability to articulate their reasoning and justification. I asked them to believe something that their own constructions did not allow them to inductively believe, and I did not ask them to deductively make sense of it either. The next day I told them,
Here’s the part I want to spend some time with today because you did such a great job of explaining things with the perpendicular bisectors. Then I made you kind of eyeball something, and it wasn’t that convincing to you. So let’s work this out. Angle bisectors were the second thing we did. We bisected the angles, and I think everyone got a point of concurrency which is called what?” Several students answer, “The incenter”.

I reviewed how they had tried to inscribe the triangle with the incircle. “So you didn’t quite buy it which I appreciate and don’t blame you. Here’s one sort of, not deductive proof, but it is a little more accurate than [me saying], “Oh try to figure out the shortest distance to a side and try that for [the radius of] a circle and see if it works.” I constructed the incircle using Geometer’s Sketchpad and projected it. “It does [work],” I say as I change the triangle dynamically to demonstrate the circle remains inscribed.

“Hopefully this is a little, somewhat more convincing than otherwise. You can make a circle that is not big enough, and you can make a circle that is too big. I think that was easy to do yesterday. Now what I want you to do is see if you can figure out why [a circle inscribed in a triangle] works in the same way you did with the circumcenter before. Why are those three lengths going to be the same?” [AC, AB, AD]

To small groups of four students per table, I offer, “It’s ok to know these are right angles ( ∠ABE, ∠ACF, ∠ADG ). How come the red lengths have to be equal?” (AC, AB, AD)

One student tried to justify to his group AC=AB=AD because “it’s the radius of the circle and the incenter is the center of the circle and the radius is always the same”.
So I said, “What I am wanting you to justify is why would the circle be possible?” That table continued to justify “because the distances are all the same like the radius of the circle is always the same”.

The students at another table stood up, leaned forward and pointed toward the projected figure. One student stood practically upside down looking at the figure. Two and one half minutes later, the group talked like this while one student was out of his seat pointing directly on the figure.

W: I got it!!! And you know these two angles are the same because it’s an angle bisector \( \angle DFA, \angle CFA \), and I think these two angles \( \angle DAF, \angle CAF \) are the same because it is a bisector as well.

J: And they have the same side \( \overline{AF} \).

W: They have the same side.

J: That makes sense.

W: So ASA

B: Oh, yeah.

M: But they are different shapes.

W: No, they are not. That’s a whole kite right there. (ACFD)

M, J, Q: Oh. Oh! I was looking at that wrong. OK. That makes sense.

Across the room another student justified this way,

B: You can’t know the other side. Right now you have the middle to itself \( \overline{AF} \) and you have 90 degree angles \( \angle DAF, \angle CAF \). You don’t know one of the other sides is the same as the other.

T: So you got an angle and a side, you need either another side or angle. Keep looking.

B found an additional pair of corresponding angles congruent, the pair formed by the angle bisector of \( \angle DFG \). She concluded AC=AD because \( \triangle ADF \cong \triangle ACF \) by the
angle, angle, side triangle congruence theorem. The students’ engaged in lively conversations with students across the room rather than within the tables so the whole class listened while W and B repeated their justifications. W already did not believe he could justify $\angle DAF \cong \angle CAF$. When asked why $\angle DAF \cong \angle CAF$, W said, “That was my question. I don’t know that’s right now.” He asked B to clarify, “How do we know that those are 90 degree angles?” A couple of students convinced him, and he responded, “Well then we will just go with that.”

T: I think Will and Melanie are convinced. Anybody else?

J: That’s congruent?

T: Yeah. Can you describe it for the rest of us?

J: But they already did.

T: I know but now somebody is willing to listen again. Can you do it with a different pair of triangles?

J: You could do it with B, C triangles. The ones in the upper right corner. Up at the top where the angle is divided, that’s a bisector. And the lines, where A to B, that angle had to be 90 degrees. That would be perpendicular. And the same with A to C. And they have the same side in common so it would be just angle, angle, side.

W: I agree with that.

Q: Exactly.

Seven minutes from when we started, students tried to justify, modified their thinking, and revoiced why a circle can always be inscribe in a triangle. “I really appreciate you challenging me on that yesterday and not being so convinced with the kinda/sorta eyeball it way. That was fantastic.” (transcript 04/09/08)
I stumbled upon something fascinating that I believe will help me learn more about my students and how they perceive the mathematics in an activity. While I confirmed each small group connected $x$-intercepts, solutions, and the determinant in the quadratic activity, I unknowingly recorded how the individuals in one of the groups understood. I will observe more closely what happens within a small group next to better understand students. My interest is in the roles within the group and who assumes those roles. Who says what kinds of things? In this tape, all spoke. The low performing student talked only about nonmathematical things. Why? The struggling but motivated student asked for help and clarification. Did he have enough confidence and mathematical power to participate even though it meant acknowledging he understood less? One student talked really fast as if she were just proclaiming her knowledge rather than sharing it to help others. The fourth students talked in support and confirmation at a more learnable tempo.

I made strides in my focus class and added a technology rich activity per unit that has a specific mathematical goal. I listen carefully to student responses to be convinced by them, overcoming my fear of being patronizing. I am not content with the level of questioning yet, but made strides by using the small group activity structure. My own awareness of calculator button pushing has changed. I hear conceptual understanding of the mathematics that is aided by the use of the technology.

I recognize a performance gap. I know what I believe, and I observe what actually happens. I can bridge the performance gap. I have tools, vocabulary, support from research, and desire to bridge the gap between my ideal and the actual. I learned more
about me than I anticipated and realized during the process. I continue the same journey and start another one. I perceived more accurately what occurs in my classroom; I must continue to learn about my students. I view my place in the school and my role as an educator through a larger lens. I understand that one does not digest an entire library about discourse to learn isolated techniques to enable better classroom talk. Better talk means better math understanding for more people and fairer learning. I must continue to develop what is at the heart of my teaching. The gap between the perceptions of my students and me is my new performance gap to bridge.

Research suggests both discourse and technology can and should provide opportunities for students to explore, make conjectures, and deepen their understanding of mathematical concepts. Both have done so for me as a student. I received feedback from my own students to support this. They, through the Conceptions of Mathematics Inventory, agreed or strongly agreed being able to explain why something is correct is as important as getting a correct answer. Students also agreed or strongly agreed when their methods of solving a problem are different from their teacher’s method, their method can be as correct as the teacher’s. Many students found proofs the most interesting part of the course. It was the first place students were absolutely unable to take a teacher’s or textbook’s word for what they were learning. Proofs became the place where students realized that math was not going to be the same. Mathematics, not just proofs, involves thinking, understanding why, communicating, and justifying. Many wrote similarly to this. “I used to just do math without thinking why or how. Proofs made me think deeper of why and how to solve problems, instead of just doing.” “I think it’s cool that instead of just accepting things in the mathematic world. You can actually prove them.” “Proofs,
because they teach you to explain EVERYTHING and they teach you that everything has
to be explained to base it off other stuff. It also teaches you to use other things to figure
what something else is.” (students’ reflections, 5/20/08 and 5/27/08)

My students know that they must understand mathematics. I believe the emphasis
on technology and discourse allows them to take that on in a unintimidating and
satisfying way.

I decided to become a mathematics teacher in part because of a mathematical
image by Scott Burns. I wanted my students appreciate the beauty of mathematics, so I
attended to my teaching practice through the discourse project. I acquired an image by
Scott Burns while I was acquiring an understanding of the underlying mathematics for
both the fractal image and the algorithm. The symmetry of that mathematical narrative is
satisfying. I simultaneously got to do the hard, intense but gratifying work of learning
more mathematics and do the challenging but necessary work of examining my practice.
This creative component is a signpost in my mathematical journey. I believe I will better
help students along their mathematical journeys too.


Burrill et al. (2002). Handheld Technology at the Secondary Level: Research Findings and Implications for the Classroom


