

An equivalence of categories

by

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Introduction

In this paper, we first explore an interesting construction called the split extension. The split extension is a way of constructing another ring from an arbitrary ring R and an R -module M . This new split extension ring contains an ideal which corresponds naturally to the module M : but, we will be primarily interested in the split extension because it gives us a nice way of obtaining a new ring from an R -module. It is this correspondence that will be exploited in the material to follow.

Some category theory basics will be necessary knowledge from the perspective of this paper, although no truly advanced results will be invoked. The reader should, of course, be familiar with the definition of a category and the notation $Ob(\underline{C})$, $Mor(\underline{C})$ for the set of objects, and set of morphisms of the category \underline{C} . Beyond this, the important concepts will be that of: terminal object, slice category (comma category), functor, pullback, power, natural transformation/isomorphism, and the definition of an abelian group in a category. These concepts are discussed briefly in the next section (except abelian group in a category, which is discussed in the last paragraph of this page). Other definitions may be invoked once or twice, but will be explained where necessary. One can also consult [2], [3] for any of these definitions and for more elaborate discussions of them. Also, see [4] or [5] for any algebraic definitions or theorems which are invoked.

Now for a word on notation. Composition of morphisms will be written using postfix notation, so that composites will be read in a "left-to-right" fashion. I.e., if we have $f : A \rightarrow B$, $g : B \rightarrow C$, then the composite will be written $fg : A \rightarrow C$ and read as "f followed by g". This notation is clearly less cumbersome than the $g \circ f$ notation. Also,

we shall employ a particularly convenient notation to express the mapping of a function. We will write $(f : A \rightarrow B ; a \mapsto b)$ to denote that “ f is a function from A to B , and f is defined by $f(a) = b$ ”.

One of the primary concepts appearing in this paper is the notion of an abelian group in a category; or more generally, a group in a category. The diagram version of the definition of a group in a category as given in [1] or [5] is merely a specific instance of a more general concept. Namely, the concept of an “interpretation of an equational algebraic theory” in a category. In particular, the notion of a group in a category arises via an interpretation of the equational theory of groups. A detailed discussion of interpretations of algebraic theories is given in [7]. In particular, the bibliography of this paper contains some references which might be of interest.

Also, it should be noted that all modules will be considered as left R -modules unless otherwise stated.

Review of Categorical Concepts

1. A **terminal object** “ t ” in a category $\underline{\underline{\mathbb{C}}}$ is an object of $\underline{\underline{\mathbb{C}}}$ with the property that $\forall a \in Ob(\underline{\underline{\mathbb{C}}})$ there is a unique morphism $(f : a \rightarrow t)$. In other words, $\underline{\underline{\mathbb{C}}}(a, t)$ is a one-point set.
2. A morphism $(f : c \rightarrow d) \in Mor(\underline{\underline{\mathbb{C}}})$ is an **isomorphism** in the categorical sense if there is a morphism $(g : d \rightarrow c)$ such that $fg = id_c$ and $gf = id_d$.
3. Given a category $\underline{\underline{\mathbb{C}}}$ and $c \in Ob(\underline{\underline{\mathbb{C}}})$, we can form a new category $\underline{\underline{\mathbb{C}/c}}$ called the “**slice category of objects over c** ” or “**comma category of objects over c** ” where

$$(a) \quad Ob(\underline{\underline{\mathbb{C}/c}}) = \left\{ (f : a \rightarrow c) \mid a \in Ob(\underline{\underline{\mathbb{C}}}) \right\}.$$

I.e. the objects of $\underline{\underline{\mathbb{C}/c}}$ are the morphisms of $\underline{\underline{\mathbb{C}}}$ with codomain c .

- (b) A morphism of $\underline{\underline{\mathbb{C}/c}}$ between $(f : a \rightarrow c)$ and $(g : b \rightarrow c)$ is a morphism $(\phi : a \rightarrow b)$ of $\underline{\underline{\mathbb{C}}}$ such that

$$\begin{array}{ccc} a & \xrightarrow{\phi} & b \\ & \searrow f & \swarrow g \\ & & c \end{array}$$

commutes.

So $Mor(\underline{\underline{\mathbb{C}/c}})$ consists of all the morphisms of $\underline{\underline{\mathbb{C}}}$ that satisfy the above conditions.

For example, if $\underline{\text{Ring}}$ denotes the category of rings and ring homomorphisms, and R is a ring, then the slice category over R will be denoted by $\underline{\text{Ring}}/R$.

4. A **functor** F between two small categories $\underline{\text{C}}$ and $\underline{\text{D}}$ is a pair of functions $\langle F_o, F_m \rangle$ where $(F_o : \text{Ob}(\underline{\text{C}}) \rightarrow \text{Ob}(\underline{\text{D}}))$, and $(F_m : \text{Mor}(\underline{\text{C}}) \rightarrow \text{Mor}(\underline{\text{D}}))$ are the “object function” and “morphism function” respectively (we normally write these functions without their subscripts and the context should make it clear whether we are talking about the “object” or “morphism” function) and they satisfy:

$$(a) \quad \forall (f : c \rightarrow d) \in \text{Mor}(\underline{\text{C}}) \text{ we have } (F(f) : F(c) \rightarrow F(d)).$$

$$(b) \quad \forall c \in \text{Ob}(\underline{\text{C}}), F(id_c) = id_{F(c)}.$$

$$(c) \quad \text{For any two composable morphisms } f, g \in \text{Ob}(\underline{\text{C}}), \text{ we have } F(fg) = F(f)F(g).$$

Loosely speaking, a functor is a “morphism of categories”. I.e., a correspondence between them which preserves the categorical structure (the identity morphisms, and composites).

5. A **natural transformation** τ between two functors $(F : \underline{\text{C}} \rightarrow \underline{\text{D}})$ and $(G : \underline{\text{C}} \rightarrow \underline{\text{D}})$ denoted by $(\tau : F \rightarrow G)$ is a function $(\tau : \text{Ob}(\underline{\text{C}}) \rightarrow \text{Mor}(\underline{\text{D}}))$ (where $\tau(c)$ is usually denoted by τ_c) that satisfies:

$$(a) \quad \forall c \in \text{Ob}(\underline{\text{C}}), \tau_c \text{ is a morphism from } F(c) \text{ to } G(c).$$

$$(b) \quad \forall (f : c \rightarrow d) \in \text{Mor}(\underline{\text{C}})$$

$$\begin{array}{ccc} F(c) & \xrightarrow{\tau_c} & G(c) \\ F(f) \downarrow & & \downarrow G(f) \\ F(d) & \xrightarrow{\tau_d} & G(d) \end{array}$$

commutes.

A natural transformation τ can also be thought of as a collection $\{\tau_c\}_{c \in \text{Ob}(\underline{\mathbb{C}})}$ of morphisms of $\underline{\mathbb{D}}$ indexed by $\text{Ob}(\underline{\mathbb{C}})$, that satisfies the above conditions. Loosely speaking, a natural transformation is a “morphism of functors”. The morphisms τ_c are called the “components” of the natural transformation, and τ is a **natural isomorphism** if every component τ_c is an isomorphism in the categorical sense.

6. A **product** of $c, d \in \text{Ob}(\underline{\mathbb{C}})$ is an object $c \times d$ of $\underline{\mathbb{C}}$ with two morphisms

$$c \xleftarrow{\pi_1} c \times d \xrightarrow{\pi_2} d$$

(called the “projections of the product”) with the “universal” property that for any $a \in \text{Ob}(\underline{\mathbb{C}})$ and morphisms $(f : a \rightarrow c)$ and $(g : a \rightarrow d)$ there is a unique morphism $(f \times g : a \rightarrow c \times d)$ such that

$$\begin{array}{ccccc} c & \xleftarrow{\pi_1} & c \times d & \xrightarrow{\pi_2} & d \\ & \swarrow f & \uparrow f \times g & \searrow g & \\ & & a & & \end{array}$$

commutes. This notion can be easily extended to finitely many factors, or even arbitrarily many factors. A **power** is a product in which all of the terms are the same.

7. A **pullback** of $(f : a \rightarrow c), (g : b \rightarrow c) \in \text{Mor}(\underline{\mathbb{C}})$ is an object $b \times_c a$ with morphisms

$$\begin{array}{ccc} b \times_c a & \xrightarrow{p} & a \\ \downarrow q & & \downarrow f \\ b & \xrightarrow{g} & c \end{array}$$

such that the diagram commutes and such that the following universal property holds:

For every object d and morphisms

$$\begin{array}{ccc} d & \xrightarrow{h} & a \\ k \downarrow & & \downarrow f \\ b & \xrightarrow{g} & c \end{array}$$

such that the diagram commutes, there exists a unique morphism $(\eta : d \rightarrow b \times_c a)$

such that the triangles in

$$\begin{array}{ccccc} & & & & \\ & & & & \\ & & & & \\ d & \xrightarrow{\eta} & b \times_c a & \xrightarrow{p} & a \\ & \searrow k & \downarrow q & & \downarrow f \\ & & b & \xrightarrow{g} & c \end{array}$$

commute.

The concepts of product and pullback are particular instances of a more general concept. Namely, that of a limit of a functor. The “universal” properties of products and pullbacks are expressions of the essentially “canonical” nature of limits of functors. Limits of functors are discussed in great detail (and great clarity) in [2](Chapter 6).

The Split Extension

In this section we are going to define a construction that will be needed later. We will also explore an important property of a morphism associated with this construction.

Definition 1. *Given a commutative ring R with identity $1_R \neq 0_R$ and a left R -module M , we can form another ring called the “split extension of M ” (denoted by $M \rtimes R$) as follows:*

The underlying set is $M \times R$ (cartesian product of sets), with operations defined as:

1. $(m, r) + (n, s) \equiv (m + n, r + s)$
2. $(m, r)(n, s) \equiv (s \cdot m + r \cdot n, rs)$, where “ \cdot ” denotes the action of R on M .

It can be verified that with operations defined as above, $M \rtimes R$ is a commutative ring with identity element $(0_M, 1_R)$. In general it also interesting to note that the split extension $M \rtimes R$ contains an ideal $M \times \{0_R\}$ that corresponds naturally to the module M ; although we will not be exploiting this fact in this paper.

It is easily seen that $(id_R : R \rightarrow R)$ is a terminal object in the slice category $\underline{\underline{\text{Ring}}}/R$. Also, the category $\underline{\underline{\text{Ring}}}$ has pullbacks, and pullbacks in $\underline{\underline{\text{Ring}}}$ can be used to construct products in $\underline{\underline{\text{Ring}}}/R$. The preceding facts are all well-known and can be generalized to other categories. For a discussion of this, see [2] (pg 140 Example 21.2(3), pg 142 Corollary 21.6, and pg 147 Exercise 21.C).

With these facts in mind, let $E \equiv M \rtimes R$ and consider the morphism

$$\left(\pi : E \rightarrow R; (m, r) \mapsto r \right) \in \text{Ob}(\underline{\underline{\text{Ring}/R}})$$

Let us determine the power π^2 in the slice category. First of all, we have the following pullback square in Ring:

$$\begin{array}{ccc} E \times_R E & \xrightarrow{\pi_2} & E \\ \pi_1 \downarrow & \searrow \varpi & \downarrow \pi \\ E & \xrightarrow{\pi} & R \end{array}$$

$$E \times_R E \equiv \left\{ \left((m_1, r_1), (m_2, r_2) \right) \mid r_1 = r_2 \right\} = \left\{ \left((m_1, r), (m_2, r) \right) \mid m_1, m_2 \in M, r \in R \right\}$$

$$\pi_1 : E \times_R E \rightarrow E; \left((m_1, r), (m_2, r) \right) \mapsto (m_1, r)$$

$$\pi_2 : E \times_R E \rightarrow E; \left((m_1, r), (m_2, r) \right) \mapsto (m_2, r)$$

where $E \times_R E$ is a ring with the restricted operations of the direct product E^2 . In other words, $E \times_R E$ is a subring of the direct product.

Thus, the power π^2 in Ring/R has object $(\varpi : E \times_R E \rightarrow R)$, and projections π_1, π_2 . The following diagram illustrates this power:

$$\begin{array}{ccccc} E & \xleftarrow{\pi_1} & E \times_R E & \xrightarrow{\pi_2} & E \\ \pi \downarrow & & \downarrow \varpi & & \downarrow \pi \\ R & \xlongequal{\quad} & R & \xlongequal{\quad} & R \end{array}$$

Note that π_1, π_2 are in fact morphisms in the slice category since the left/right squares

in the above diagram commute by definition of ϖ .

Now that we know what the power π^2 looks like, we can prove the following proposition.

Proposition 1. *The morphism $(\pi : E \rightarrow R ; (m, r) \mapsto r)$ is an abelian group in Ring/R.*

Proof. Consider the following morphisms in Ring/R:

multiplication: μ

$$\left((m_1, r), (m_2, r) \right) \mapsto (m_1 + m_2, r)$$

$$\begin{array}{ccc} E \times_R E & \xrightarrow{\mu} & E \\ \downarrow \varpi & & \downarrow \pi \\ R & & R \end{array}$$

inverse: ι

$$(m, r) \mapsto (-m, r)$$

$$\begin{array}{ccc} E & \xrightarrow{\iota} & E \\ \downarrow \pi & & \downarrow \pi \\ R & & R \end{array}$$

identity: ϕ

$$r \mapsto (0_M, r)$$

$$\begin{array}{ccc} R & \xrightarrow{\phi} & E \\ \parallel \text{id}_R & & \downarrow \pi \\ R & & R \end{array}$$

permute factors: β

$$\left((m_1, r), (m_2, r) \right) \mapsto \left((m_2, r), (m_1, r) \right)$$

$$\begin{array}{ccc} E \times_R E & \xrightarrow{\beta} & E \times_R E \\ \downarrow \varpi & & \downarrow \varpi \\ R & & R \end{array}$$

As a quick aside observe that π is the unique morphism from π to id_R in Ring/R given that

$$\begin{array}{ccc} E & \xrightarrow{\pi} & R \\ \pi \downarrow & & \parallel \text{id}_R \\ R & \xrightarrow{\text{id}_R} & R \end{array}$$

commutes. (Recall that id_R is the terminal object in Ring/R)

Note also that $\mu, \iota, \phi, \beta \in \text{Mor}(\underline{\underline{\text{Ring}}})$ by construction, and that β is characterized as the following product:

$$\begin{array}{ccccc}
 E & \xleftarrow{\pi_1} & E \times_R E & \xrightarrow{\pi_2} & E \\
 & & \uparrow \beta & & \\
 & & E \times_R E & & \\
 & \swarrow \pi_2 & & \searrow \pi_1 & \\
 & & & &
 \end{array}$$

With the above observations firmly in our minds, we can proceed with the proof at hand. Firstly, we need to explicitly record a few product objects in $\underline{\underline{\text{Ring}/R}}$ for future reference. In particular, we have the following product objects in $\underline{\underline{\text{Ring}/R}}$ (I.E. pullbacks in $\underline{\underline{\text{Ring}}}$)

Note: these are not really the product objects, but the product object domains. The actual product object is, in each case, the “projection onto R” morphism.

$$(\pi^3) \quad E \times_R E \times_R E = \left\{ \left((m_1, r), (m_2, r), (m_3, r) \right) \mid m_i \in M, r \in R \right\}$$

$$(id_R \times \pi) \quad R \times_R E = \left\{ \left(r, (m, r) \right) \mid r \in R, m \in M \right\}$$

$$(\pi \times id_R) \quad E \times_R R = \left\{ \left((m, r), r \right) \mid m \in M, r \in R \right\}$$

where each of the above objects is a ring with operations given by the restrictions of the operations of the respective direct product. Note that the construction of π^3 requires the use of a multiple pullback.

With product objects as above, we can easily verify that π is an abelian group by chasing elements around the following commutative diagrams:

(a.) associativity

$$\begin{array}{ccc}
 E \times_R E \times_R E & \xrightarrow{(id_E, \mu)} & E \times_R E \\
 (\mu, id_E) \downarrow & & \downarrow \mu \\
 E \times_R E & \xrightarrow{\mu} & E
 \end{array}$$

(b.) identity

$$\begin{array}{ccccc}
 R \times_R E & \xrightarrow{(\phi, id_E)} & E \times_R E & \xleftarrow{(id_E, \phi)} & E \times_R R \\
 \uparrow iso & & \downarrow \mu & & \downarrow iso \\
 E & \xlongequal{id_E} & E & \xlongequal{id_E} & E
 \end{array}$$

(“iso” denotes that two objects are isomorphic in the categorical sense)

(c.) inverse

$$\begin{array}{ccc}
 E & \xrightarrow{(id_E, \iota)} & E \times_R E \\
 \downarrow \pi & & \downarrow \mu \\
 R & \xrightarrow{\phi} & E
 \end{array}$$

(d.) commutativity

$$\begin{array}{ccc}
 E \times_R E & \xrightarrow{\mu} & E \\
 \downarrow \beta & & \parallel id_E \\
 E \times_R E & \xrightarrow{\mu} & E
 \end{array}$$

This completes the proof. □

The Equivalence of Two Categories of Interest

Using the split-extension construction and Proposition 1, we can proceed to prove an interesting equivalence of categories. Before we begin, we need a nice name for the category of abelian groups in $\underline{\text{Ring}}/\underline{\text{R}}$. Let $\underline{\underline{\text{C}}}$ denote this category. For the definition of the morphisms of this category $\underline{\underline{\text{C}}}$, see Definition 4 in Appendix A. The result we wish to prove is that $\underline{\underline{\text{C}}}$ and $\underline{\underline{\text{R-Mod}}}$ are equivalent.

Recall that two categories $\underline{\underline{\text{A}}}$, $\underline{\underline{\text{B}}}$ are equivalent in the categorical sense if there are functors

$$F : \underline{\underline{\text{A}}} \longrightarrow \underline{\underline{\text{B}}} \quad , \quad G : \underline{\underline{\text{B}}} \longrightarrow \underline{\underline{\text{A}}}$$

such that

$$FG \cong 1_{\underline{\underline{\text{A}}}} \quad \text{and} \quad GF \cong 1_{\underline{\underline{\text{B}}}}$$

where \cong denotes a natural isomorphism and $1_{\underline{\underline{\text{A}}}}$ denotes the identity functor.

So, let M, N be R -modules, and $f : M \rightarrow N \in \text{Mor}(\underline{\underline{\text{R-Mod}}})$.

Definition 2. Consider $F : \underline{\underline{\text{R-Mod}}} \rightarrow \underline{\underline{\text{C}}}$ defined by:

1. $M \mapsto M \rtimes R$ (on objects)
2. $(f : M \rightarrow N) \mapsto (\bar{f} : M \rtimes R \rightarrow N \rtimes R; (m, r) \mapsto (f(m), r))$ (on morphisms)

In order for the above definition to make sense, we really should verify the following proposition.

Proposition 2. The function \bar{f} is a ring homomorphism, and a morphism of abelian groups.

Proof. 1. $\bar{f}((m, r) + (n, s))$

$$\begin{aligned}
&= \bar{f}(m + n, r + s) \\
&\equiv (f(m + n), r + s) \\
&= (fm + fn, r + s) \\
&= (fm, r) + (fn, s) \\
&\equiv \bar{f}(m, r) + \bar{f}(n, s)
\end{aligned}$$

2. $\bar{f}((m, r)(n, s))$

$$\begin{aligned}
&= \bar{f}(s \cdot m + r \cdot n, rs) \\
&\equiv (f(s \cdot m + r \cdot n), rs) \\
&= (s \cdot fm + r \cdot fn, rs) \\
&= (fm, r)(fn, s) \\
&\equiv \bar{f}(m, r)\bar{f}(n, s)
\end{aligned}$$

We have shown that $\bar{f} \in \text{Mor}(\underline{\text{Ring}})$.

3. It remains to be shown that \bar{f} is a morphism of abelian groups (I.E., $\bar{f} \in \text{Mor}(\underline{\text{C}})$).

To this end, we need to show that $\mu_{M \times R} \bar{f} = \bar{f} \bar{f} \mu_{N \times R}$ where $\bar{f} \bar{f}$ is the product map

$$\begin{array}{ccccc}
N \times R & \longleftarrow & (N \times R)^2 & \longrightarrow & N \times R \\
& \swarrow & \uparrow \bar{f} \bar{f} & \searrow & \\
& & (M \times R)^2 & & \\
& \swarrow \pi_1^{(M \times R)^2} \bar{f} & & \searrow \pi_2^{(M \times R)^2} \bar{f} & \\
& & & &
\end{array}$$

Recall that in the section on the split extension, explicit descriptions of the product objects $(_\times R)^2$ and the “multiplication” morphisms $\mu_{_\times R}$ were given. With these descriptions in mind we have:

(a)

$$(M \rtimes R)^2 \xrightarrow{\mu_{M \rtimes R}} M \rtimes R \xrightarrow{\bar{f}} N \rtimes R$$

$$((m_1, r), (m_2, r)) \longmapsto (m_1 + m_2, r) \longmapsto (f(m_1 + m_2), r)$$

(b)

$$(M \rtimes R)^2 \xrightarrow{\bar{f}\bar{f}} (N \rtimes R)^2 \xrightarrow{\mu_{N \rtimes R}} N \rtimes R$$

$$((m_1, r), (m_2, r)) \longmapsto ((f(m_1), r), (f(m_2), r)) \longmapsto (f(m_1) + f(m_2), r)$$

(It is easily seen that the product map $\bar{f}\bar{f}$ must act as asserted in (b.))

But f is a module homomorphism, therefore $(f(m_1 + m_2), r) = (f(m_1) + f(m_2), r)$.

Clearly then, $\mu_{M \rtimes R} \bar{f} = \bar{f}\bar{f} \mu_{N \rtimes R}$ and we are done.

□

Notice that in Definition 2 the split extensions $M \rtimes R$ and $N \rtimes R$ are being considered as objects in the slice category $\underline{\text{Ring}}/\mathbb{R}$, but we have dropped explicit mention of the morphisms associated with these abelian groups. In fact the morphisms we are concerned with (but are not mentioning) are the “projection morphisms onto R ”. I.e., the respective version of the morphism π as described in Prop.1. It is these morphisms that are actually the abelian groups in $\underline{\text{Ring}}/\mathbb{R}$, but we are using $M \rtimes R$ and $N \rtimes R$ to represent them.

We can now proceed with the following proposition.

Proposition 3. F is a functor.

Proof. 1. Observe that under F :

$$(id_M : M \rightarrow M) \longmapsto (\overline{id_M} : M \rtimes R \rightarrow M \rtimes R; (m, r) \mapsto (m, r))$$

$$\therefore F(id_M) = \overline{id_M} = id_{M \rtimes R} = id_{F(M)}$$

2. If we have $f : M \rightarrow N$ and $g : N \rightarrow P \in \text{Mor}(\underline{\text{R-Mod}})$ then under F :

$$\begin{aligned}
(fg : M \rightarrow P) &\mapsto (\overline{fg} : M \rtimes R \rightarrow P \rtimes R; (m, r) \mapsto (g(fm), r)) \\
(f : M \rightarrow N) &\mapsto (\overline{f} : M \rtimes R \rightarrow N \rtimes R; (m, r) \mapsto (fm, r)) \\
(g : N \rightarrow P) &\mapsto (\overline{g} : N \rtimes R \rightarrow P \rtimes R; (n, r) \mapsto (gn, r))
\end{aligned}$$

$$\text{So } \overline{f\overline{g}} : M \rtimes R \rightarrow P \rtimes R; (m, r) \mapsto (fm, r) \mapsto (g(fm), r)$$

$$\therefore F(fg) = \overline{fg} = \overline{f\overline{g}} = FfFg$$

It follows that F is a functor. □

We now have one of the functors we need to prove the equivalence. The other functor we require must map from the category of abelian groups in Ring/R to the category of modules over R . In particular, on the object level, it must map an abelian group to an R -module. So, before trying to define a functor we need to investigate how to obtain an R -module from an arbitrary abelian group.

Let $(\pi : E \rightarrow R)$ be an abelian group object in Ring/R. Then in particular, $\pi \in \text{Mor}(\underline{\text{Ring}})$. Therefore $\pi^{-1}\{0_R\} = \text{Ker } \pi \equiv M_\pi$ is an ideal of E . In particular M_π is a ring, so it is an additive abelian group in the usual group-theoretic sense.

Now, let ϕ

$$\begin{array}{ccc}
R & \xrightarrow{\phi} & E \\
\text{\scriptsize } id_R \parallel & & \downarrow \pi \\
R & \xlongequal{\quad} & R
\end{array}$$

denote the “identity” of the abelian group $(\pi : E \rightarrow R)$. We are going to use ϕ to define an action of R on M_π that will give us the desired R -module.

So, define an action $R \times M_\pi \rightarrow M_\pi$ by:

$$r \cdot m \equiv \phi(r)m \quad (*)$$

where $\phi(r)m$ represents multiplication in E . Note that we know $\phi(r)m \in M_\pi$ since M_π is an ideal of E .

Proposition 4. $(*)$ defines a unital action of R on M_π .

Proof. 1. $r \cdot (s \cdot m) \equiv r \cdot \phi(s)m = [\phi(r)\phi(s)]m = \phi(rs)m = rs \cdot m$.

2. $1_R \cdot m \equiv \phi(1_R)m = 1_S m = m$.

3. $r \cdot (m + n) \equiv \phi(r)(m + n) = \phi(r)m + \phi(r)n = r \cdot m + r \cdot n$

4. $(r + s) \cdot m \equiv \phi(r + s)m = [\phi(r) + \phi(s)]m = \phi(r)m + \phi(s)m = r \cdot m + s \cdot m$.

Thus, $(*)$ defines a ring action.

□

With the above action, we see that $M_\pi \in \text{Ob}(\underline{\mathbf{R-Mod}})$. With this M_π construction in mind, we can proceed to define the required functor.

Definition 3. Consider $G : \underline{\mathbf{C}} \rightarrow \underline{\mathbf{R-Mod}}$ defined by:

1. $(\pi : E \rightarrow R) \mapsto \pi^{-1}\{O_R\} = M_\pi$ (on objects)

2. $(f : (\pi : E \rightarrow R) \rightarrow (\psi : F \rightarrow R)) \mapsto (\bar{f} : M_\pi \rightarrow M_\psi; m \mapsto f(m))$ (on morphisms)

As with the previous definition, we need to verify some technical details to ensure that our definition makes sense.

Proposition 5. Let \bar{f} be defined as above.

1. \bar{f} takes values in M_ψ .

2. \bar{f} is a morphism of R -modules. I.e., it is R -linear.

Proof. 1. $m \in M_\pi = \text{Ker}(\pi) \implies \psi(f(m)) = \pi(m) = 0_R \implies f(m) \in M_\psi = \text{Ker}(\psi)$.

2. (a) $\bar{f}(m+n) = \bar{f}(m) + \bar{f}(n)$, since $f \in \text{Mor}(\underline{\text{Ring}})$

(b) $M_\pi \subseteq E$, $M_\psi \subseteq F$ and

$$\begin{aligned}
 \bar{f}(r \cdot m) &\equiv f(r \cdot m) \\
 &= f(\phi(r)m) \\
 &= f(\phi(r))f(m) \\
 &= \phi'(r)f(m) \quad (**) \\
 &= r \cdot \bar{f}(m).
 \end{aligned}$$

Where ϕ'

$$\begin{array}{ccc}
 R & \xrightarrow{\phi'} & F \\
 \parallel \textit{id}_R & & \downarrow \psi \\
 R & & R
 \end{array}$$

is the “identity” of the abelian group $(\psi : F \rightarrow R)$. Note that the equivalence of ϕf and ϕ' asserted in $(**)$ follows from Lemma 2 which is proved later in this paper (For an explicit discussion of this, see Appendix A.2).

□

We can now proceed to prove the following proposition.

Proposition 6. G is a functor.

Proof. 1. If $(\pi : E \rightarrow R) \in \text{Ob}(\underline{\underline{\mathbb{C}}})$, then the identity morphism on π is

$$id_E : (\pi : E \rightarrow R) \rightarrow (\pi : E \rightarrow R).$$

Now, under G

$$id_E \mapsto \left(\overline{id}_E : M_\pi \rightarrow M_\pi ; m \mapsto id_E(m) = m \right)$$

$$\therefore G(id_E) \equiv \overline{id}_E = id_{M_\pi} = id_{G(E)}.$$

2. If we have the following diagram in $\underline{\underline{\mathbb{C}}}$

$$\begin{array}{ccccc} E & \xrightarrow{f} & F & \xrightarrow{g} & G \\ \pi \downarrow & & \downarrow \psi & & \downarrow \theta \\ R & \xlongequal{\quad} & R & \xlongequal{\quad} & R \end{array}$$

then we have the composite morphism fg

$$\begin{array}{ccc} E & \xrightarrow{fg} & G \\ \pi \downarrow & & \downarrow \theta \\ R & \xlongequal{\quad} & R \end{array}$$

Now under G we have the following mappings

$$(fg : E \rightarrow G) \mapsto \left(\overline{fg} : M_\pi \rightarrow M_\theta ; m \mapsto g(f(m)) \right)$$

$$(f : E \rightarrow F) \mapsto \left(\overline{f} : M_\pi \rightarrow M_\psi ; m \mapsto f(m) \right)$$

$$(g : F \rightarrow G) \mapsto \left(\overline{g} : M_\psi \rightarrow M_\theta ; n \mapsto g(n) \right)$$

So

$$\overline{f\bar{g}} : M_\pi \rightarrow M_\theta ; m \mapsto f(m) \mapsto g(f(m))$$

$$\therefore \overline{f\bar{g}} = \overline{f\bar{g}}$$

$$\text{I.E., } G(fg) = GfGg.$$

It follows that G is a functor.

□

We have the two functors F, G . We can now use these functors to show that $\mathbf{R-Mod}$ and \mathbf{C} are equivalent.

Proposition 7. *The composite functor FG and the identity functor $1_{\underline{\mathbf{R-Mod}}}$ are naturally isomorphic.*

Proof. If M is an \mathbf{R} module, then

$$G(FM) = G(M \rtimes R)$$

where

$$\pi : M \rtimes R \rightarrow R ; (m, r) \mapsto r$$

is an abelian group in $\mathbf{Ring/R}$.

$$\begin{aligned} \therefore G(M \rtimes R) &\equiv \pi^{-1}\{O_R\} \\ &= \{(m, 0_R) \mid m \in M\} \end{aligned}$$

which we will denote by $M \times 0$.

To define the natural transformation, consider $\tau : FG \xrightarrow{\cdot} 1_{\underline{\mathbf{R-Mod}}}$ defined on objects in $\mathbf{R-Mod}$ by

$$\tau_M : G(FM) \rightarrow 1(M) ; (m, 0_R) \mapsto m$$

$$\text{I.E. } \tau_M : M \times 0 \rightarrow M ; (m, 0_R) \mapsto m$$

For this function to work as our natural transformation, we need to know that for every module M the function τ_M is an R -module isomorphism.

Claim. *As defined τ_M is a module isomorphism.*

Proof. It should be obvious from the definition that τ_M is bijective. To show that it is R -linear we need to recall the definitions of the ring operations of $M \rtimes R$, the operations of $M \times 0$ as an R -module, and the definition of the “identity” morphism ϕ of the abelian group $(\pi : M \rtimes R \rightarrow R)$. With these definitions in mind, we can proceed to verify linearity.

1.

$$\begin{aligned} \tau_M\left((m, 0_R) + (n, 0_R)\right) &= \tau_M(m + n, 0_R) \\ &= m + n \\ &= \tau_M(m, 0_R) + \tau_M(n, 0_R) \end{aligned}$$

2.

$$\begin{aligned}
\tau_M\left(r \cdot (m, 0_R)\right) &= \tau_M\left(\phi(r)(m, 0_R)\right) \\
&= \tau_M\left((0_M, r)(m, 0_R)\right) \\
&= \tau_M(r \cdot m, 0_R) \\
&\equiv r \cdot m \\
&= r \cdot \tau_M(m, 0_R)
\end{aligned}$$

$\therefore \tau_M$ is linear, so it is an R -module isomorphism.

□

We can now proceed to show that τ is a natural transformation. So, $\forall (f : M \rightarrow N) \in \text{Mor}(\underline{\mathbf{R-Mod}})$. We have the following diagram:

$$\begin{array}{ccccc}
M & & M \times 0 & \xrightarrow{\tau_M} & M & & (m, 0) & \longmapsto & m \\
\downarrow f & & \downarrow G(Ff) & & \downarrow f & & \downarrow & & \downarrow \\
N & & N \times 0 & \xrightarrow{\tau_N} & N & & (fm, 0) & \longmapsto & fm
\end{array}$$

(the mappings of morphisms are indicated by the figure on the right)

Clearly then, $\tau_M f = G(Ff) \tau_N$ and the above diagram is commutative. Therefore, τ is a natural transformation and since each component function is an R -module isomorphism, τ is a natural isomorphism. I.E., $FG \cong 1_{\underline{\mathbf{R-Mod}}}$. □

Proposition 8. *The composite functor GF and the identity functor $1_{\underline{\mathbf{C}}}$ are naturally isomorphic.*

Proof. If $(\pi : E \rightarrow R) \in \text{Ob}(\underline{\mathbf{C}})$, then $F(G(E)) = F(M_\pi) = M_\pi \rtimes R$.

(recall that $M_\pi = \text{Ker}\pi$ considered as a module over R - see Prop. 4).

To define the natural transformation consider $\sigma : GF \rightarrow 1_{\underline{\underline{C}}}$ defined by

$$\sigma_E : M_\pi \rtimes R \rightarrow E ; (m, r) \mapsto m + \phi(r)$$

where

$$\begin{array}{ccc} R & \xrightarrow{\phi} & E \\ id_R \parallel & & \downarrow \pi \\ R & \xlongequal{\quad} & R \end{array}$$

is the “identity” of the abelian group $(\pi : E \rightarrow R)$. Note that, technically, σ_E is a morphism between the abelian group objects $(p : M_\pi \rtimes R \rightarrow R)$ and $(\pi : E \rightarrow R)$ yet we will usually omit mention of the morphisms p, π and the ring R , and refer to σ_E as above.

We need to verify the following claim.

Claim. *As defined, σ_E is an isomorphism. (I.e., a bijective ring homomorphism)*

Proof. 1. To show σ_E is bijective, we need only construct its inverse.

To this end, we have the function

$$\sigma_E : M_\pi \rtimes R \rightarrow E ; (m, r) \mapsto m + \phi(r)$$

now consider the function

$$\sigma'_E : E \rightarrow M_\pi \rtimes R ; e \mapsto \left(-[\phi(\pi(e))] + e, \pi(e) \right)$$

Then we have

$$E \xrightarrow{\sigma'_E} M_\pi \rtimes R \xrightarrow{\sigma_E} E$$

$$e \mapsto \left(-[\phi(\pi(e))] + e, \pi(e) \right) \mapsto \left(-[\phi(\pi(e))] + e + \phi(\pi(e)) \right) = e$$

$$\therefore \sigma_E(\sigma'_E(e)) = e, \forall e \in E.$$

$$\text{So } \sigma'_E \sigma_E = id_E.$$

This is half of the desired conclusion. For the other half, observe that we also have

$$M_\pi \rtimes R \xrightarrow{\sigma_E} E \xrightarrow{\sigma'_E} M_\pi \rtimes R$$

$$\begin{aligned} (m, r) \mapsto m + \phi(r) \mapsto & \left(-[\phi(\pi(m + \phi(r)))] + m + \phi(r), \pi(m + \phi(r)) \right) \\ & = \left(-[\phi(\pi(m)) + \phi(\pi(\phi(r)))] + m + \phi(r), \pi(m) + \pi(\phi(r)) \right) \\ & = \left(-[\phi(0_R) + \phi(r)] + m + \phi(r), 0_R + r \right) \\ & = \left(-\phi(r) + m + \phi(r), r \right) \\ & = (m, r) \end{aligned}$$

$$\therefore \sigma'_E(\sigma_E(m, r)) = (m, r), \forall (m, r) \in M_\pi \rtimes R.$$

$$\text{So } \sigma_E \sigma'_E = id_{M_\pi \rtimes R}.$$

It follows that σ_E is a bijection.

2. The function σ_E is also a ring homomorphism since

(a)

$$\begin{aligned}
\sigma_E((m, r) + (n, s)) &= \sigma_E((m + n, r + s)) \\
&\equiv m + n + \phi(r + s) \\
&= m + n + \phi(r) + \phi(s) \\
&= m + \phi(r) + n + \phi(s) \\
&= \sigma_E((m, r)) + \sigma_E((n, s))
\end{aligned}$$

(b)

$$\begin{aligned}
\sigma_E((m, r)(n, s)) &= \sigma_E((s \cdot m + r \cdot n, rs)) \\
&\equiv s \cdot m + r \cdot n + \phi(rs) \\
&= mn + s \cdot m + r \cdot n + \phi(rs) \quad (\dagger) \\
&= (m + \phi(r))(n + \phi(s)) \\
&= \sigma_E((m, r))\sigma_E((n, s))
\end{aligned}$$

Note that the equality asserted in (\dagger) follows from Lemma 1, which is proved at the end of this section.

$\therefore \sigma_E$ is a ring isomorphism as claimed.

□

Now that we know σ_E is in fact a ring homomorphism, we can proceed to show that σ is a natural transformation. So, let $(f : (\pi : E \rightarrow R) \rightarrow (\psi : K \rightarrow R)) \in \text{Mor}(\underline{\mathbf{C}})$. Then we have the following diagram:

$$\begin{array}{ccc}
E & M_\pi \rtimes R & \xrightarrow{\sigma_E} & E & (m, r) & \mapsto & \phi(r) + m \\
\downarrow f & \downarrow F(Gf) & & \downarrow f & \downarrow & & \downarrow \\
K & M_\psi \rtimes R & \xrightarrow{\sigma_K} & K & (fm, r) & \mapsto & f(\phi(r)) + f(m)
\end{array}$$

where $F(Gf) : M_\pi \rtimes R \rightarrow M_\psi \rtimes R$; $(m, r) \mapsto (Gf(m), r) = (fm, r)$.

It is important to note that $f(\phi(r)) + f(m) = \phi'(r) + f(m)$ given that $\phi f = \phi'$, where ϕ' is the “identity” of the abelian group $(\psi : K \rightarrow R)$. As mentioned before, this equality follows from Lemma 2 which is proved in Appendix A.

To complete the proof, an easy diagram chase reveals $\sigma_E f = F(Gf) \sigma_K$ and the above diagram is commutative.

$\therefore \sigma$ is a natural transformation and since each component function is a ring isomorphism, σ is a natural isomorphism. I.E., $GF \cong \underline{1}_{\underline{C}}$. \square

Proposition 7 showed that $FG \cong \underline{1}_{\underline{R-Mod}}$ and Proposition 8 showed that $GF \cong \underline{1}_{\underline{C}}$. We have therefore shown that $\underline{R-Mod}$ and \underline{C} are equivalent categories. Given that the two categories in questions are equivalent, we know they must have isomorphic skeletons. An interesting, albeit probably very difficult, problem would be to attempt to provide explicit skeletons for these categories. See [2](§14) for a discussion of skeletons.

The following lemma was cited above in the proof of the Claim in Proposition 8.

Lemma 1. *For every $m, n \in M_\pi$, $mn = 0_E$.*

Proof. Since $\pi : E \rightarrow R$ is an abelian group in $\underline{Ring/R}$, all the operations of the ring E are ring homomorphisms. In particular the addition operation

$$+ : E \times E \rightarrow E$$

is a ring homomorphism, so the restriction of $+$ to $M_\pi \times M_\pi$ is also a ring homomorphism.

Thus $\forall a, b, c, d \in M_\pi$ we have

$$\begin{aligned} +\left((a, b)(c, d)\right) &= +\left((a, b)\right) + \left((c, d)\right) \\ \Leftrightarrow +\left((ac, bd)\right) &= (a + b)(c + d) \\ \Leftrightarrow ac + bd &= ac + ad + bc + bd \end{aligned}$$

$$\therefore ad + bc = 0_E.$$

So if we let $a = d = 0_E$ then we have $bc = 0_E, \forall b, c \in M_\pi$. □

Appendix A

The preceding results depended upon the following lemma which is well known, but will be proved for completeness' sake. The content of the lemma is essentially the equivalence of the two standard definitions of an abelian group in a category. See [1] for the definitions in question. Before proceeding, the reader should recall that an abelian group in $\underline{\text{Set}}$ (as defined in the categorical way) is merely an abelian group in the usual group-theoretic sense.

Appendix A.1

Lemma 2. *Let $\underline{\underline{C}}$ be a category with finite products and a terminal object “ t ”. Then:*

1. $c \in \text{Ob}(\underline{\underline{C}})$ is an abelian group in $\underline{\underline{C}} \Leftrightarrow \forall a \in \text{Ob}(\underline{\underline{C}})$ the set of morphisms $\underline{\underline{C}}(a, c)$ is an abelian group in $\underline{\underline{\text{Set}}}$.
2. $(f : b \rightarrow c)$ is a morphism of abelian groups in $\underline{\underline{C}} \Leftrightarrow \forall a \in \text{Ob}(\underline{\underline{C}})$

$$\left(\hat{f}_a : \underline{\underline{C}}(a, b) \rightarrow \underline{\underline{C}}(a, c) ; h \mapsto hf \right)$$

is a morphism of abelian groups in $\underline{\underline{\text{Set}}}$ (a group-theoretic homomorphism).

Proof.

Proof of 1.

Let $c \in \text{Ob}(\underline{\underline{C}})$ be an abelian group, and let

1. $\mu : c \times c \rightarrow c$ denote the multiplication morphism of c .

2. $\phi : t \rightarrow c$ denote the identity of c .
3. $\iota : c \rightarrow c$ denote the inverse of c .
4. $\beta : c \times c \rightarrow c \times c$ denote the “permutation of factors” morphism.

If $a \in \text{Ob}(\underline{\mathbb{C}})$, then $\underline{\mathbb{C}}(a, c) \in \text{Ob}(\underline{\text{Set}})$. We want to show that $\underline{\mathbb{C}}(a, c)$ is an abelian group in $\underline{\text{Set}}$. To do this, we need to define the required set functions (I.E. morphisms in $\underline{\text{Set}}$) and show that the relevant diagrams are commutative. To this end, consider the following functions (keep in mind that products in $\underline{\text{Set}}$ are merely cartesian products):

multiplication:

$$\bar{\mu} : \underline{\mathbb{C}}(a, c)^2 \rightarrow \underline{\mathbb{C}}(a, c) ; (f, g) \mapsto (f \times g)\mu$$

where $(f \times g : a \rightarrow c \times c)$ is the product map of f, g .

identity:

$$\bar{\phi} : \{\star\} \rightarrow \underline{\mathbb{C}}(a, c) ; \star \mapsto \eta\phi$$

where $(\eta : a \rightarrow t)$ is the unique morphism from “a” to the terminal object “t”, and $\{\star\}$ denotes a one point set, which is terminal in $\underline{\text{Set}}$. (this set would perhaps most commonly be called the ordinal $\mathbf{1}$, but I shall shun convention!)

inverse:

$$\bar{\iota} : \underline{\mathbb{C}}(a, c) \rightarrow \underline{\mathbb{C}}(a, c) ; f \mapsto f\iota$$

Now that we have some morphisms at hand, we can show that the necessary diagrams are commutative. We will omit cumbersome notation where possible to streamline the presentation. For example, the image of f, g under $\bar{\mu}$ will be written as $fg\mu$ instead of $(f \times g)\mu$. A careful examination of the context should make it apparent what notation is being omitted, thus eliminating any ambiguity.

1. associativity:

$$\begin{array}{ccc}
 \underline{\underline{C}}(a, c)^3 & \xrightarrow{(id, \bar{\mu})} & \underline{\underline{C}}(a, c)^2 \\
 \downarrow (\bar{\mu}, id) & & \downarrow \bar{\mu} \\
 \underline{\underline{C}}(a, c)^2 & \xrightarrow{\bar{\mu}} & \underline{\underline{C}}(a, c)
 \end{array}
 \qquad
 \begin{array}{ccc}
 (f, g, h) & \dashrightarrow & (f, gh\mu) \\
 \downarrow & & \downarrow \\
 (fg\mu, h) & \dashrightarrow & fg\mu h\mu, fgh\mu\mu
 \end{array}$$

Proposition 9. *The above diagram is commutative.*

Proof. For this diagram to be commutative we need to verify that $fg\mu h\mu = fgh\mu\mu$. This equality will follow from properties of the morphism μ . In particular, we have the following commuting diagram in $\underline{\underline{C}}$:

$$\begin{array}{ccc}
 c^3 & \xrightarrow{(id, \mu)} & c^2 & (*) \\
 \swarrow fgh & & \searrow f(gh\mu) & \\
 & a & & \\
 \swarrow (fg\mu)h & & \searrow fgh\mu\mu & \\
 c^2 & \xrightarrow{\mu} & c & \\
 \downarrow (\mu, id) & & \downarrow \mu &
 \end{array}$$

(See Appendix B.1 for an explanation of the origin of this diagram.)

We deduce that

$$\begin{aligned}
 fgh\mu\mu &= fgh(id, \mu)\mu \\
 &= fgh(\mu, id)\mu && \text{(follows from associativity of } \mu) \\
 &= fg\mu h\mu
 \end{aligned}$$

\therefore the diagram commutes. □

2. commutativity:

$$\begin{array}{ccc}
 \underline{\underline{C}}(a, c)^2 & \xrightarrow{\bar{\mu}} & \underline{\underline{C}}(a, c) & (f, g) \mapsto & fg\mu \\
 \downarrow \bar{\beta} & & \parallel id & \downarrow & \\
 \underline{\underline{C}}(a, c)^2 & \xrightarrow{\bar{\mu}} & \underline{\underline{C}}(a, c) & (g, f) \mapsto & gf\mu
 \end{array}$$

Proposition 10. *The above diagram is commutative.*

Proof. For this diagram to be commutative we need to verify that $fg\mu = gf\mu$. To this end, we have the following commuting diagram:

$$\begin{array}{ccc}
 c^2 & \xrightarrow{\beta} & c^2 & (*) \\
 \swarrow fg & & \searrow gf & \\
 & a & & \\
 \swarrow \mu & \downarrow fg\mu & \searrow \mu & \\
 & c & &
 \end{array}$$

(See Appendix B.2 for an explanation of the origin of this diagram.)

We deduce that

$$\begin{aligned}
 gf\mu &= fg\beta\mu \\
 &= fg\mu,
 \end{aligned}$$

where the last equality follows from the commutativity of the outside triangle in (*).

\therefore the above diagram commutes. □

3. inverse:

$$\begin{array}{ccc}
 \underline{\underline{C}}(a, c) & \xrightarrow{(id, \bar{\iota})} & \underline{\underline{C}}(a, c)^2 \\
 \downarrow & & \downarrow \bar{\mu} \\
 \{\star\} & \xrightarrow{\bar{\phi}} & \underline{\underline{C}}(a, c)
 \end{array}
 \qquad
 \begin{array}{ccc}
 f & \longmapsto & (f, f\iota) \\
 \downarrow & & \downarrow \\
 \star & \longmapsto & \eta\phi, ff\iota\mu
 \end{array}$$

Proposition 11. *The above diagram is commutative.*

Proof. For this diagram to be commutative, we need to verify that $\eta\phi = ff\iota\mu$. To this end, we have the following commuting diagram:

$$\begin{array}{ccc}
 c & \xrightarrow{id \iota} & c^2 & (*) \\
 \swarrow f & & \searrow ff\iota & \\
 & a & & \\
 \nwarrow \eta & & \nearrow ff\iota\mu & \\
 t & \xrightarrow{\phi} & c & \\
 \downarrow \alpha & & \downarrow \mu &
 \end{array}$$

(See Appendix B.3 for an explanation of the origin of this diagram.)

We deduce that

$$\begin{aligned}
 \eta\phi &= f\alpha\phi \\
 &= fid\iota\mu \\
 &= ff\iota\mu,
 \end{aligned}$$

where the second equality follows from the commutativity of the outside square in (*).

\therefore the above diagram commutes. □

4. identity:

$$\begin{array}{ccc}
 \{\star\} \times \underline{\underline{C}}(a, c) & \xrightarrow{(\bar{\phi}, id)} & \underline{\underline{C}}(a, c)^2 \\
 \swarrow iso & & \swarrow \bar{\mu} \\
 & & \underline{\underline{C}}(a, c)
 \end{array}$$

$$\begin{array}{ccc}
 (\star, f) & \xrightarrow{\quad} & (\eta\phi, f) \\
 \swarrow & & \swarrow \\
 & & f, \eta\phi f\mu
 \end{array}$$

Proposition 12. *The above diagram is commutative.*

Proof. For this diagram to be commutative we need to verify that $f = \eta\phi f\mu$. We have the following commuting diagram:

$$\begin{array}{ccc}
 t \times c & \xrightarrow{(\phi, id)} & c^2 & (*) \\
 \swarrow \eta f & & \swarrow \eta\phi f & \\
 & a & & \\
 \swarrow \gamma & \downarrow f & \swarrow \mu & \\
 & c & & \\
 \swarrow & \downarrow \eta\phi f\mu & \swarrow & \\
 & c & &
 \end{array}$$

(See Appendix B.4 for an explanation of the origin of this diagram.)

We deduce that

$$\begin{aligned}
 \eta\phi f\mu &= \eta f(\phi, id)\mu \\
 &= \eta f\pi_2^{t \times c} \\
 &= f
 \end{aligned}$$

where the second equality follows from commutativity of the outside triangle in (*).

\therefore the above diagram commutes. \square

Commutativity of

$$\begin{array}{ccc}
 \underline{\underline{\mathbb{C}}}(a, c)^2 & \xleftarrow{(id, \bar{\phi})} & \underline{\underline{\mathbb{C}}}(a, c) \times \{\star\} \\
 & \searrow \bar{\mu} & \swarrow iso \\
 & \underline{\underline{\mathbb{C}}}(a, c) &
 \end{array}$$

$$\begin{array}{ccc}
 (f, \eta\phi) & \xleftarrow{\quad} & (f, \star) \\
 & \searrow & \swarrow \\
 & f\eta\phi\mu, f &
 \end{array}$$

follows similarly.

We have shown that the diagrams (a) . . . (d) commute. Therefore, $\langle \underline{\underline{\mathbb{C}}}(a, c), \bar{\mu}, \bar{\iota}, \bar{\phi} \rangle$ is an abelian group in $\underline{\underline{\text{Set}}}$, and this is one half of the desired result. To prove the converse, let $c \in Ob(\underline{\underline{\mathbb{C}}}) \ni \forall a \in Ob(\underline{\underline{\mathbb{C}}})$, $\underline{\underline{\mathbb{C}}}(a, c)$ is an abelian group in $\underline{\underline{\text{Set}}}$. In other words, $\underline{\underline{\mathbb{C}}}(a, c)$ is an abelian group in the standard group-theoretic sense. We will write the groups $\underline{\underline{\mathbb{C}}}(_, c)$ additively. Also, in the following arguments we will have several groups of the form $\underline{\underline{\mathbb{C}}}(_, c)$. Each of these groups has an associated “multiplication”, “identity”, and “inverse” function. For the sake of clarity, we need to introduce a simple notation to keep track of these functions. So, for $a \in Ob(\underline{\underline{\mathbb{C}}})$ let

1. $\mu_a : \underline{\underline{\mathbb{C}}}(a, c)^2 \rightarrow \underline{\underline{\mathbb{C}}}(a, c)$ denote the multiplication function of $\underline{\underline{\mathbb{C}}}(a, c)$.
2. $\iota_a : \underline{\underline{\mathbb{C}}}(a, c) \rightarrow \underline{\underline{\mathbb{C}}}(a, c)$ denote the inverse function of $\underline{\underline{\mathbb{C}}}(a, c)$.
3. $\phi_a : \{\star\} \rightarrow \underline{\underline{\mathbb{C}}}(a, c)$ denote the identity function of $\underline{\underline{\mathbb{C}}}(a, c)$.

$$4. \beta_a : \underline{\underline{\mathbb{C}}}(a, c)^2 \rightarrow \underline{\underline{\mathbb{C}}}(a, c)^2 ; (f, g) \mapsto (g, f)$$

We are attempting to show that c is an abelian group in $\underline{\underline{\mathbb{C}}}$. The first thing we need to do is define the necessary morphisms. To this end, consider the following morphisms in $\underline{\underline{\mathbb{C}}}$:

multiplication:

$$\pi_1^{c^2} + \pi_2^{c^2} : c^2 \rightarrow c$$

where $\pi_1^{c^2}, \pi_2^{c^2} \in \underline{\underline{\mathbb{C}}}(c^2, c)$ are the projections of the product

$$c \xleftarrow{\pi_1^{c^2}} c^2 \xrightarrow{\pi_2^{c^2}} c$$

and $\pi_1^{c^2} + \pi_2^{c^2}$ is their sum in the group $\underline{\underline{\mathbb{C}}}(c^2, c)$.

identity:

$$\bar{\phi} : t \rightarrow c.$$

where $\bar{\phi} \in \underline{\underline{\mathbb{C}}}(t, c)$ is the identity of the group.

inverse:

$$\bar{\iota} : c \rightarrow c.$$

where, $\bar{\iota} \equiv -id_c \in \underline{\underline{\mathbb{C}}}(c, c)$ denotes the inverse of id_c with respect to the binary operation of $\underline{\underline{\mathbb{C}}}(c, c)$.

Now that we have some morphisms to work with, we can verify that the necessary diagrams commute. The following diagrams will be diagrams in the general category $\underline{\underline{\mathbb{C}}}$, whereas the diagrams which were employed in the first half of this proof were diagrams in the nice category $\underline{\underline{\text{Set}}}$. Since we were working in $\underline{\underline{\text{Set}}}$ above, we could work with our diagrams on the object level. We have no such luxury in this case, and proving commutativity of diagrams will be necessarily more involved. In fact, to prove commutativity of the following diagrams, we will have to invoke a result pertaining to a functor called

the Yoneda Embedding. A careful description of this functor, and the way in which it will be used, is given in Appendix C.

1. associativity:

$$\begin{array}{ccc}
 c^3 & \xrightarrow{(id, \pi_1^{c^2} + \pi_2^{c^2})} & c^2 \\
 (\pi_1^{c^2} + \pi_2^{c^2}, id) \downarrow & & \downarrow \pi_1^{c^2} + \pi_2^{c^2} \\
 c^2 & \xrightarrow{\pi_1^{c^2} + \pi_2^{c^2}} & c
 \end{array}$$

where

$$\begin{aligned}
 (id, \pi_1^{c^2} + \pi_2^{c^2}) &= \pi_1^{c^3} \times ((\pi_2^{c^3} \times \pi_3^{c^3})(\pi_1^{c^2} + \pi_2^{c^2})) \\
 (\pi_1^{c^2} + \pi_2^{c^2}, id) &= ((\pi_1^{c^3} \times \pi_2^{c^3})(\pi_1^{c^2} + \pi_2^{c^2})) \times \pi_3^{c^3}
 \end{aligned}$$

Proposition 13. *The above diagram is commutative.*

Proof. Applying the Yoneda embedding to the above diagram we obtain

$$\begin{array}{ccc}
 F_{c^3} & \xrightarrow{\widehat{(id, \pi_1^{c^2} + \pi_2^{c^2})}} & F_{c^2} & (*) \\
 (\widehat{\pi_1^{c^2} + \pi_2^{c^2}}, id) \downarrow & & \downarrow \widehat{\pi_1^{c^2} + \pi_2^{c^2}} & \\
 F_{c^2} & \xrightarrow{\widehat{\pi_1^{c^2} + \pi_2^{c^2}}} & F_c &
 \end{array}$$

in $\underline{\underline{\text{Set}}}^{\underline{\underline{\mathbb{C}}}}^{\text{op}}$. Now (*) is a diagram involving functors and natural transformations.

So, the lemma in Appendix D tells us that (*) commutes if and only if $\forall a \in \text{Ob}(\underline{\underline{\mathbb{C}}})$

$$\begin{array}{ccc}
\underline{\underline{\mathbb{C}}}(a, c^3) \xrightarrow{\widehat{(id, \pi_1^{c^2} + \pi_2^{c^2})}} \underline{\underline{\mathbb{C}}}(a, c^2) & (**) & \\
\downarrow \widehat{(\pi_1^{c^2} + \pi_2^{c^2}, id)_a} & & \downarrow \widehat{\pi_1^{c^2} + \pi_2^{c^2}}_a \\
\underline{\underline{\mathbb{C}}}(a, c^2) \xrightarrow{\widehat{\pi_1^{c^2} + \pi_2^{c^2}}_a} \underline{\underline{\mathbb{C}}}(a, c) & &
\end{array}$$

commutes in Set. Where the mappings of the component functions are given by composition on the right (see Appendix C). Recalling that $\underline{\underline{\mathbb{C}}}(a, c)$ is an abelian group, we would like to use commutativity of

$$\begin{array}{ccc}
\underline{\underline{\mathbb{C}}}(a, c)^3 \xrightarrow{(id, \mu_a)} \underline{\underline{\mathbb{C}}}(a, c)^2 & (\dagger) & \\
\downarrow (\mu_a, id) & & \downarrow \mu_a \\
\underline{\underline{\mathbb{C}}}(a, c)^2 \xrightarrow{\mu_a} \underline{\underline{\mathbb{C}}}(a, c) & &
\end{array}$$

to show that $(**)$ commutes. We could accomplish this by showing that the corresponding functions in (\dagger) and $(**)$ “act in the same way”. Doing this is rather involved and requires some care, and the author of this paper was unable to completely figure it out before “press time”. Thus, we arrive at the fundamental incompleteness in this paper. At this point, I will say it can be done, but I don’t quite know how to do it. The answer to this unresolved matter can probably be found in [6] once one completely understands [6].

Assuming we have shown that (commutativity of (\dagger)) \implies (commutativity of $(**)$), we would have finished the proof of the proposition since then:

(commutativity of $(**)$) \implies (commutativity of $(*)$) \implies (commutativity of (1)).

□

2. inverse:

$$\begin{array}{ccc}
 c & \xrightarrow{id_c \times \bar{\iota}} & c^2 \\
 \alpha \downarrow & & \downarrow \pi_1^{c^2} + \pi_2^{c^2} \\
 t & \xrightarrow{\bar{\phi}} & c
 \end{array}$$

Proposition 14. *The above diagram is commutative.*

Proof. Applying the Yoneda Embedding to the above diagram we obtain

$$\begin{array}{ccc}
 F_c & \xrightarrow{\widehat{id_c \times \bar{\iota}}} & F_{c^2} & (*) \\
 \hat{\alpha} \downarrow & & \downarrow \widehat{\pi_1^{c^2} + \pi_2^{c^2}} & \\
 F_t & \xrightarrow{\widehat{\phi}} & F_c &
 \end{array}$$

in $\underline{\underline{\text{Set}}}^{\text{C}^{\text{op}}}$. Now, $(*)$ commutes if and only if $\forall a \in \text{Ob}(\underline{\underline{\text{C}}})$

$$\begin{array}{ccc}
 \underline{\underline{\text{C}}}(a, c) & \xrightarrow{\widehat{id_c \times \bar{\iota}}_a} & \underline{\underline{\text{C}}}(a, c^2) & (**) \\
 \hat{\alpha}_a \downarrow & & \downarrow \widehat{\pi_1^{c^2} + \pi_2^{c^2}}_a & \\
 \underline{\underline{\text{C}}}(a, t) & \xrightarrow{\widehat{\phi}_a} & \underline{\underline{\text{C}}}(a, c) &
 \end{array}$$

commutes in $\underline{\underline{\text{Set}}}$. Where the mappings of the component functions are given by composition on the right. As described in (1), we can use commutativity of

$$\begin{array}{ccc}
 \underline{\underline{\text{C}}}(a, c) & \xrightarrow{id \times \iota_a} & \underline{\underline{\text{C}}}(a, c)^2 \\
 \downarrow & & \downarrow \mu_a \\
 \{\star\} & \xrightarrow{\phi_a} & \underline{\underline{\text{C}}}(a, c)
 \end{array}$$

to show commutativity of (**), and finish the proof of the proposition. \square

3. commutativity:

$$\begin{array}{ccc}
 c^2 & \xrightarrow{\pi_1^{c^2} + \pi_2^{c^2}} & c \\
 \beta \downarrow & & \parallel \\
 c^2 & \xrightarrow{\pi_1^{c^2} + \pi_2^{c^2}} & c
 \end{array}$$

Proposition 15. *The above diagram is commutative.*

Proof. Applying the Yoneda Embedding to the above diagram we obtain

$$\begin{array}{ccc}
 F_{c^2} & \xrightarrow{\widehat{\pi_1^{c^2} + \pi_2^{c^2}}} & F_c \quad (*) \\
 \widehat{\beta} \downarrow & & \parallel \\
 F_{c^2} & \xrightarrow{\widehat{\pi_1^{c^2} + \pi_2^{c^2}}} & F_c
 \end{array}$$

in $\underline{\underline{\text{Set}}}^{\text{C}^{\text{op}}}$. Now, (*) commutes if and only if $\forall a \in \text{Ob}(\underline{\underline{\text{C}}})$

$$\begin{array}{ccc}
 \underline{\underline{\text{C}}}(a, c^2) & \xrightarrow{\widehat{\pi_1^{c^2} + \pi_2^{c^2}}_a} & \underline{\underline{\text{C}}}(a, c) \quad (**) \\
 \widehat{\beta}_a \downarrow & & \parallel \\
 \underline{\underline{\text{C}}}(a, c^2) & \xrightarrow{\widehat{\pi_1^{c^2} + \pi_2^{c^2}}_a} & \underline{\underline{\text{C}}}(a, c)
 \end{array}$$

commutes in $\underline{\underline{\text{Set}}}$. Where the mappings of the component functions are given by

composition on the right. As described in (1), we can use commutativity of

$$\begin{array}{ccc}
 \underline{\underline{\mathbb{C}}}(a, c)^2 & \xrightarrow{\mu_a} & \underline{\underline{\mathbb{C}}}(a, c) \\
 \downarrow \beta_a & & \parallel \\
 \underline{\underline{\mathbb{C}}}(a, c)^2 & \xrightarrow{\mu_a} & \underline{\underline{\mathbb{C}}}(a, c)
 \end{array}$$

to show commutativity of (**), and complete the proof of the proposition. \square

4. identity:

$$\begin{array}{ccc}
 t \times c & \xrightarrow{(\bar{\phi}, id)} & c^2 \\
 \swarrow iso & & \swarrow \pi_1^{c^2} + \pi_2^{c^2} \\
 & c &
 \end{array}$$

Proposition 16. *The above diagram is commutative.*

Proof. Applying the Yoneda embedding to the above diagram we obtain

$$\begin{array}{ccc}
 F_{t \times c} & \xrightarrow{\widehat{(\bar{\phi}, id)}} & F_{c^2} & (*) \\
 \swarrow iso & & \swarrow \widehat{\pi_1^{c^2} + \pi_2^{c^2}} & \\
 & F_c & &
 \end{array}$$

in $\underline{\underline{\text{Set}}}^{\underline{\underline{\mathbb{C}}}\text{op}}$. Now, (*) commutes if and only if $\forall a \in \text{Ob}(\underline{\underline{\mathbb{C}}})$

$$\begin{array}{ccc}
 \underline{\underline{\mathbb{C}}}(a, t \times c) & \xrightarrow{\widehat{(\bar{\phi}, id)}_a} & \underline{\underline{\mathbb{C}}}(a, c^2) & (**) \\
 \swarrow iso & & \swarrow \widehat{\pi_1^{c^2} + \pi_2^{c^2}}_a & \\
 & \underline{\underline{\mathbb{C}}}(a, c) & &
 \end{array}$$

commutes in $\underline{\underline{\text{Set}}}$. Where the mappings of component functions are given by com-

position on the right. As described in (1) we can use commutativity of

$$\begin{array}{ccc}
 \{\star\} \times \underline{\underline{C}}(a, c) & \xrightarrow{(\phi_a, id)} & \underline{\underline{C}}(a, c)^2 \\
 & \searrow & \swarrow \mu_a \\
 & \underline{\underline{C}}(a, c) &
 \end{array}$$

to show commutativity of (**), and finish the proof of the proposition. \square

This proves 1.

Proof of 2.

We have as yet not explicitly defined the term “morphism of abelian groups” which occurs in the statement of 2. So before proceeding, we will start this section with a definition.

Definition 4. Let $b, c \in Ob(\underline{\underline{C}})$ be abelian groups in $\underline{\underline{C}}$. Then $(f : b \rightarrow c) \in Mor(\underline{\underline{C}})$ is a morphism of abelian groups if $\mu_b f = f f \mu_c$. Where $\mu_{(-)}$ denotes the respective multiplication map, and $f f$ is the product map

$$\begin{array}{ccccc}
 c & \longleftarrow & c^2 & \longrightarrow & c \\
 & \swarrow & \uparrow & \searrow & \\
 & \pi_1^{b^2} f & f f & \pi_2^{b^2} f & \\
 & & b^2 & &
 \end{array}$$

Now that we know what a morphism of abelian groups is, we can proceed with the proof at hand. So, let $b, c \in Ob(\underline{\underline{C}})$ be abelian groups, and $(f : b \rightarrow c)$ a morphism between them. Assume f is a morphism of abelian groups. Then $\forall a \in Ob(\underline{\underline{C}})$ we have the set function $(\hat{f}_a : \underline{\underline{C}}(a, b) \rightarrow \underline{\underline{C}}(a, c); h \mapsto hf)$. We just proved in Part 1 that $\underline{\underline{C}}(a, b), \underline{\underline{C}}(a, c)$ are abelian groups in the usual sense. We now want to prove the following proposition.

Proposition 17. With notation as above, the function $(\hat{f}_a : \underline{\underline{C}}(a, b) \rightarrow \underline{\underline{C}}(a, c); h \mapsto hf)$

Proof. Consider the following commutative diagrams in $\underline{\underline{\mathbb{C}}}$, where unlabelled arrows are projections:

1.

$$\begin{array}{ccccc}
 c & \longleftarrow & c^2 & \longrightarrow & c \\
 \uparrow f & & \uparrow \text{ff} & & \uparrow f \\
 b & \longleftarrow & b^2 & \longrightarrow & b \\
 & \swarrow h & \uparrow hk & \searrow k & \\
 & & a & &
 \end{array}$$

2.

$$\begin{array}{ccccc}
 c & \longleftarrow & c^2 & \longrightarrow & c \\
 & \swarrow hf & \uparrow hfkf & \searrow kf & \\
 & & a & &
 \end{array}$$

Commutativity of 1 implies that

$$\begin{array}{ccccc}
 c & \longleftarrow & c^2 & \longrightarrow & c \\
 & \swarrow hf & \uparrow hkkff & \searrow kf & \\
 & & a & &
 \end{array}$$

is commutative. Thus, it follows from uniqueness of product maps that $hf \times kf = (h \times k)ff$.

$$\therefore (h \times k)ff\mu_c = (hf \times kf)\mu_c.$$

In other words, $(*) = (**)$.

This proves the claim. □

We have shown that $\forall h, k \in \underline{\underline{\mathbb{C}}}(a, b)$, $\hat{f}_a(h + k) = \hat{f}_a(h) + \hat{f}_a(k)$. Thus, \hat{f}_a is a group homomorphism.

This proves the proposition. □

We have shown if f is a morphism of abelian groups then $\forall a \in \text{Ob}(\underline{\underline{\mathbb{C}}})$, \hat{f}_a is an abelian group homomorphism. This is one half of the desired conclusion. To prove the converse, assume $\forall a \in \text{Ob}(\underline{\underline{\mathbb{C}}})$, $(\hat{f}_a : \underline{\underline{\mathbb{C}}}(a, b) \rightarrow \underline{\underline{\mathbb{C}}}(a, c) ; h \mapsto hf)$ is a homomorphism of abelian groups. In particular, we are assuming that $\forall a \in \text{Ob}(\underline{\underline{\mathbb{C}}})$, $\underline{\underline{\mathbb{C}}}(a, b)$ and $\underline{\underline{\mathbb{C}}}(a, c)$ are abelian groups. Yet, if $\underline{\underline{\mathbb{C}}}(a, b)$, $\underline{\underline{\mathbb{C}}}(a, c)$ are abelian groups $\forall a \in \text{Ob}(\underline{\underline{\mathbb{C}}})$, then b, c are abelian groups in $\underline{\underline{\mathbb{C}}}$ (Part 1). With these observations in mind, we can now proceed to prove the following proposition.

Proposition 18. *With notation as above, $(f : b \rightarrow c)$ is a morphism of abelian groups.*

Proof. We need to show that $\mu_b f = f f \mu_c$ (as in Definition 4). To this end, consider the function $(\hat{f}_{b^2} : \underline{\underline{\mathbb{C}}}(b^2, b) \rightarrow \underline{\underline{\mathbb{C}}}(b^2, c) ; h \mapsto hf)$ which is, by assumption, a group homomorphism. Then, $\mu_b \in \underline{\underline{\mathbb{C}}}(b^2, b)$, and in fact (Part 1)

$$\mu_b = \pi_1^{b^2} + \pi_2^{b^2}. \quad (*)$$

where $+$ represents the binary operation of $\underline{\underline{\mathbb{C}}}(b^2, b)$.

With $(*)$ in mind, we can compute

$$\begin{aligned} \hat{f}_{b^2}(\mu_b) &= \hat{f}_{b^2}(\pi_1^{b^2} + \pi_2^{b^2}) \\ &= \hat{f}_{b^2}(\pi_1^{b^2}) + \hat{f}_{b^2}(\pi_2^{b^2}) \\ &= \pi_1^{b^2} f + \pi_2^{b^2} f \\ &= (\pi_1^{b^2} f \times \pi_2^{b^2} f) \mu_c \\ &= f f \mu_c. \end{aligned}$$

Yet we also know that

$$\hat{f}_{b^2}(\mu_b) \equiv \mu_b f.$$

$$\therefore \mu_b f = f f \mu_c.$$

This proves the proposition. □

We have shown that if $\forall a \in \text{Ob}(\underline{\underline{\mathbb{C}}})$, \hat{f}_a is an abelian group homomorphism, then f is a morphism of abelian groups in $\underline{\underline{\mathbb{C}}}$.

This proves 2.

We have proven Parts 1 and 2 thus, we have proven Lemma 2. □

Appendix A.2

Lemma 2 is invoked in Proposition 5 (and elsewhere) to prove that $\phi f = \phi'$. The way in which this equality follows from Lemma 2 is discussed in this section.

Recall, in Section 4 we have abelian groups $(\pi : E \rightarrow R)$ and $(\psi : F \rightarrow R)$ in $\underline{\underline{\text{Ring/R}}}$. We also have a morphism of abelian groups $(f : (\pi : E \rightarrow R) \rightarrow (\psi : F \rightarrow R))$, and ϕ is the “identity” of $(\pi : E \rightarrow R)$, ϕ' the identity of $(\psi : F \rightarrow R)$. Since f is a morphism of abelian groups and $id_R \in Ob(\underline{\underline{\text{Ring/R}}})$, from Lemma 2 we have the group homomorphism

$$\begin{array}{ccc} \underline{\underline{\text{Ring/R}}}(id_R, \pi) & \xrightarrow{\hat{f}_{id_R}} & \underline{\underline{\text{Ring/R}}}(id_R, \psi) \\ h & \longmapsto & hf \end{array}$$

If we can show that ϕ is the group identity of $\underline{\underline{\text{Ring/R}}}(id_R, \pi)$ and ϕ' is the group identity of $\underline{\underline{\text{Ring/R}}}(id_R, \psi)$ then we will be done since then

$$\begin{aligned} \hat{f}_{id_R}(\phi) &\equiv \phi f \\ &= \phi' \end{aligned}$$

because \hat{f}_{id_R} is a group hom. and must preserve the identity elements.

Recall from part 1 of Lemma 2 that the group identity of $\underline{\underline{\text{Ring/R}}}(id_R, \pi)$ is $\eta\phi$, where $(\eta : id_R \rightarrow id_R)$ is the unique morphism to the terminal object id_R . In this case, $\eta = id_R$. Therefore, $\eta\phi = \phi$ and ϕ is the group identity element. A similar argument shows that ϕ' is the group identity of $\underline{\underline{\text{Ring/R}}}(id_R, \psi)$.

Appendix B

For the purposes of appendices B.1-B.4, most diagrams will have morphisms written without any infix notation. But for the sake of clarity during arguments, we will usually write morphisms with all necessary notation when attempting to prove something. For example, “ \times ” will be used to denote a product map, and juxtaposition will denote composition in postfix notation.

Appendix B.1

The following commuting diagram is used in the proof of proposition 1.

$$\begin{array}{ccc}
 c^3 & \xrightarrow{(id, \mu)} & c^2 \\
 \swarrow fgh & & \searrow f(gh\mu) \\
 & a & \\
 \swarrow (fg\mu)h & & \searrow fgh\mu\mu \\
 c^2 & \xrightarrow{\mu} & c
 \end{array}
 \quad (*)$$

where

$$(id, \mu) = \pi_1^{c^3} \times ((\pi_2^{c^3} \times \pi_3^{c^3})\mu)$$

$$(\mu, id) = ((\pi_1^{c^3} \times \pi_2^{c^3})\mu) \times \pi_3^{c^3}$$

This diagram is, in turn, derived from the commuting diagram

$$\begin{array}{ccc}
 c^3 & \xrightarrow{(id, \mu)} & c^2 \\
 (\mu, id) \downarrow & & \downarrow \mu \\
 c^2 & \xrightarrow{\mu} & c
 \end{array}$$

which expresses the “associativity” of the morphism μ .

The lower-right arrow in (*) merely represents the two composites as indicated. Therefore the commutativity of the bottom and right-hand triangles should be obvious. Yet, the commutativity of the left-hand and top triangles requires some justification. To this end, we need to show that both triangles in the diagram

$$\begin{array}{ccc}
 c^3 & \xrightarrow{(id, \mu)} & c^2 & (**) \\
 (\mu, id) \downarrow & \swarrow fgh & \nearrow f(gh\mu) & \\
 & a & & \\
 & \swarrow (fg\mu)h & & \\
 c^2 & & &
 \end{array}$$

commute.

Proposition 19. *With notation as above, we have:*

1. $fgh(id, \mu) = f(gh\mu)$
2. $fgh(\mu, id) = (fg\mu)h$

Proof. The proof of this proposition uses the important fact that “composition distributes over products on the left ” when composition is written using postfix notation. I.E. $f(g \times h) = (fg \times fh)$ when defined.

1.

$$\begin{aligned}
(f \times g \times h)(id, \mu) &= (f \times g \times h)(\pi_1^{c^3} \times (\pi_2^{c^3} \times \pi_3^{c^3})\mu) \\
&= (f \times g \times h)\pi_1^{c^3} \times (f \times g \times h)((\pi_2^{c^3} \times \pi_3^{c^3})\mu) \\
&= f \times ((f \times g \times h)(\pi_2^{c^3} \times \pi_3^{c^3}))\mu \\
&= f \times ((f \times g \times h)\pi_2^{c^3} \times (f \times g \times h)\pi_3^{c^3})\mu \\
&= f \times (g \times h)\mu
\end{aligned}$$

2.

$$\begin{aligned}
(f \times g \times h)(\mu, id) &= (f \times g \times h)((\pi_1^{c^3} \times \pi_2^{c^3})\mu \times \pi_3^{c^3}) \\
&= (f \times g \times h)((\pi_1^{c^3} \times \pi_2^{c^3})\mu) \times (f \times g \times h)\pi_3^{c^3} \\
&= ((f \times g \times h)(\pi_1^{c^3} \times \pi_2^{c^3}))\mu \times h \\
&= ((f \times g \times h)\pi_1^{c^3} \times (f \times g \times h)\pi_2^{c^3})\mu \times h \\
&= (f \times g)\mu \times h
\end{aligned}$$

□

We have shown that both triangles in (**) commute. Therefore, all triangles in (*) commute.

Appendix B.2

The following commuting diagram is used in the proof of Proposition 2:

$$\begin{array}{ccc}
 c^2 & \xrightarrow{\beta} & c^2 \\
 \swarrow fg & & \searrow gf \\
 & a & \\
 \swarrow \mu & \downarrow fg\mu \quad gf\mu & \searrow \mu \\
 & c &
 \end{array} \quad (*)$$

Where the morphism β is the product map

$$\begin{array}{ccccc}
 c & \xleftarrow{\pi_1} & c^2 & \xrightarrow{\pi_2} & c \\
 & \swarrow \pi_2 & \uparrow \beta & \searrow \pi_1 & \\
 & & c^2 & &
 \end{array}$$

The diagram $(*)$ is in turn derived from the commuting diagram

$$\begin{array}{ccc}
 c^2 & \xrightarrow{\beta} & c^2 \\
 \swarrow \mu & & \searrow \mu \\
 & c &
 \end{array}$$

which expresses the “commutativity” of the morphism μ .

The lower-middle arrow in $(*)$ represents the two composite morphisms as indicated. Thus, the commutativity of the lower two triangles should be obvious. Yet, the commutativity of the top triangle requires some justification. So, we need to show that the triangle

$$\begin{array}{ccc}
 c^2 & \xrightarrow{\beta} & c^2 \\
 \swarrow fg & & \searrow gf \\
 & a &
 \end{array} \quad (**)$$

commutes.

Proposition 20. *With notation as above, we have: $fg\beta = gf$.*

Proof. We need to show that $(f \times g)\beta$ is the product map $g \times f$.

1.

$$\begin{aligned} ((f \times g)\beta)\pi_1^{c^2} &= (f \times g)(\beta\pi_1^{c^2}) \\ &= (f \times g)\pi_2^{c^2} \\ &= g \end{aligned}$$

2.

$$\begin{aligned} ((f \times g)\beta)\pi_2^{c^2} &= (f \times g)(\beta\pi_2^{c^2}) \\ &= (f \times g)\pi_1^{c^2} \\ &= f \end{aligned}$$

So, $(f \times g)\beta$ is a morphism from a to c^2 and it satisfies the properties of the product map $g \times f$.

Thus, the uniqueness of product maps implies that $(f \times g)\beta = g \times f$. □

We have shown that the triangle $(**)$ commutes. Therefore, all triangles in $(*)$ commute.

Appendix B.3

The following commuting diagram is used in the proof of Proposition 3:

$$\begin{array}{ccc}
 c & \xrightarrow{id \iota} & c^2 \\
 \alpha \downarrow & \swarrow f & \nearrow ff\iota \\
 & a & \\
 & \nwarrow \eta & \searrow ff\iota\mu \\
 t & \xleftarrow{\phi} & c \\
 & & \downarrow \mu
 \end{array}
 \quad (*)$$

where α, η are unique morphisms to the terminal object t .

This diagram is, in turn, derived from the commuting diagram

$$\begin{array}{ccc}
 c & \xrightarrow{id \iota} & c^2 \\
 \alpha \downarrow & & \downarrow \mu \\
 t & \xrightarrow{\phi} & c
 \end{array}$$

which expresses the fact that ι is an “inverse” morphism w.r.t the “multiplication” μ .

The bottom-right arrow in $(*)$ represents the two composites as indicated. Therefore the commutativity of the bottom and right triangles should be obvious. Yet, the commutativity of the top and left triangles requires some justification. So, we need to show that the two triangles in

$$\begin{array}{ccc}
 c & \xrightarrow{id \iota} & c^2 \\
 \alpha \downarrow & \swarrow f & \nearrow ff\iota \\
 & a & \\
 & \nwarrow \eta & \\
 t & &
 \end{array}
 \quad (**)$$

commute.

Proposition 21. *With notation as above, we have:*

1. $f\alpha = \eta$

2. $f(id\iota) = ff\iota$

Proof.

1. Since $f\alpha$ is a morphism from a to t , and t is terminal in $\underline{\underline{\mathbf{C}}}$, it follows from uniqueness of morphisms to the terminal object that $f\alpha = \eta$.
2. As in the proof in Appendix B.1, we have

$$\begin{aligned} f(id \times \iota) &= (f id) \times (f\iota) \\ &= f \times f\iota \end{aligned}$$

□

We have shown that the triangles in (**) commute. Therefore, all triangles in (*) commute.

Appendix B.4

The following commuting diagram is used in the proof of Proposition 4:

$$\begin{array}{ccc}
 t \times c & \xrightarrow{(\phi, id)} & c^2 \\
 \eta f \swarrow & & \searrow \eta \phi f \\
 & a & \\
 \gamma \swarrow & & \searrow \mu \\
 & c & \\
 f \downarrow & & \downarrow \eta \phi f \mu
 \end{array} \quad (*)$$

where $(\phi, id) = (\pi_1^{t \times c} \phi) \times \pi_2^{t \times c}$ and γ denotes the isomorphism between $t \times c$ and c . Explicitly, this isomorphism is $\pi_2^{t \times c}$ from $t \times c$ to c , and the product morphism $\alpha \times id_c$ (where α is the unique morphism from c to t) from c to $t \times c$.

This diagram is, in turn, derived from the commuting diagram

$$\begin{array}{ccc}
 t \times c & \xrightarrow{(\phi, id)} & c^2 \\
 \gamma \searrow & & \swarrow \mu \\
 & c &
 \end{array}$$

which expresses the “identity” property of the morphism ϕ .

The bottom arrow in $(*)$ represents the composites as indicated. Therefore the commutativity of the bottom triangles should be obvious. Yet, we need to verify that the top triangle commutes. So, we need to show that

$$\begin{array}{ccc}
 t \times c & \xrightarrow{(\phi, id)} & c^2 \\
 \eta f \swarrow & & \searrow \eta \phi f \\
 & a &
 \end{array} \quad (**)$$

commutes.

Proposition 22. *With notation as above, we have: $\eta f(\phi, id) = \eta\phi f$.*

Proof. We are again going to use the “distributive property” used in Appendix B.1.

$$\begin{aligned}
 \eta f(\phi, id) &= (\eta \times f)((\pi_1^{t \times c} \phi) \times \pi_2^{t \times c}) \\
 &= (\eta \times f)(\pi_1^{t \times c} \phi) \times (\eta \times f)\pi_2^{t \times c} \\
 &= ((\eta \times f)\pi_1^{t \times c})\phi \times f \\
 &= \eta\phi \times f
 \end{aligned}$$

□

We have shown that the triangle (***) commutes. Therefore all triangles in (*) commute.

Appendix C

The following functor is employed in the proof of the second half of part 1. The functor maps from the category $\underline{\underline{\mathbb{C}}}$ to the category of covariant functors $\underline{\underline{\text{Set}}}^{\underline{\underline{\mathbb{C}}^{\text{op}}}}$. One can consult [5](pg 376) for more information on this functor.

The Yoneda Embedding

Consider the functor $\underline{\underline{\mathbb{C}}} \rightarrow \underline{\underline{\text{Set}}}^{\underline{\underline{\mathbb{C}}^{\text{op}}}}$ defined by:

$$1. \ c \longmapsto F_c \quad (\text{on objects})$$

Where $F_c : \underline{\underline{\mathbb{C}}}^{\text{op}} \rightarrow \underline{\underline{\text{Set}}}$ is the covariant functor defined by:

$$(a) \ b \longmapsto \underline{\underline{\mathbb{C}}}(b, c) \quad (\text{on objects})$$

$$(b) \ (f : b \longrightarrow d) \longmapsto \left(\hat{f} : \underline{\underline{\mathbb{C}}}(d, c) \longrightarrow \underline{\underline{\mathbb{C}}}(b, c) ; g \longmapsto fg \right) \quad (\text{on morphisms})$$

$$2. \ (g : b \longrightarrow d) \longmapsto (\hat{g} : F_b \longrightarrow F_d) \quad (\text{on morphisms})$$

Where \hat{g} is the natural transformation with component functions defined $\forall x \in \text{Ob}(\underline{\underline{\mathbb{C}}})$ by:

$$\hat{g}_x : \underline{\underline{\mathbb{C}}}(x, b) \longrightarrow \underline{\underline{\mathbb{C}}}(x, d) ; h \longmapsto hg$$

Technically, it should be proven that the Yoneda embedding is in fact a functor, and that the various morphisms in its definition are truly functors and natural transformations. All of these things are fairly straightforward to show, and their verification will

be left to (if I may borrow, and then alter a phrase from Dr. Howard Levine) “that elusive creature, the meticulous reader.”

It is an important fact that the Yoneda embedding is a faithful functor (see [3] (pg 61-62) for the definition of “faithful”). In fact, the term “embedding”, used in reference to a functor, means that the functor is faithful. Faithful functors have the nice property that “commutativity of image diagrams” implies “commutativity of domain diagrams”. That is, if $F : \underline{\underline{\mathbb{C}}} \rightarrow \underline{\underline{\mathbb{D}}}$ is a faithful functor and

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ & \searrow h & \swarrow g \\ & & c \end{array} \quad (*)$$

is a diagram in $\underline{\underline{\mathbb{C}}}$, then commutativity of

$$\begin{array}{ccc} Fa & \xrightarrow{Ff} & Fb \\ & \searrow Fh & \swarrow Fg \\ & & Fc \end{array}$$

in $\underline{\underline{\mathbb{D}}}$, implies commutativity of $(*)$ in $\underline{\underline{\mathbb{C}}}$. In fact, this property follows immediately from the definition of faithful. For if $FfFg = Ffg = Fh$ then faithfulness of F implies that $fg = h$.

The faithfulness of the Yoneda embedding is the property that will be of use to us in this paper. We are going to use the embedding to translate diagrams in the general category $\underline{\underline{\mathbb{C}}}$ into diagrams in the category $\underline{\underline{\text{Set}}}^{\underline{\underline{\mathbb{C}}\text{op}}}$. We will then proceed to show that the diagrams obtained in $\underline{\underline{\text{Set}}}^{\underline{\underline{\mathbb{C}}\text{op}}}$ commute, thereby demonstrating commutativity of the original diagrams in $\underline{\underline{\mathbb{C}}}$.

Appendix D

The following technical lemma is needed in the proof of part 1 of Lemma 2.

Lemma. *Let $F, G, H, K : \underline{\underline{C}} \longrightarrow \underline{\underline{D}}$ be functors. Then the diagram*

$$\begin{array}{ccc} F & \xrightarrow{\sigma} & G \\ \tau \downarrow & & \downarrow \tau' \\ H & \xrightarrow{\sigma'} & K \end{array}$$

commutes if and only if $\forall a \in \text{Ob}(\underline{\underline{C}})$

$$\begin{array}{ccc} Fa & \xrightarrow{\sigma_a} & Ga \\ \tau_a \downarrow & & \downarrow \tau'_a \\ Ha & \xrightarrow{\sigma'_a} & Ka \end{array}$$

commutes in $\underline{\underline{D}}$.

Proof. Assume $\sigma\tau' = \tau\sigma'$. This means that $\forall a \in \text{Ob}(\underline{\underline{C}})$ the component functions $\sigma\tau'_a$ and $\tau\sigma'_a$ are equal. Yet,

$$\sigma\tau'_a = \sigma_a\tau'_a$$

$$\tau\sigma'_a = \tau_a\sigma'_a$$

$$\therefore \sigma_a\tau'_a = \tau_a\sigma'_a, \forall a \in \text{Ob}(\underline{\underline{C}}).$$

This is one half of the desired result. Yet at this point, the proof of the converse implication should be obvious. □

BIBLIOGRAPHY

- [1] S.I. Gelfand, Y.I. Manin, *Methods of Homological Algebra*, (Springer, Berlin 1996) 82-83.
- [2] H. Herrlich, G.E. Strecker, *Category Theory: An Introduction*, 2 ed., Sigma Series in Pure Mathematics 1, (Heldermann Verlag, Berlin 1979).
- [3] S. MacLane, *Categories for the Working Mathematician*, 2 ed., Graduate Texts in Mathematics 5, (Springer, New York 1978).
- [4] D.S. Dummit, R.M. Foote, *Abstract Algebra* 2 ed., (John Wiley and Sons, New York 1999).
- [5] J.D.H. Smith, A.B. Romanowska, *Post-Modern Algebra*, (John Wiley and Sons, New York 1999).
- [6] J.M. Beck, *Triples, Algebra and Cohomology*, Ph.D. thesis, Columbia University 1967.
- [7] P. Freyd, *algebra valued functors in general and tensor products in particular*, Colloquium Mathematicum **14** (1966), 89-106.