

**Results on minimum skew rank
of matrices described by a graph**

by

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DEDICATION

This thesis is dedicated to my parents, Vaughn and Gayle, for instilling in me the belief that I could accomplish whatever I set out to do and for teaching me the work ethic to do so.

TABLE OF CONTENTS

LIST OF FIGURES	iv
ACKNOWLEDGEMENTS	v
ABSTRACT	vi
1 INTRODUCTION	1
1.1 Graph Theory: Definitions and notation	3
1.2 Literature Survey: The minimum (symmetric) rank problem	4
1.3 Known results on matching and skew-symmetric matrices	7
2 CUT-VERTEX REDUCTION FOR SKEW-SYMMETRIC MATRICES	8
3 SKEW ZERO FORCING NUMBER	12
4 KNOWN RESULTS ON MINIMUM SKEW RANK	14
5 MINIMUM SKEW RANK OF SELECTED FAMILIES OF GRAPHS .	18
5.1 Known minimum skew ranks	18
5.2 New computations of minimum skew rank	18
5.2.1 Hypercube and Pineapple	18
5.2.2 Coronas	20
BIBLIOGRAPHY	26

LIST OF FIGURES

Figure 1.1	Forbidden subgraphs of G if F is an infinite field with $\text{char}(F) \neq 2$ and $\text{mr}(F, G) \leq 2$	5
Figure 4.1	The paw	14
Figure 4.2	Calculating the Matching Number of a Tree T on 23 Vertices	16
Figure 4.3	Calculating the Matching Number: Induced subgraph of the tree T after edge $\{1, 3\}$ is removed	16
Figure 4.4	Calculating the Matching Number: Induced subgraph T'	17
Figure 5.1	Graph $P_{4,3}$, the 4, 3-pineapple	20
Figure 5.2	Graph $C_5 \circ K_2$, the corona of C_5 with K_2	21
Figure 5.3	Graph $P_2 \circ K_3$, the 3-barbell	21
Figure 5.4	Graph $P_2 \circ K_t$, the t -barbell	22
Figure 5.5	Graphs $K_s \circ K_t$ and $K_s \circ K_t - v$	23
Figure 5.6	The connected components of $K_s \circ K_t$ including the cut vertex v	23
Figure 5.7	Graph $P_{s+1} \circ K_t$ with white cut vertex $v = s$	24

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ABSTRACT

The minimum skew rank of a finite, simple, undirected graph G over a field F of characteristic not equal to 2 is defined to be the minimum possible rank of all skew-symmetric matrices over F whose i, j -entry is nonzero if and only if there exists an edge $\{i, j\}$ in the graph G . The problem of determining the minimum skew rank of a graph arose after extensive study of the minimum (symmetric) rank problem.

This thesis gives a background of techniques used to find minimum skew rank first developed by the IMA-ISU research group on minimum rank [9], proves cut-vertex reduction of a graph realized by a skew-symmetric matrix, and proves there is a bound for minimum skew rank created by the skew zero forcing number. The result of cut-vertex reduction is used to calculate the minimum skew ranks of families of coronas, and the minimum skew ranks of multiple other families of graphs are also computed.

1 INTRODUCTION

The minimum skew rank problem, to calculate the minimum rank of skew-symmetric matrices which realize a graph, arose after extensive study of the minimum (symmetric) rank problem. The minimum (symmetric) rank problem is to determine the minimum possible rank of all real symmetric matrices that realize a graph G [7]. This problem has been modified to consider all fields [4],[5],[7],[8] and to consider graphs with loops and multiple edges [11]. The problem has also been altered to consider positive definite matrices, Hermitian matrices, Hermitian positive semidefinite matrices and other non-symmetric matrices that realize a graph G [7],[9].

Since determining the minimum rank is an equivalent problem to determining the maximum nullity or maximum geometric multiplicity of the zero eigenvalue for a family of matrices, motivation for this problem came from the Inverse Eigenvalue Problem of a Graph (IEPG). The IEPG is to determine the possible eigenvalues of a real symmetric matrix that realizes a given graph G . Section 1.2 will give a more detailed background on the minimum (symmetric) rank problem and existing results.

Let $M_n(F)$ be the set of all square $n \times n$ matrices with entries from the field F . A matrix A is symmetric if $A = A^T$, and A is skew-symmetric if $A = -A^T$. The graph of a symmetric or skew-symmetric matrix $A \in M_n(F)$, denoted $\mathcal{G}(A)$, is the graph with vertices $\{1, \dots, n\}$ having an edge $\{i, j\}$ if and only if $a_{ij} \neq 0$. A symmetric or skew-symmetric matrix $A = [a_{ij}] \in M_n(F)$ is said to realize a graph G of order n if $\mathcal{G}(A) = G$. The set of symmetric matrices with entries from a field F which realize the graph G is

$$\mathcal{S}(F, G) = \{A \in M_n(F) : A = A^T, \mathcal{G}(A) = G\}.$$

The set of skew-symmetric matrices with entries from F which realize the graph G is

$$\mathcal{S}^-(F, G) = \{A \in M_n(F) : A = -A^T, \mathcal{G}(A) = G\}.$$

We denote the minimum (symmetric) rank and maximum (symmetric) multiplicity of a graph G over the field F , respectively, as

$$\text{mr}(F, G) = \min\{\text{rank}(B) : B \in \mathcal{S}(F, G)\}, \text{ and}$$

$$M(F, G) = \max\{\text{mult}_B(\lambda) : \lambda \in \mathbb{R}, B \in \mathcal{S}(F, G)\}$$

where $\text{mult}_B(\lambda)$ is the geometric multiplicity of λ if λ is an eigenvalue of B , and $\text{mult}_B(\lambda) = 0$ otherwise [7]. The maximum multiplicity is also referred to as the maximum nullity since the maximum multiplicity of any eigenvalue is the same by translation of the matrix by a scalar matrix [7].

The minimum skew rank of a finite, simple, undirected graph G is defined to be the minimum possible rank of all skew-symmetric matrices over a field F whose i, j -entry is nonzero if and only if there exists an edge $\{i, j\}$ in the graph G , that is,

$$\text{mr}^-(F, G) = \min\{\text{rank}(A) : A \in \mathcal{S}^-(F, G)\}.$$

The corresponding maximum skew nullity of a graph G over the field F is

$$M^-(F, G) = \max\{\text{null}(A) : A \in \mathcal{S}^-(F, G)\}.$$

We also define the maximum skew rank of a finite, simple, undirected graph G to be

$$\text{MR}^-(F, G) = \max\{\text{rank}(A) : A \in \mathcal{S}^-(F, G)\}.$$

In this thesis, the field F will never have characteristic two since in a field of characteristic two the minimum (symmetric) rank problem and the minimum skew rank problem are the same. The graph $G = (V_G, E_G)$ will be finite, simple, and undirected, that is, there will be a finite number of vertices and neither loops nor multiple edges are allowed.

When calculating the minimum (symmetric) rank, a matrix $A \in \mathcal{S}(F, G)$ can have zero or nonzero diagonal entries; the diagonal is unconstrained. In the skew-symmetric case, for $A \in \mathcal{S}^-(F, G)$ each diagonal entry $a_{ii} = -a_{ii}$, and thus each diagonal entry must be zero.

1.1 Graph Theory: Definitions and notation

This section contains graph theory terms that may be necessary to review before continuing to read this thesis. These concepts and notations will be used throughout the paper.

A graph is a pair (V_G, E_G) , where V_G is the set of vertices of G and E_G is the edge set of G . Each edge is a two-vertex set $\{i, j\}$. If $\{i, j\} \in E_G$, then we say i is adjacent to j . The order of a graph G , denoted $|G|$, is the number of vertices in V_G .

A subgraph H of G is a graph that has a subset of V_G as its set of vertices and a subset of E_G as its set of edges. An induced subgraph, denoted $G[W]$, has vertex set $W \subseteq V_G$ and edges $\{i, j\} \in E_G$ where $i, j \in W$. If A is a matrix that realizes G , then the principal submatrix that is indexed by the vertex set W , $A[W]$, is a matrix that realizes $G[W]$. If we are interested in deleting a set W from the indices of the matrix A or deleting set W from the vertex set V_G , it is denoted $A(W)$ or $G(W)$, respectively. If $W = \{k\}$, then $A(W)$ is denoted $A(k)$ and $G(W) = G - k$.

A path of order n , denoted P_n , is a graph with vertex set $\{v_1, \dots, v_n\}$ and edge set $\{\{v_i, v_{i+1}\} : 1 \leq i \leq n - 1\}$. A cycle of order n , C_n , is a graph with vertex set $\{v_1, \dots, v_n\}$ and edge set $\{\{v_i, v_{i+1}\} : 1 \leq i \leq n - 1\} \cup \{v_n, v_1\}$.

A graph, G , is connected if any two vertices of G may be joined by a path of edges in E_G ; otherwise, G is disconnected. A subgraph H of G is a connected component if it is a maximal connected subgraph. If G is the disjoint union of connected components H_1, \dots, H_k and each A_i represents a realization of H_i , then the direct sum $A = A_1 \oplus \dots \oplus A_k$ is a matrix that realizes G . Also, the rank of A is the sum of the ranks of A_i for $1 \leq i \leq k$.

A complete graph of order n , denoted K_n , has vertex set $\{v_1, \dots, v_n\}$ and edge set $E_{K_n} = \{\{v_i, v_j\} : i \neq j, \text{ for all } 1 \leq i, j \leq n\}$. The complement of a graph $G = (V, E)$ with order n , denoted \overline{G} , has vertex set V and edge set \overline{E} , the set of all edges in $E_{K_n} \setminus E$.

A bipartite graph has a disjoint union of two sets for a vertex set, $V = V_1 \dot{\cup} V_2$ such that each edge $\{i, j\}$ has one element in V_1 and the other in V_2 ; no edge may have both endpoints in the same V_i . A complete bipartite graph has vertex set $V = V_1 \dot{\cup} V_2$, and $E = \{\{v, w\} : v \in V_1, w \in V_2\}$, i.e., there must be an edge $\{v, w\}$ for every combination of vertices $v \in V_1, w \in V_2$.

If $|V_1| = p$ and $|V_2| = q$, then the complete bipartite graph is denoted $K_{p,q}$. A complete multipartite graph has vertex set $V_1 \dot{\cup} V_2 \dot{\cup} \dots \dot{\cup} V_h$ where $h \geq 3$, and an edge $\{v_i, v_j\}$ must occur for every $i \neq j$, $v_i \in V_i, v_j \in V_j$.

A matching of the graph $G = (V_G, E_G)$ is a set of edges in E_G where no two edges share an endpoint. A perfect matching is one which includes every vertex in V_G . A maximum matching of a graph G is a matching which includes the largest number of edges over all matchings of G . The number of edges, or cardinality of the maximum matching, is called the matching number of the graph G , denoted $\text{match}(G)$.

1.2 Literature Survey: The minimum (symmetric) rank problem

The study of the minimum (symmetric) rank problem began with the exploration of the minimum ranks of graphs known as trees. A tree is a connected, acyclic graph, i.e., a connected graph without any cycles as subgraphs. The minimum ranks of all simple trees, i.e., trees without loops, are known [11],[12]. Algorithms to compute the minimum rank of a tree can be found in the survey on minimum rank by S. Fallat and L. Hogben [7].

The following observations include well-known facts from linear algebra that can be applied toward all graphs.

Observation 1.1. [7, Observations 3.5, 3.8]

1. $M(F, G) + \text{mr}(F, G) = |G|$.
2. $\text{mr}(F, G) \leq |G| - 1$.
3. $\text{mr}(F, P_n) = n - 1$.
4. For $n \geq 2$, $\text{mr}(F, K_n) = 1$. If G is connected and $\text{mr}(F, G) = 1$, then $G = K_{|G|}$.
5. $\text{mr}(F, K_{p,q}) = 2$.
6. If the connected components of the graph G are G_1, \dots, G_s , then $\text{mr}(F, G) = \sum_{i=1}^s \text{mr}(F, G_i)$.
7. $\text{mr}(F, G) \leq |G| - c$, where c is the number of connected components in the graph G .

8. If H is an induced subgraph of G , then $\text{mr}(F, H) \leq \text{mr}(F, G)$.

Observation 1.2. [7] If F is an infinite field and $G = \bigcup_{i=1}^h G_i$, then $\text{mr}(F, G) \leq \sum_{i=1}^h \text{mr}(F, G_i)$.

The following observation is true for the field of real numbers, and is stated this way in [7]. However, clearly it holds for any field F .

Observation 1.3. [7, Observation 1.6] Let $A \in M_n(F)$ be symmetric, and let G be a graph.

1. $\text{rank}(A) - 2 \leq \text{rank}(A(k)) \leq \text{rank}(A)$ for any $k \in \{1, \dots, n\}$.
2. For any vertex $v \in V_G$, $0 \leq \text{mr}(F, G) - \text{mr}(F, G - v) \leq 2$.
3. Adding or removing an edge from G can change minimum rank by at most 1.
4. If $R \subseteq \{1, \dots, n\}$ and $k \in \{1, \dots, n\}$, then $\mathcal{G}(A[R]) = \mathcal{G}(A)[R]$ and $\mathcal{G}(A(k)) = \mathcal{G}(A) - k$.

For connected graphs, we have the following theorem to distinguish which graphs have $\text{mr}(F, G) \leq 2$.

Theorem 1.4. [4] Let G be a connected graph, and let F be an infinite field with characteristic not equal to 2. Then $\text{mr}(F, G) \leq 2$ if and only if G does not contain any graph from Figure 1.1 as an induced subgraph.

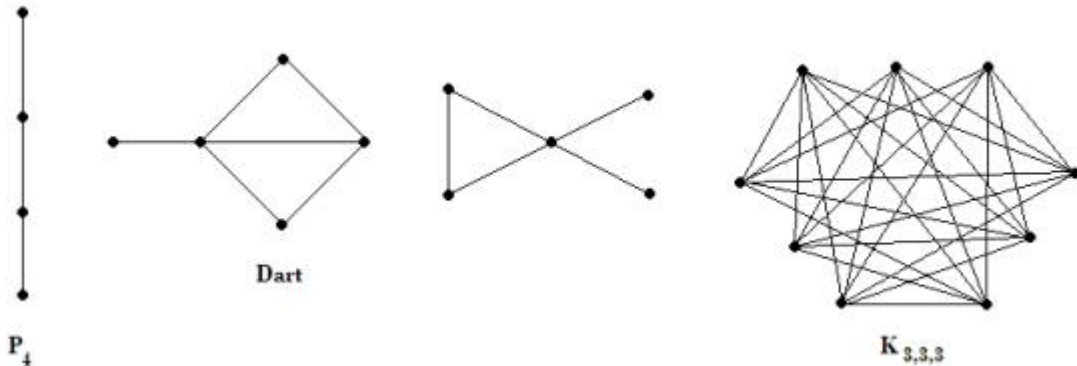


Figure 1.1 Forbidden subgraphs of G if F is an infinite field with $\text{char}(F) \neq 2$ and $\text{mr}(F, G) \leq 2$

There is a result for computing the minimum rank of a graph with a cut vertex which is described in [3]. A *cut vertex* is a vertex v such that when v and the edges incident to v are removed from the graph the number of connected components increases. The *rank-spread* of a graph G at vertex v over the field F , denoted $r_v(F, G)$, is the difference $\text{mr}(F, G) - \text{mr}(F, G - v)$. For any vertex v , $0 \leq r_v(F, G) \leq 2$ since $\text{rank}(A(k)) \leq \text{rank}(A) \leq \text{rank}(A(k)) + 2$ over any field F .

Theorem 1.5. [7, Theorem 3.12],[3] *Let G have a cut vertex v . For $i = 1, \dots, h$, let $W_i \subseteq V_G$ be the vertices of the i^{th} component of $G - v$, and let G_i be the subgraph induced by $\{v\} \cup W_i$. Then $r_v(F, G) = \min\{\sum_{i=1}^h r_v(F, G_i), 2\}$, and thus*

$$\text{mr}(F, G) = \sum_{i=1}^h \text{mr}(F, G_i - v) + \min \left\{ \sum_{i=1}^h r_v(F, G_i), 2 \right\}.$$

Using cut-vertex reduction, we can reduce the problem of determining the minimum rank of a graph G to computing the minimum ranks of multiple smaller graphs.

Another strategy to bound the minimum rank was defined in [1] by the AIM Minimum Rank Special Graphs Work Group. The zero forcing number, defined below, creates an upper bound for the maximum nullity of a graph, giving us a lower bound for minimum rank.

Definition 1.6. [1, Definition 2.1]

1. The *color-change rule* states that if a graph G has all vertices colored either black or white, u is a black vertex of G , and exactly one neighbor, v , of u is white, then change the color of v to black.
2. Given that each vertex in V_G is colored black or white, the *derived coloring* of G is the unique set of black vertices resulting from applying the color-change rule until no further changes may occur.
3. A *zero forcing set*, Z , of a graph G is a set of vertices $Z \subseteq V_G$ such that if Z is the set that is initially colored black, then the derived coloring is the entire set V_G .
4. The *zero forcing number*, $Z(G)$, is the minimum $|Z|$ over all zero forcing sets $Z \subseteq V_G$.

Proposition 1.7. [1, Proposition 2.4] *Let $G = (V_G, E_G)$ be a graph and let $Z \subseteq V_G$ be a zero forcing set. Then $M(F, G) \leq |Z|$, and thus $M(F, G) \leq Z(G)$.*

Consequently, $\text{mr}(F, G) \geq |G| - Z(G)$.

These techniques have been used to compute the minimum (symmetric) ranks of over fifty families of graphs. The minimum ranks can be found in the online AIM minimum rank graph catalog [2].

1.3 Known results on matching and skew-symmetric matrices

Lastly, we state some well-known results on the rank of skew-symmetric matrices which follow from results on matchings, and properties of skew-symmetric matrices, submatrices and connected components. These will frequently be used in subsequent chapters. Let F be a field with characteristic not equal to 2.

Theorem 1.8. [9, Theorem 1.1] *Let $A \in M_n(F)$ be skew-symmetric. Then*

$$\text{rank}(A) = \max\{|S| : \det(A[S]) \neq 0\}.$$

Observation 1.9. [9, Observation 1.7]

1. *The rank of a skew-symmetric matrix is always even; $\text{mr}^-(F, G)$ and $\text{MR}^-(F, G)$ are always even.*
2. *If there exists a unique perfect matching for the graph G , then G has full minimum rank, i.e., $\text{mr}^-(F, G) = |G|$.*
3. *If H is an induced subgraph of G , then $\text{mr}^-(F, H) \leq \text{mr}^-(F, G)$.*
4. *$\text{mr}^-(G) = 0$ if and only if G is discrete, i.e., $E_G = \emptyset$.*
5. *If a graph G has connected components G_1, \dots, G_h , then $\text{mr}^-(F, G) = \sum_{i=1}^h \text{mr}^-(F, G_i)$.*

2 CUT-VERTEX REDUCTION FOR SKEW-SYMMETRIC MATRICES

One way to find the minimum skew rank of a graph G is by examining the induced subgraphs of G . Cut-vertex reduction calculates the minimum skew rank of a graph by finding the minimum skew ranks of multiple induced subgraphs. A similar concept aided in calculating minimum (symmetric) rank and the details are displayed in [7] and the Literature Survey, Section 1.2.

Definition 2.1. Let the *skew rank-spread of G at vertex v* be defined as

$$r_v^-(F, G) = \text{mr}^-(F, G) - \text{mr}^-(F, G - v).$$

For the symmetric case, we know $\text{mr}(F, G - v) \leq \text{mr}(F, G) \leq \text{mr}(F, G - v) + 2$ [12]. The rank-spread and the skew rank-spread of a graph G must be between 0 and 2 since $A(v)$ is obtained from A by deleting one row and one column. Since skew-symmetric matrices have even rank, the difference between the rank of G and the rank of $G - v$ cannot be 1. Therefore, $r_v^-(F, G) \in \{0, 2\}$.

Definition 2.2. Let G be a graph of order n , and let v be the first vertex. Define $R_v(F, G)$ to be the subset of skew-symmetric matrices $A = [a_{ij}]$ of $M_n(F)$, which satisfy the following properties:

1. $\mathcal{G}(A) = G$, and
2. $\mathbf{b} = [a_{21}, \dots, a_{n1}]^T$ is in the range of $A' := A(1)$.

Clearly, $\mathcal{G}(A') = G - v$, and since $A \in R_v(F, G)$ is skew-symmetric, $a_{11} = 0$. Thus the elements of $R_v(F, G)$ are of the form:

$$A = \begin{bmatrix} 0 & -\mathbf{b}^T \\ \mathbf{b} & A' \end{bmatrix} \tag{2.1}$$

Proposition 2.3. For any matrix $A \in R_v(F, G)$, $\text{rank}(A) = \text{rank}(A')$.

Proof. Since \mathbf{b} is in the range of A' , $\mathbf{b} = A'\mathbf{x}$ for some vector $\mathbf{x} \in F^{n-1}$. It is clear from the skew-symmetric property of A' that

$$-\mathbf{b}^T \mathbf{x} = -(A'\mathbf{x})^T \mathbf{x} = -\mathbf{x}^T (A')^T \mathbf{x} = -\mathbf{x}^T (-A') \mathbf{x} = \mathbf{x}^T A' \mathbf{x} = \mathbf{x}^T \mathbf{b} = \mathbf{b}^T \mathbf{x}.$$

Thus, $\mathbf{b}^T \mathbf{x} = 0$, and this proves the first column of A is in the range of the submatrix $\begin{bmatrix} -\mathbf{b}^T \\ A' \end{bmatrix}$.
Therefore $\text{rank}(A) = \text{rank}(A')$. □

In Section 1.2, we defined a cut vertex. An equivalent characterization of a cut vertex of a connected graph G follows. The vertex v is a cut vertex of G if and only if $G = \bigcup_{i=1}^h G_i$ where G_1, \dots, G_h ($h \geq 2, |G_i| \geq 2$) are connected graphs and $\bigcap_{i=1}^h V_{G_i} = v$. If v is removed from the graph, then the number of connected components increases. The connected components will be $G_i - v$.

Theorem 2.4. Let G be a graph with cut vertex v , where $G = \bigcup_{i=1}^h G_i$ and $\bigcap_{i=1}^h V_{G_i} = v$. Then

$$r_v^-(F, G) = \min \left\{ \sum_{i=1}^h r_v^-(F, G_i), 2 \right\}.$$

Consequently, $\text{mr}^-(F, G) = \sum_{i=1}^h \text{mr}^-(F, G_i - v) + \min \left\{ \sum_{i=1}^h r_v^-(F, G_i), 2 \right\}$.

Proof. If necessary, relabel the vertices such that v is in the first position. From the above characterization of a cut vertex, $G - v$ is the set of disjoint graphs $G_i - v$. Hence $A \in \mathcal{S}^-(F, G)$ can be written

$$A = \begin{bmatrix} 0 & -\mathbf{b}^T \\ \mathbf{b} & A' \end{bmatrix} = \begin{bmatrix} 0 & -\mathbf{b}_1^T & \dots & -\mathbf{b}_h^T \\ \mathbf{b}_1 & A'_1 & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{b}_h & \mathbf{0} & \dots & A'_h \end{bmatrix} \quad (2.2)$$

where $A' \in \mathcal{S}^-(F, G - v)$, and $A'_i \in \mathcal{S}^-(F, G_i - v)$, for $1 \leq i \leq h$.

We will show the following are equivalent:

1. $r_v^-(F, G) = 0$
2. $\min\{\text{rank}(A') : A \in R_v(F, G)\} = \text{mr}^-(F, G - v)$
3. $\sum_{i=1}^h r_v^-(F, G_i) = 0$

(1 \Rightarrow 2) Let $\text{mr}^-(F, G) = \text{mr}^-(F, G - v)$. There exists an optimal matrix $A \in \mathcal{S}^-(F, G)$ with $\text{rank}(A) = \text{mr}^-(F, G)$. Let $A' = A(1)$, remember $v = 1$, hence $A' \in \mathcal{S}^-(F, G - v)$. Then $\text{mr}^-(F, G - v) \leq \text{rank}(A') \leq \text{rank}(A) = \text{mr}^-(F, G) = \text{mr}^-(F, G - v)$. Thus, $\text{rank}(A) = \text{rank}(A') = \text{mr}^-(F, G - v)$. Hence \mathbf{b} must be in the range of A' because if it were not, then $\text{rank}(A) > \text{rank}(A')$. This would be a contradiction. Therefore, if $\text{mr}^-(F, G) = \text{mr}^-(F, G - v)$, then any optimal matrix will be in the family $R_v(F, G)$ and $\text{rank}(A') = \text{mr}^-(F, G - v)$. Thus, $\min\{\text{rank}(A') : A \in R_v(F, G)\} = \text{mr}^-(F, G - v)$.

(2 \Rightarrow 3) Suppose there exists a matrix $A \in R_v(F, G)$, a cut vertex v , and a submatrix $A' = A(v)$ such that $\text{rank}(A') = \text{mr}^-(F, G - v)$. Since A' is the direct sum of block matrices A'_1, \dots, A'_h , $\text{rank}(A') = \sum_{i=1}^h \text{rank}(A'_i)$, and since $G_i - v$ are the connected components of $G - v$, $\text{mr}^-(F, G - v) = \sum_{i=1}^h \text{mr}^-(F, G_i - v)$. Since $\text{rank}(A') = \text{mr}^-(F, G - v)$, $\sum_{i=1}^h \text{rank}(A'_i) = \sum_{i=1}^h \text{mr}^-(F, G_i - v)$. Hence,

$$\sum_{i=1}^h (\text{rank}(A'_i) - \text{mr}^-(F, G_i - v)) = 0.$$

By the definition of minimum skew rank and because $A'_i \in \mathcal{S}^-(F, G_i - v)$, $\text{rank}(A'_i) \geq \text{mr}^-(F, G_i - v)$. Thus, $\text{rank}(A'_i) - \text{mr}^-(F, G_i - v) \geq 0$, i.e., each term in the sum above is nonnegative. Assume for some j , $\text{rank}(A'_j) - \text{mr}^-(F, G_j - v) > 0$, then for at least one k , $\text{rank}(A'_k) < \text{mr}^-(F, G_k - v)$. This is a contradiction. Hence $\text{rank}(A'_i) = \text{mr}^-(F, G_i - v)$, for $i = 1, \dots, h$.

Next, let $\mathbf{b} = A'\mathbf{x}$, as in the form of Equation 2.2, and partition the vector \mathbf{x} into subvectors \mathbf{x}_i that match the size of each \mathbf{b}_i . Clearly, \mathbf{b}_i is in the range of A'_i . Therefore, Proposition 2.3 shows $\text{rank} \left(\begin{bmatrix} 0 & -\mathbf{b}_i^T \\ \mathbf{b}_i & A'_i \end{bmatrix} \right) = \text{rank}(A'_i)$ and $r_v^-(F, G_i) = 0$ for $i = 1, \dots, h$.

(3 \Rightarrow 1) If each $r_v^-(F, G_i) = 0$, with the same argument as (1 \Rightarrow 2), we can find matrices

$$\begin{bmatrix} 0 & -\mathbf{b}_i^T \\ \mathbf{b}_i & A'_i \end{bmatrix} \in R_v(F, G_i) \text{ where } \text{rank}(A'_i) = \text{mr}^-(F, G_i - v). \text{ Then we may construct } A \text{ as in}$$

Equation (2.2) over the field F with A' equal to the direct sum of A'_1, \dots, A'_h and $\text{rank}(A') = \sum_{i=1}^h \text{rank}(A'_i) = \sum_{i=1}^h \text{mr}^-(F, G_i - v) = \text{mr}^-(F, G - v)$. Clearly, if each vector \mathbf{b}_i is in the range of A'_i , then $\mathbf{b} = [\mathbf{b}_1^T, \mathbf{b}_2^T, \dots, \mathbf{b}_h^T]^T$ is in the range of A' . Therefore, Proposition 2.3 shows $\text{rank}(A) = \text{mr}^-(F, G - v)$, and thus $r_v^-(F, G) = 0$. \square

3 SKEW ZERO FORCING NUMBER

In efforts to compute a bound for the maximum (symmetric) nullity and, in turn, minimum (symmetric) rank, an AIM minimum rank work group [1] defined and applied a parameter that is an upper bound for $M(F, G)$. The zero forcing number, $Z(G)$, was defined for symmetric matrices over any field F . This gives a lower bound for the minimum (symmetric) rank, $\text{mr}(F, G) \geq |G| - Z(G)$. The IMA-ISU research group on minimum skew rank [9] introduced a modified version of the zero forcing number to give a lower bound for $\text{mr}^-(F, G)$. Here, the details of the proof that the skew zero forcing number is indeed an upper bound for $M^-(F, G)$ are provided.

Definition 3.1. Let G be a graph.

1. A subset of vertices $Z \subseteq V_G$ is an *initial coloring* when all vertices in Z are colored black, and the vertices $V_G \setminus Z$ are colored white.
2. The *skew color change rule* says if any vertex v in V_G has exactly one white neighbor, w , then we change the color of w to black. In this case, it is said that v *forces* w .
3. The *skew derived set* of the initial coloring Z is the set of black vertices resulting after the skew color change rule cannot be applied further.
4. A *skew zero forcing set* is a subset of the vertices $Z \subseteq V_G$ such that the skew derived set of Z is the entire set V_G .
5. The *skew zero forcing number*, $Z^-(G)$, is the minimum $|Z|$ over all skew zero forcing sets Z for the graph G .

The idea is that each black vertex corresponds to a zero entry in a vector and a white vertex corresponds to a zero or nonzero entry. Forcing a vertex to be colored black is nothing more than the corresponding entry of the vector being forced to equal zero if the vector is in the kernel of $A \in \mathcal{S}^-(F, G)$. For more detail on the derivation of this process and these definitions see [1].

Proposition 3.2. [1, Proposition 2.2] *If F is a field, $A \in M_n(F)$, and the nullity of A is greater than k , then there is a nonzero vector $\mathbf{x} \in \ker(A)$ vanishing at any k specified positions. In other words, if W is a set of k indices, then there is a nonzero vector $\mathbf{x} \in \ker(A)$ such that the intersection of W and the set of indices, i , where $x_i \neq 0$, is empty.*

Proposition 3.3. *If G is a graph and F is any field, then $M^-(F, G) \leq Z^-(G)$. Thus, $\text{mr}^-(F, G) \geq |G| - Z^-(G)$.*

Proof. Let Z be an optimal skew zero forcing set of size $Z^-(G)$. Assume $M^-(F, G) > Z^-(G)$. Let $A \in \mathcal{S}^-(F, G)$ such that the nullity of A is greater than $|Z|$. By Proposition 3.2, there exists a nonzero vector \mathbf{x} such that $A\mathbf{x} = \mathbf{0}$ where the set of indices of nonzero entries and the set of indices of vertices in Z are disjoint.

Since Z is a proper subset of V_G and Z is a skew zero forcing set, we must be able to perform at least one round of the color change rule. Hence, there exists a vertex u such that u has exactly one white neighbor w . The u entry $(A\mathbf{x})_u$ is zero, and $(A\mathbf{x})_u = a_{uw}x_w$. Since a_{uw} is nonzero, we must have $x_w = 0$. Each round of the color change rule will require another entry of the vector \mathbf{x} to be zero, so $\mathbf{x} = \mathbf{0}$. Thus we reach a contradiction. \square

The following connection between the zero forcing number and the skew zero forcing number was also discovered in [9].

Proposition 3.4. [9, Proposition 3.6] *Let G be a graph. Then $Z^-(G) \leq Z(G)$. Consequently, if $\text{mr}(F, G) = |G| - Z(G)$ for a field F , then $\text{mr}^-(F, G) \geq \text{mr}(F, G)$.*

4 KNOWN RESULTS ON MINIMUM SKEW RANK

This chapter summarizes results that have already been submitted for publication [9] by the IMA-ISU research group on minimum skew rank, which includes the author of this thesis.

Proposition 4.1. [9, Proposition 3.3] *Let $G = \bigcup_{i=1}^h G_i$. If the edge sets of each pair of subgraphs G_i, G_j are disjoint, i.e., $E_{G_i} \cap E_{G_j} = \emptyset$, for all $i \neq j$, or if F is an infinite field, then $\text{mr}^-(F, G) \leq \sum_{i=1}^h \text{mr}^-(F, G_i)$.*

Theorem 4.2. [9, Theorem 2.1] *If G is a connected graph with order greater than 1 and F is an infinite field, then the following statements are equivalent:*

1. $\text{mr}^-(F, G) = 2$.
2. G is a complete multipartite graph, K_{n_1, n_2, \dots, n_m} , with at least two nonempty, disjoint partite sets. (This includes all complete graphs K_n for $n \geq 2$ since $K_n \cong K_{1, 1, \dots, 1}$.)
3. The graphs P_4 and the paw (see Figure 4.1) are not induced subgraphs of G .

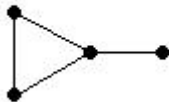


Figure 4.1 The paw

Corollary 4.3. [9, Theorem 2.1] *If G is disconnected and F is an infinite field, $\text{mr}^-(F, G) = 2$ if and only if G is the disjoint union of a complete multipartite graph and isolated vertices.*

Note that this result does not hold for finite fields, as is illustrated in the following example.

Example 4.4. [9, Example 2.2] $\text{mr}^-(\mathbb{Z}_3, K_5) = 4$, which can be found by computing the minimum skew rank for all 2^{10} possible matrices with entries in \mathbb{Z}_3 that realize K_5 .

Theorem 4.5. [9, Theorem 2.5] Let G be a graph and F be any field with $\text{char}(F) \neq 2$. Then $\text{MR}^-(F, G) = 2 \cdot \text{match}(G)$, and for all even numbers in the interval $[\text{mr}^-(F, G), \text{MR}^-(F, G)]$ there exists a matrix of that rank which realizes G .

Theorem 4.6. [9, Theorem 2.6] *Let G be a graph and F be any field with $|F| \geq 5$ and $\text{char}(F) \neq 2$. Then G has full rank if and only if $|G|$ is even and there exists a unique perfect matching for G .*

Corollary 4.7. [9, Corollary 2.7] *Let G be a graph with a matching M that covers k edges, and let F be any field with $\text{char}(F) \neq 2$. If M is the only perfect matching for the subgraph induced by the $2k$ vertices in M , then $\text{mr}^-(F, G) \geq 2k$.*

In any graph, we refer to a single vertex of degree one and its incident edge as a leaf; if we have more than one, they are referred to as leaves. The process described in the next theorem is illustrated in Example 4.9.

Theorem 4.8. [9, Theorem 2.8] *If T is a tree and F is any field with $\text{char}(F) \neq 2$, then $\text{mr}^-(F, T) = 2 \cdot \text{match}(T) = \text{MR}^-(F, T)$. The matching number, $\text{match}(T)$, can be determined by removing leaves and the vertices with which they are adjacent from the tree one by one.*

Example 4.9. To find the matching number of a tree T , we construct a matching \mathcal{M} in the following way. Remove a degree-one vertex and the single vertex with which it is adjacent. Since a tree with order no less than 2 always has a leaf, we know this process can be done. Repeat this step by removing another edge $\{v_i, v_j\}$ with a degree-one vertex v_i . Continue this process until you are left with only isolated vertices. The edges removed will form the matching \mathcal{M} , and the matching number will be $\text{match}(T) = |\mathcal{M}|$.

Let us find $\text{match}(T)$ for the tree T in Figure 4.2. We begin by searching for a degree-one vertex. In T , we find the set $\{1, 2, 6, 7, 9, 10, 13, 14, 15, 23\}$ are all degree one vertices. If we choose to begin with vertex 1, the edge $\{1, 3\}$ is removed first. Once we have removed vertices 1 and 3, vertex 2 becomes isolated, i.e., degree-zero. See Figure 4.3.

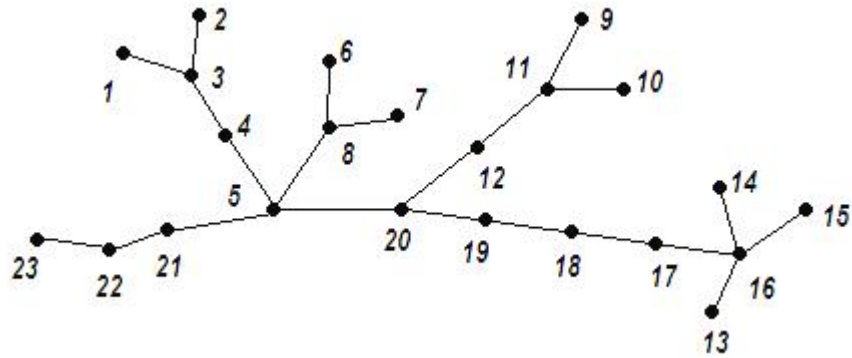


Figure 4.2 Calculating the Matching Number of a Tree T on 23 Vertices

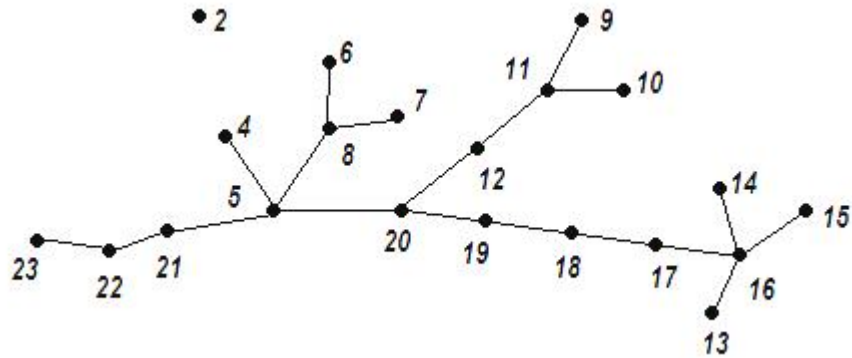


Figure 4.3 Calculating the Matching Number: Induced subgraph of the tree T after edge $\{1, 3\}$ is removed

Next, we choose another degree-one vertex, and repeat the process. After removing the edges $\{6, 8\}$, $\{9, 11\}$, $\{13, 16\}$ and $\{22, 23\}$, in order, we are left with a new graph T' (see Figure 4.4), which now contains many degree-zero vertices.

We continue to form the matching \mathcal{M} by removing the edges $\{4, 5\}$, $\{12, 20\}$, and $\{17, 18\}$, in order, which leaves us with only isolated vertices. Thus,

$$\mathcal{M} = \{\{1, 3\}, \{6, 8\}, \{9, 11\}, \{13, 16\}, \{22, 23\}, \{4, 5\}, \{12, 20\}, \{17, 18\}\},$$

$\text{match}(T) = 8$, and $\text{mr}^-(F, T) = 2 \cdot 8 = 16$

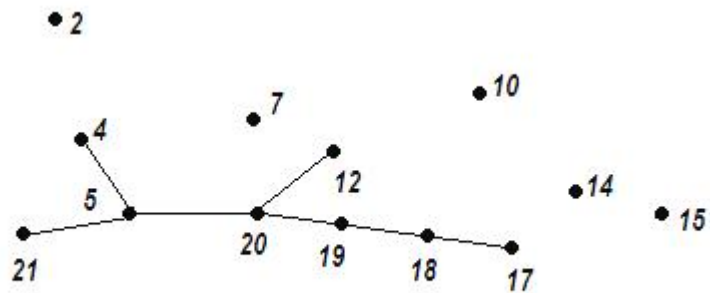


Figure 4.4 Calculating the Matching Number: Induced subgraph T'

For the proof of Theorem 4.8 in [9], it is shown that a tree T has an induced subgraph H such that $\text{mr}^-(F, T) = |H| = \text{MR}^-(F, T)$. This is not necessarily true for a graph that is not a tree. The Petersen graph is one example of a graph which does not have an induced subgraph such that the above equality holds [9, Example 2.10].

5 MINIMUM SKEW RANK OF SELECTED FAMILIES OF GRAPHS

5.1 Known minimum skew ranks

The minimum skew ranks of a path and a cycle were calculated by the IMA-ISU research group on minimum rank [9]. The minimum skew ranks of additional graphs have also been computed and appear in [9].

Theorem 5.1. [9, Propositions 4.1, 4.2] *Let F be any field with $\text{char}(F) \neq 2$.*

1. *For the path on n vertices, $\text{mr}^-(F, P_n) = \begin{cases} n & : \text{if } n \text{ is even} \\ n - 1 & : \text{if } n \text{ is odd} \end{cases}$.*
2. *For the cycle on n vertices, $\text{mr}^-(F, C_n) = \begin{cases} n - 1 & : \text{if } n \text{ is odd} \\ n - 2 & : \text{if } n \text{ is even} \end{cases}$.*

5.2 New computations of minimum skew rank

5.2.1 Hypercube and Pineapple

Definition 5.2. Let G and H be graphs. The *Cartesian product* $G \square H$, has the set of vertices $V_G \times V_H$, and two vertices, (g_1, h_1) and (g_2, h_2) , in the Cartesian product are adjacent if and only if

$$g_1 = g_2 \text{ and } \{h_1, h_2\} \in E_H$$

or

$$\{g_1, g_2\} \in E_G \text{ and } h_1 = h_2.$$

Definition 5.3. The *hypercube* is defined inductively, $Q_1 = K_2$ and $Q_{n+1} = Q_n \square K_2$.

Note that $|Q_n| = 2^n$. We may prove the following theorem similarly to [6, Theorem 3.14].

Theorem 5.4. *Let F be a field of order at least 7 and characteristic not equal to 2. Then for $n > 1$, the minimum skew rank of the hypercube is $\text{mr}^-(F, Q_n) = 2^{n-1}$.*

Proof. Let α and β be nonzero scalars in a field F such that $\alpha^2 + \beta^2 = 1$. Such α, β exist for any field F with $|F| \geq 7$ and with characteristic not equal to 2; see [6]. We define the following recursive matrices. Let

$$L_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } L_n = \begin{bmatrix} \alpha L_{n-1} & \beta I \\ -\beta I & -\alpha L_{n-1} \end{bmatrix}.$$

Each $L_n \in M_{2^n}(F)$ is a skew-symmetric matrix. For $n \geq 2$, define

$$H_n = \begin{bmatrix} L_{n-1} & I \\ -I & L_{n-1} \end{bmatrix}.$$

Hence, each $H_n \in M_{2^n}(F)$ is a skew-symmetric matrix such that $H_n \in \mathcal{S}^-(F, Q_n)$. We show

by induction that $L_n^2 = -I_{2^n}$. We can see $L_1^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I_2$. Next, we assume $L_{n-1}^2 = -I_{2^{n-1}}$, and it is easy to see that

$$L_n^2 = \begin{bmatrix} \alpha L_{n-1} & \beta I \\ -\beta I & -\alpha L_{n-1} \end{bmatrix}^2 = \begin{bmatrix} \alpha^2 L_{n-1}^2 - \beta^2 I & 0 \\ 0 & -\beta^2 I + \alpha^2 L_{n-1}^2 \end{bmatrix} = \begin{bmatrix} -(\alpha^2 + \beta^2)I & 0 \\ 0 & -(\alpha^2 + \beta^2)I \end{bmatrix}.$$

Thus, $L_n^2 = -I_{2^n}$, and since

$$\begin{bmatrix} I & 0 \\ -L_{n-1} & I \end{bmatrix} \begin{bmatrix} L_{n-1} & I \\ -I & L_{n-1} \end{bmatrix} = \begin{bmatrix} L_{n-1} & I \\ -L_{n-1}^2 - I & -L_{n-1} + L_{n-1} \end{bmatrix} = \begin{bmatrix} L_{n-1} & I \\ 0 & 0 \end{bmatrix},$$

$\text{rank } H_n = 2^{n-1}$; see [1] for a similar minimum (symmetric) rank argument.

Therefore, $\text{mr}^-(F, Q_n) \leq 2^{n-1}$, and so the maximum nullity $M^-(F, Q_n) \geq 2^{n-1}$. Because Q_{n-1} is a zero forcing set for Q_n , $Z^-(Q_n) \leq 2^{n-1}$, and since the maximum nullity is bounded above by the skew zero forcing number we know

$$2^{n-1} \leq M^-(F, Q_n) \leq Z^-(Q_n) \leq 2^{n-1}.$$

Thus, $M^-(F, Q_n) = 2^{n-1}$ and $\text{mr}^-(F, Q_n) = |Q_n| - 2^{n-1} = 2^{n-1}$.

□

Definition 5.5. The m, k -pineapple $P_{m,k}$ with $m \geq 3$ and $k \geq 1$ is the graph $K_m \cup K_{1,k}$ where $K_m \cap K_{1,k}$ is the single vertex of $K_{1,k}$ with degree k .

A specific example of the pineapple where $m = 4$ and $k = 3$ can be seen in Figure 5.1. Note that in [2] an m, k -pineapple is defined to have $k \geq 2$, however in this thesis we expand the definition to include the family of $m, 1$ -pineapples.

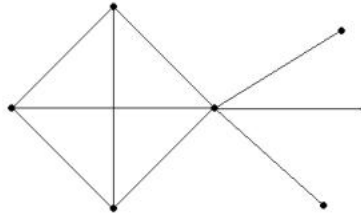


Figure 5.1 Graph $P_{4,3}$, the 4, 3-pineapple

Theorem 5.6. If F is an infinite field with $\text{char}(F) \neq 2$, and $m \geq 3, k \geq 1$, then $\text{mr}^-(F, P_{m,k}) = 4$.

Proof. Let v be the vertex of degree $m - 1 + k$. Then v is a cut vertex and $P_{m,k} - v = K_{m-1} \dot{\cup} K_1 \dot{\cup} \dots \dot{\cup} K_1$ with k copies of K_1 . Since $m \geq 3$, $r_v^-(F, K_m) = 0$ by Theorem 4.2, and $r_v^-(F, K_2) = \text{mr}^-(F, K_2) - \text{mr}^-(F, K_1) = 2 - 0 = 2$, so $r_v^-(F, P_{m,k}) = 2$. Thus,

$$\text{mr}^-(F, P_{m,k}) = \text{mr}^-(F, K_{m-1}) + k \cdot \text{mr}^-(F, K_1) + r_v^-(F, P_{m,k}) = 2 + k \cdot 0 + 2 = 4.$$

□

5.2.2 Coronas

Definition 5.7. The *corona* of G with H , denoted $G \circ H$, is the graph of order $|G||H| + |G|$ obtained by making $|G|$ copies of H , and for each $v \in V_G$ join all vertices of a copy of H to v .

Note that the order in the notation is very important; $G \circ H \neq H \circ G$ in most cases. For a specific example of a corona, see Figure 5.2 for the corona of C_5 with K_2 .

Theorem 5.8. Let G be any graph, and let F be any field. Then $\text{mr}^-(F, G \circ K_1) = 2|G|$.

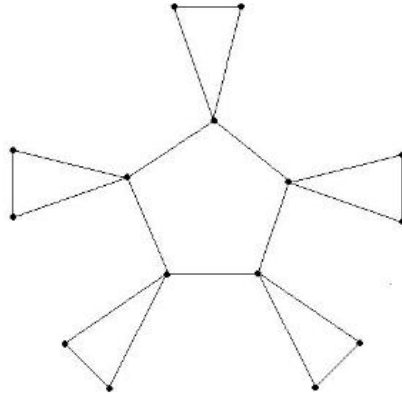


Figure 5.2 Graph $C_5 \circ K_2$, the corona of C_5 with K_2

Proof. There exists a unique perfect matching of each K_1 with the vertex $g \in G$ with which it is adjacent. By Observation 1.9, the graph $G \circ K_1$ has full rank over any field with characteristic not equal to two. \square

Definition 5.9. Define a t -barbell with $t \geq 2$ to be the graph $P_2 \circ K_t$.

The general layout of the t -barbell is shown in Figure 5.4. A specific example, where $t = 3$, is shown below in Figure 5.3.

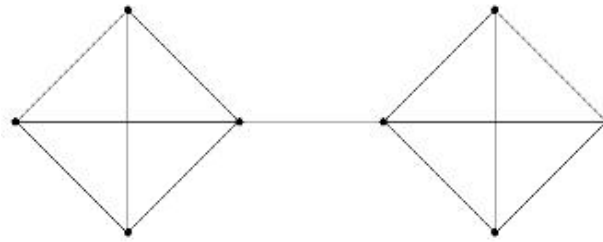


Figure 5.3 Graph $P_2 \circ K_3$, the 3-barbell

Theorem 5.10. Let F be an infinite field with $\text{char}(F) \neq 2$. Then for $t \geq 2$, any t -barbell has minimum skew rank 6, i.e., $\text{mr}^-(F, P_2 \circ K_t) = 6$.

Proof. Let K_t be one complete graph and label its vertices, $1, \dots, t$, and let K'_t be the other complete graph with vertices, $1', \dots, t'$. Label the vertices of P_2 , $t+1, t+2$ with $t+1$ adjacent to $1, \dots, t$. Let vertex $t+2$ be the cut vertex, v , to which we apply Theorem 2.4.

Let $G_1 = G[1, \dots, t, t+1, t+2]$ (component on the left in Figure 5.4 including v) and $G_2 = G[1', \dots, t', t+2]$ (component on the right including v). Then $G_2 = K_{t+1}$ and $G_2 - v = K_t$, so $r_v^-(F, G_2) = 0$ by Theorem 4.2. Also G_1 is isomorphic to a $(t+1)$, 1-pineapple and $G_1 - v = K_{t+1}$. From Theorem 5.6 above, $\text{mr}^-(F, G_1) = 4$ and by Theorem 4.2 $\text{mr}^-(F, G_1 - v) = 2$. Thus, $r_v^-(F, G_1) = 2$ and

$$\text{mr}^-(F, P_2 \circ K_t) = \text{mr}^-(F, G_1 - v) + \text{mr}^-(F, G_2 - v) + \min\{r_v^-(F, G_1) + r_v^-(F, G_2), 2\} = 6.$$

□

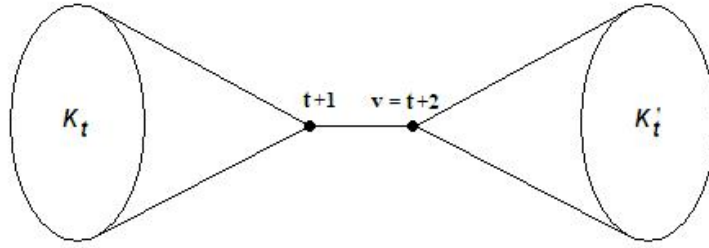


Figure 5.4 Graph $P_2 \circ K_t$, the t -barbell

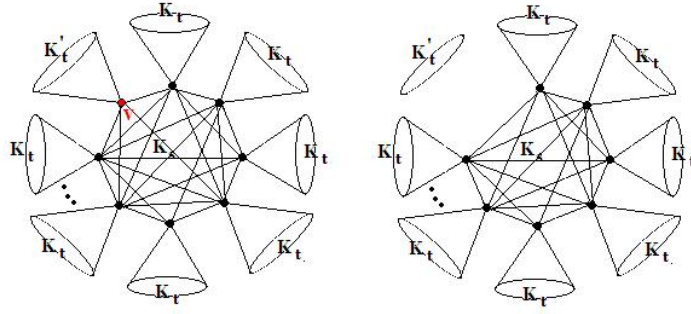
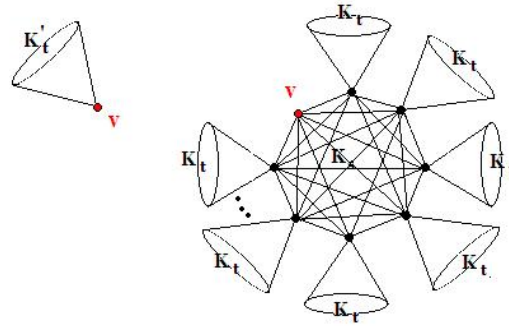
Theorem 5.11. *Let F be an infinite field with $\text{char}(F) \neq 2$. Then for $s \geq 2, t \geq 2$, $\text{mr}^-(F, K_s \circ K_t) = 2(s+1)$.*

Proof. $K_2 \circ K_t$ is isomorphic to the t -barbell. Hence, by Theorem 5.10,

$$\text{mr}^-(F, K_2 \circ K_t) = 2(2+1) = 6.$$

Assume $\text{mr}^-(F, K_{s-1} \circ K_t) = 2((s-1)+1) = 2s$ for $s \geq 3$. In the graph $K_s \circ K_t$, let v be any vertex in K_s , and let K'_t be the complete graph adjacent to v . Vertex v is a cut vertex, and $K_s \circ K_t - v = (K_{s-1} \circ K_t) \dot{\cup} K'_t$. Both graphs, $K_s \circ K_t$ and $K_s \circ K_t - v$, can be seen in Figure 5.5. Graph $K_s \circ K_t$ is pictured on the left, and the graph without cut vertex v , $K_s \circ K_t - v$, is shown on the right.

The components including the cut vertex v are K_{t+1} and $(K_{s-1} \circ K_t) \cup K_{1,s-1}$, where the vertex in $V_{K_{1,s-1}}$ of degree $s-1$ is adjacent to all vertices in K_{s-1} . See Figure 5.6. The rank-spread $r_v^-(F, K_s \circ K_t) = \min\{r_v^-(F, (K_{s-1} \circ K_t) \cup K_{1,s-1}) + r_v^-(F, K_{t+1}), 2\}$.

Figure 5.5 Graphs $K_s \circ K_t$ and $K_s \circ K_t - v$ Figure 5.6 The connected components of $K_s \circ K_t$ including the cut vertex v

By Proposition 4.1, $\text{mr}^-(F, (K_{s-1} \circ K_t) \cup K_{1,s-1}) \leq 2s$, since we can cover all the edges of the graph with s complete graphs. Since $(K_{s-1} \circ K_t) \cup K_{1,s-1}$ contains $K_{s-1} \circ K_t$ as an induced subgraph, $\text{mr}^-(F, (K_{s-1} \circ K_t) \cup K_{1,s-1}) \geq 2s$. Thus, $\text{mr}^-(F, (K_{s-1} \circ K_t) \cup K_{1,s-1}) = 2s$ and $r_v^-(F, (K_{s-1} \circ K_t) \cup K_{1,s-1}) = 0$. Also, $r_v^-(F, K_{t+1}) = \text{mr}^-(F, K_{t+1}) - \text{mr}^-(F, K_t) = 0$. Hence the minimum skew rank of $K_s \circ K_t$ is exactly the sum of the minimum skew ranks of the disjoint, connected components resulting from cut-vertex reduction, i.e.,

$$\text{mr}^-(F, K_s \circ K_t) = 2s + 2 = 2(s + 1).$$

□

Theorem 5.12. *Let F be an infinite field with $\text{char}(F) \neq 2$. Then for $s \geq 1, t \geq 2$,*

$$\text{mr}^-(F, P_s \circ K_t) = \begin{cases} 3s & : \text{if } s \text{ is even} \\ 3s - 1 & : \text{if } s \text{ is odd} \end{cases}.$$

Proof. For $s = 1$, $P_1 \circ K_t = K_{t+1}$ and by Theorem 4.2 $\text{mr}^-(F, K_{t+1}) = 2 = 3 \cdot s - 1$. For $s = 2$, $P_2 \circ K_t$ is exactly the t -barbell. Hence, $\text{mr}^-(F, P_2 \circ K_t) = 6 = 3s$ holds by Theorem 5.10.

Let $s \geq 2$, and assume $\text{mr}^-(F, P_s \circ K_t) = 3s$ if s is even and $\text{mr}^-(F, P_s \circ K_t) = 3s - 1$ if s is odd. Examine the graph $P_{s+1} \circ K_t$. Let v be a vertex on P_{s+1} adjacent to an end vertex of P_{s+1} (such as the white vertex s in Figure 5.7). Then v is a cut vertex and

$P_{s+1} \circ K_t - v = (P_{s-1} \circ K_t) \dot{\cup} K_t \dot{\cup} K_{t+1}$. Let $G_1 - v = K_{t+1}$, $G_2 - v = K_t$, and $G_3 - v = P_{s-1} \circ K_t$.

Then $G_1 = P_{t+1,1}$ and $r_v^-(F, G_1) = \text{mr}^-(F, P_{t+1,1}) - \text{mr}^-(F, K_{t+1}) = 4 - 2 = 2$.

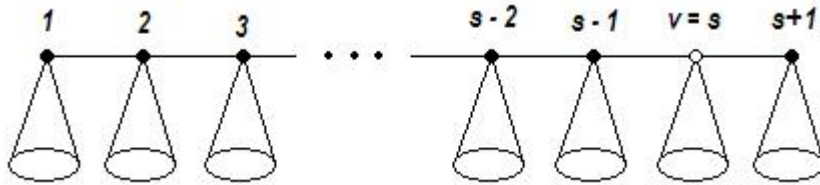


Figure 5.7 Graph $P_{s+1} \circ K_t$ with white cut vertex $v = s$

Therefore, $r_v^-(F, P_{s+1} \circ K_t) = \min\{\sum_{i=1}^3 r_v^-(F, G_i), 2\} = 2$ since it only takes one subgraph G_i with $r_v^-(F, G_i) = 2$ for the skew rank-spread of the entire graph to be 2.

The minimum skew rank of the graph follows:

$$\begin{aligned}
 \text{mr}^-(F, P_{s+1} \circ K_t) &= 2 + \sum_{i=1}^3 \text{mr}^-(F, G_i - v) \\
 &= 2 + \text{mr}^-(F, K_{t+1}) + \text{mr}^-(F, K_t) + \text{mr}^-(F, P_{s-1} \circ K_t) \\
 &= 2 + 2 + 2 + \begin{cases} 3(s-1) & : \text{if } s-1 \text{ is even} \\ 3(s-1) - 1 & : \text{if } s-1 \text{ is odd} \end{cases} \\
 &= \begin{cases} 3(s+1) & : \text{if } s+1 \text{ is even} \\ 3(s+1) - 1 & : \text{if } s+1 \text{ is odd} \end{cases} .
 \end{aligned}$$

□

Theorem 5.13. *Let F be an infinite field with $\text{char}(F) \neq 2$. Then for $s \geq 3, t \geq 2$,*

$$\text{mr}^-(F, C_s \circ K_t) = \begin{cases} 3s - 1 & : \text{if } s \text{ is odd} \\ 3s - 2 & : \text{if } s \text{ is even} \end{cases}.$$

Proof. Let v be a vertex on the cycle C_s in the graph $C_s \circ K_t$. Thus, v is a cut vertex, and when v is removed we have disconnected subgraphs $G_1 - v = K_t$ and $G_2 - v = P_{s-1} \circ K_t$, where $G_1 = K_{t+1}$ and G_2 is constructed from $P_{s-1} \circ K_t$ by joining v to the end vertices of P_{s-1} . We know $r_v^-(F, G_1) = 0$.

$$\text{Also, } \text{mr}^-(F, G_2 - v) = \begin{cases} 3(s-1) & : \text{if } s \text{ is odd} \\ 3(s-1) - 1 & : \text{if } s \text{ is even} \end{cases} = \begin{cases} 3s - 3 & : \text{if } s \text{ is odd} \\ 3s - 4 & : \text{if } s \text{ is even} \end{cases}.$$

Since G_2 can be covered by $s - 1$ copies of K_{t+1} and a cycle C_s ,

$$\text{mr}^-(F, G_2) \leq \begin{cases} 2(s-1) + (s-1) & : \text{if } s \text{ is odd} \\ 2(s-1) + (s-2) & : \text{if } s \text{ is even} \end{cases} = \begin{cases} 3s - 3 & : \text{if } s \text{ is odd} \\ 3s - 4 & : \text{if } s \text{ is even} \end{cases}.$$

Since G_2 has an induced subgraph $G_2 - v$, we know that equality must hold, i.e.,

$$\text{mr}^-(F, G_2) = \begin{cases} 3s - 3 & : \text{if } s \text{ is odd} \\ 3s - 4 & : \text{if } s \text{ is even} \end{cases}.$$

Hence $r_v^-(F, G_2) = 0$, and thus $r_v^-(F, C_s \circ K_t) = 0$.

Therefore,

$$\begin{aligned} \text{mr}^-(F, C_s \circ K_t) &= \text{mr}^-(F, P_{s-1} \circ K_t) + \text{mr}^-(F, K_t) \\ &= \begin{cases} 3(s-1) + 2 & : \text{if } s \text{ is odd} \\ 3(s-1) - 1 + 2 & : \text{if } s \text{ is even} \end{cases} \\ &= \begin{cases} 3s - 1 & : \text{if } s \text{ is odd} \\ 3s - 2 & : \text{if } s \text{ is even} \end{cases}. \end{aligned}$$

□

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