COMMUTATORS AND ASSOCIATORS IN CATALAN LOOPS

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Abstract. This paper approximates and compares the different commutators and associators in the left and right loop reducts of a Catalan loop. To within a certain order of approximation these differ, if at all, only by a sign. We investigate the structure of a specific Catalan loop which is non-commutative, but associative, that appears in the original number-theoretic application of Catalan loops.

1. Introduction

The structure of Catalan loops originated from an issue in number theory, concerning the relationship between Fermat curves and modular curves. (For a detailed description of the motivation one may refer to [1].) Catalan loops are two-sided loops offering a range of possible definitions for commutators and associators. The topic of this paper is the calculation of the commutators and associators of Catalan loops up to a certain order of approximation, and a further investigation of the structure in one of the motivating cases. In Section 2 we define the objects of interest - the commutators and associators in left and right loops. We introduce Catalan loops in Section 3. Section 4 covers the formal calculation of the different commutators and associators. In Section 5 we make approximations in order to compare the different commutators and associators in Section 6. Then, Section 7 focuses on one of the smaller motivating cases where the Catalan loop actually turns out to be associative and, hence, a group. Finally, we discuss directions for future work on this topic in Section 8.

2. Commutators and Associators in Left Loops and Right Loops

A *right quasigroup* $(Q, \cdot, /)$ is a set Q together with binary operations of *multiplication* (denoted by $x{\cdot}y$ or juxtaposition xy) and *right division*

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x/y such that

$$(x \cdot y)/y = x = (x/y) \cdot y$$

for x, y in Q. A *right loop* $(Q, \cdot, /, 1)$ is a right quasigroup $(Q, \cdot, /)$ with an *identity element* 1 such that

$$1 \cdot x = x = x \cdot 1$$

for x in Q. We define a *right loop commutator* to be

$$[x, y]_R = (xy)/(yx)$$

for x, y in Q. Further, we set *right loop associators* to be

$$(x, y, z)_R = (xy \cdot z)/(x \cdot yz)$$

and

$$(x, y, z)_R^* = (x \cdot yz)/(xy \cdot z)$$

for x, y, z in Q.

Similarly, a *left quasigroup* $(U, \cdot, \backslash)$ is a set U together with binary operations of *multiplication* (denoted by $x \cdot y$ or juxtaposition xy) and *left division* $y \backslash x$ such that

$$y\backslash(y \cdot x) = x = y \cdot (y\backslash x)$$

for x, y in U. A *left loop* $(U, \cdot, \backslash, 1)$ is a left quasigroup $(U, \cdot, \backslash)$ with an *identity element* 1 such that

$$1 \cdot x = x = x \cdot 1$$

for x in U. We define a *left loop commutator* to be

$$[x, y]_L = (yx)\backslash(xy)$$

for x, y in U. Further, we set *left loop associators* to be

$$(x, y, z)_L = (x \cdot yz)\backslash(xy \cdot z)$$

and

$$(x, y, z)_L^* = (xy \cdot z)\backslash(x \cdot yz)$$

for x, y, z in U. Note that a right loop is commutative if and only if

$$[x, y]_R = 1$$

for all x, y. Also, a right loop is associative if and only if

$$(x, y, z)_R = 1$$

for all x, y, z. Or equivalently,

$$(x, y, z)_R^* = 1$$

for all x, y, z. The analogous results also hold in a left loop. Lastly, notice that in a group (with $x/y = xy^{-1}$ and $y\backslash x = y^{-1}x$), we have

$$[x, y]_R = xyx^{-1}y^{-1}$$

and

$$[x,y]_L = x^{-1}y^{-1}xy\,.$$

3. Catalan Loops

Let R be a commutative, unital ring, with a topologically nilpotent element e, and let E be the annihilator of e in R. Let H be the subgroup of diagonal matrices in $\mathrm{SL}(2,R)$. Consider the set

$$Q' = \left\{ \begin{bmatrix} 1 & ex \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ ex' & 1 \end{bmatrix} \;\middle|\; x, x' \in R \right\}.$$

Define $G = HQ'$. By [1, Prop. 4.2, 4.3], the set Q' forms a loop transversal to the subgroup H in G. The characterization as a loop transversal yields a right loop structure on Q', a so-called *Catalan loop*. Further, by [1, Cor. 5.2; Th. 5.3] the Catalan loop on Q' is two-sided and may alternatively be represented on $(R/E)^2$, as exhibited in the next paragraph. We use the alternative representation, since it proves to be helpful for our calculations.

Let Q be $(R/E)^2$. Then the three binary operations in the *Catalan loop* $(Q,\cdot,/,\backslash,(0,0))$ are given as follows. Multiplication:

$$\langle x,x'\rangle \cdot \langle y,y'\rangle = \left\langle x\lambda_m^2 + y\lambda_m, x'\lambda_m^{-1} + y'\right\rangle \tag{1}$$

with $\lambda_m = \lambda_m(\mathbf{x},\mathbf{y}) = 1 + e^2\,(yx')$. Right division:

$$\langle x,x'\rangle \,/\, \langle y,y'\rangle = \left\langle x\lambda_r^2 - y\lambda_r, x'\lambda_r^{-1} - y'\lambda_r^{-1}\right\rangle \tag{2}$$

with $\lambda_r = \lambda_r(\mathbf{x},\mathbf{y}) = 1 - e^2 y\,(x'-y')$. Left division:

(3)

$$\langle x,x'\rangle \backslash \langle y,y'\rangle = \left\langle dy - d^{-1}x, \left(d^{-1}y' - dx'\right) - e^2x'y'\left(dy - d^{-1}x\right)\right\rangle,$$

where $d = d(\mathbf{x},\mathbf{y})$ is the unique recursive solution

$$d = 1 + e^2 \cdot x'\,(x-y) + O\left(e^4\right)$$

to the equation

$$d = \left(1 + e^2xx'\right) - e^2d^2x'y.$$

The multipliers λ_m, λ_r, d are known as *fudge factors.*

We will denote a typical pair $\left\langle x,x'\right\rangle$ as $\mathbf{x}$. For more complicated pairs $\mathbf{x}$ we will refer to the first component by $[\mathbf{x}]_1$. Now, consider the following two remarks about the right and left division which prove to be useful.

Remark 1. If the respective fudge factors $\lambda_r(\mathbf{x},\mathbf{y}), \lambda_r(\mathbf{y},\mathbf{x})$ are equal to 1, which means

$$e^2y\,(x'-y') = e^2x\,(y'-x') = 0\,,$$

we have

$$\mathbf{x}/\mathbf{y} = \langle x - y, x' - y' \rangle = -\mathbf{y}/\mathbf{x}.$$

Remark 2. If the respective fudge factors $d(\mathbf{y},\mathbf{x})$ and $\lambda_r(\mathbf{x},\mathbf{y})$ are equal to 1 and

$$e^2 y'x' \left(d(\mathbf{y},\mathbf{x})x - d(\mathbf{y},\mathbf{x})^{-1} y \right) = 0\,,$$

then

$$\mathbf{x}/\mathbf{y} = \mathbf{y}\backslash\mathbf{x}.$$

4. Formal Calculations

The following mainly serves the purpose of introducing the different fudge factors we will have to deal with later on. Having the respective formulae handy will make the approximations easier.

$$\mathbf{xy} = \left\langle x\lambda_1^2 + y\lambda_1, x'\lambda_1^{-1} + y' \right\rangle \text{ with } \lambda_1 = 1 + e^2 yx',$$

and

$$\mathbf{yx} = \left\langle y\bar{\lambda}_1^2 + x\bar{\lambda}_1, y'\bar{\lambda}_1^{-1} + x' \right\rangle \text{ with } \bar{\lambda}_1 = 1 + e^2 xy'.$$

4.1. **The commutators.** First we will formally calculate the left and right commutator. The right commutator $[\mathbf{x},\mathbf{y}]_R$ is

$$\begin{aligned} &\Big\langle \left(x\lambda_1^2 + y\lambda_1\right)\lambda_R^2 - \left(y\bar{\lambda}_1^2 + x\bar{\lambda}_1\right)\lambda_R, \\ &\left(x'\lambda_1^{-1} + y'\right)\lambda_R^{-1} - \left(y'\bar{\lambda}_1^{-1} + x'\right)\lambda_R^{-1}\Big\rangle \\ &= \Big\langle x\left(\lambda_1^2\lambda_R^2 - \bar{\lambda}_1\lambda_R\right) + y\left(\lambda_1\lambda_R^2 - \bar{\lambda}_1^2\lambda_R\right), \\ &x'\left(\lambda_1^{-1}\lambda_R^{-1} - \lambda_R^{-1}\right) + y'\left(\lambda_R^{-1} - \bar{\lambda}_1^{-1}\lambda_R^{-1}\right)\Big\rangle \end{aligned}$$

with λ_R equal to

$$\begin{aligned} &1 - e^2\left(y\bar{\lambda}_1^2 + x\bar{\lambda}_1\right)\left[\left(x'\lambda_1^{-1} + y'\right) - \left(y'\bar{\lambda}_1^{-1} + x'\right)\right] \\ &= 1 - e^2\left(y\bar{\lambda}_1^2 + x\bar{\lambda}_1\right)\left[x'\left(\lambda_1^{-1} - 1\right) - y'\left(\bar{\lambda}_1^{-1} - 1\right)\right]. \end{aligned}$$

Further, the left commutator $[\mathbf{x},\mathbf{y}]_L$ is

$$\begin{aligned} &\Big\langle \lambda_L\left(x\lambda_1^2 + y\lambda_1\right) - \lambda_L^{-1}\left(y\bar{\lambda}_1^2 + x\bar{\lambda}_1\right), \\ &\left[\lambda_L^{-1}\left(x'\lambda_1^{-1} + y'\right) - \lambda_L\left(y'\bar{\lambda}_1^{-1} + x'\right)\right] - \\ &- e^2\left(y'\bar{\lambda}_1^{-1} + x'\right)\left(x'\lambda_1^{-1} + y'\right)\left([[\mathbf{x},\mathbf{y}]_L]_1\right)\Big\rangle, \end{aligned}$$

with λ_L equal to

$$1 + e^2 \cdot \left(y'\bar{\lambda}_1^{-1} + x'\right)\left[\left(y\bar{\lambda}_1^2 + x\bar{\lambda}_1\right) - \left(x\lambda_1^2 + y\lambda_1\right)\right] + O\left(e^4\right).$$

4.2. **The right associators.** Let us now have a look at the associators in the right loop reduct. Firstly, $\mathbf{xy}\cdot\mathbf{z}$ is

$$\left\langle (x\lambda_1^2+y\lambda_1)\lambda_2^2+z\lambda_2, (x'\lambda_1^{-1}+y')\lambda_2^{-1}+z'\right\rangle$$

with $\lambda_2 = 1+e^2z(x'\lambda_1^{-1}+y')$. Secondly,

$$\mathbf{yz} = \left\langle y\lambda_3^2+z\lambda_3, y'\lambda_3^{-1}+z'\right\rangle \text{ with } \lambda_3 = 1+e^2zy',$$

and thus $\mathbf{x}\cdot\mathbf{yz}$ is

$$\left\langle x\lambda_4^2+(y\lambda_3^2+z\lambda_3)\lambda_4, x'\lambda_4^{-1}+(y'\lambda_3^{-1}+z')\right\rangle$$

with $\lambda_4 = 1+e^2(y\lambda_3^2+z\lambda_3)x'$. Finally, $(\mathbf{x},\mathbf{y},\mathbf{z})_R$ is equal to

$$\begin{aligned}
&\Big\langle \left[\left(x\lambda_1^2+y\lambda_1\right)\lambda_2^2+z\lambda_2\right]\lambda_5^2-\left[x\lambda_4^2+\left(y\lambda_3^2+z\lambda_3\right)\lambda_4\right]\lambda_5,\\
&\quad\left[\left(x'\lambda_1^{-1}+y'\right)\lambda_2^{-1}+z'\right]\lambda_5^{-1}-\left[x'\lambda_4^{-1}+\left(y'\lambda_3^{-1}+z'\right)\right]\lambda_5^{-1}\Big\rangle\\
&=\Big\langle x\left(\lambda_1^2\lambda_2^2\lambda_5^2-\lambda_4^2\lambda_5\right)+y\left(\lambda_1\lambda_2^2\lambda_5^2-\lambda_3^2\lambda_4\lambda_5\right)+z\left(\lambda_2\lambda_5^2-\lambda_3\lambda_4\lambda_5\right),\\
&\quad x'\left(\lambda_1^{-1}\lambda_2^{-1}\lambda_5^{-1}-\lambda_4^{-1}\lambda_5^{-1}\right)+y'\left(\lambda_2^{-1}\lambda_5^{-1}-\lambda_3^{-1}\lambda_5^{-1}\right)\Big\rangle,
\end{aligned}$$

where λ_5 is

$$\begin{aligned}
&1-e^2\left[x\lambda_4^2+\left(y\lambda_3^2+z\lambda_3\right)\lambda_4\right]\cdot\left[\left(x'\lambda_1^{-1}+y'\right)\lambda_2^{-1}+z'-x'\lambda_4^{-1}-y'\lambda_3^{-1}-z'\right]\\
&=1-e^2\left[x\lambda_4^2+\left(y\lambda_3^2+z\lambda_3\right)\lambda_4\right]\cdot\left[x'\left(\lambda_1^{-1}\lambda_2^{-1}-\lambda_4^{-1}\right)+y'\left(\lambda_2^{-1}-\lambda_3^{-1}\right)\right].
\end{aligned}$$

Similarly, we get $(\mathbf{x},\mathbf{y},\mathbf{z})_R^* =$

$$\begin{aligned}
&=\Big\langle \left[x\lambda_4^2+\left(y\lambda_3^2+z\lambda_3\right)\lambda_4\right]\lambda_5^{*^2}-\left[\left(x\lambda_1^2+y\lambda_1\right)\lambda_2^2+z\lambda_2\right]\lambda_5^*,\\
&\quad-\left[\left[\left(x'\lambda_1^{-1}+y'\right)\lambda_2^{-1}+z'\right]\lambda_5^{*^{-1}}-\left[x'\lambda_4^{-1}+\left(y'\lambda_3^{-1}+z'\right)\right]\lambda_5^{*^{-1}}\right]\Big\rangle\\
&=\Big\langle x\left(-\lambda_1^2\lambda_2^2\lambda_5^*+\lambda_4^2\lambda_5^{*^2}\right)+y\left(-\lambda_1\lambda_2^2\lambda_5^*+\lambda_3^2\lambda_4\lambda_5^{*^2}\right)+z\left(-\lambda_2\lambda_5^*+\lambda_3\lambda_4\lambda_5^{*^2}\right),\\
&\quad-\left[x'\left(\lambda_1^{-1}\lambda_2^{-1}\lambda_5^{*^{-1}}-\lambda_4^{-1}\lambda_5^{*^{-1}}\right)+y'\left(\lambda_2^{-1}\lambda_5^{*^{-1}}-\lambda_3^{-1}\lambda_5^{*^{-1}}\right)\right]\Big\rangle,
\end{aligned}$$

where λ_5^* is

$$\begin{aligned}
&1-e^2\left[(x\lambda_1^2+y\lambda_1)\lambda_2^2+z\lambda_2\right]\cdot\left[-\left[\left(x'\lambda_1^{-1}+y'\right)\lambda_2^{-1}+z'-x'\lambda_4^{-1}-y'\lambda_3^{-1}-z'\right]\right]\\
&=1+e^2\left[(x\lambda_1^2+y\lambda_1)\lambda_2^2+z\lambda_2\right]\cdot\left[x'\left(\lambda_1^{-1}\lambda_2^{-1}-\lambda_4^{-1}\right)+y'\left(\lambda_2^{-1}-\lambda_3^{-1}\right)\right].
\end{aligned}$$

4.3. **The left associators.** In the left loop reduct $(\mathbf{x},\mathbf{y},\mathbf{z})_L$ is

$$\begin{aligned}
&\Big\langle d\left[(x\lambda_1^2+y\lambda_1)\lambda_2^2+z\lambda_2\right]-d^{-1}\left[x\lambda_4^2+(y\lambda_3^2+z\lambda_3)\lambda_4\right],\\
&\quad\left[d^{-1}\left[(x'\lambda_1^{-1}+y')\lambda_2^{-1}+z'\right]-d\left[x'\lambda_4^{-1}+(y'\lambda_3^{-1}+z')\right]\right]-\\
&\quad-e^2\left[(x'\lambda_1^{-1}+y')\lambda_2^{-1}+z'\right]\left[x'\lambda_4^{-1}+(y'\lambda_3^{-1}+z')\right]\cdot\left([(\mathbf{x},\mathbf{y},\mathbf{z})_L]_1\right)\Big\rangle,
\end{aligned}$$

where d is

$$1 - e^2 \left[x'\lambda_4^{-1} + (y'\lambda_3^{-1} + z')\right] \times$$
$$\times \left[x\left(\lambda_1^2\lambda_2^2 - \lambda_4^2\right) + y\left(\lambda_1\lambda_2^2 - \lambda_3^2\lambda_4\right) + z\left(\lambda_2 - \lambda_3\lambda_4\right)\right] + O\left(e^4\right).$$

Similarly, $(\mathbf{x}, \mathbf{y}, \mathbf{z})_L^*$ is equal to

$$\Big\langle d^* \left[x\lambda_4^2 + (y\lambda_3^2 + z\lambda_3)\lambda_4\right] - d^{*^{-1}} \left[(x\lambda_1^2 + y\lambda_1)\lambda_2^2 + z\lambda_2\right],$$
$$\Big[d^{*^{-1}} \left[x'\lambda_4^{-1} + (y'\lambda_3^{-1} + z')\right] - d^* \left[(x'\lambda_1^{-1} + y')\lambda_2^{-1} + z'\right]\Big]$$
$$- e^2 \left[(x'\lambda_1^{-1} + y')\lambda_2^{-1} + z'\right] \left[x'\lambda_4^{-1} + (y'\lambda_3^{-1} + z')\right] \cdot \left([(\mathbf{x}, \mathbf{y}, \mathbf{z})_L^*]_1\right) \Big\rangle,$$

where d^* is

$$1 + e^2 \left[(x'\lambda_1^{-1} + y')\lambda_2^{-1} + z'\right] \times$$
$$\times \left[\left[(x\lambda_1^2 + y\lambda_1)\lambda_2^2 + z\lambda_2\right] - \left[x\lambda_4^2 + (y\lambda_3^2 + z\lambda_3)\lambda_4\right]\right] + O\left(e^4\right)$$
$$= 1 + e^2 \left(x'\lambda_1^{-1}\lambda_2^{-1} + y'\lambda_2^{-1} + z'\right) \times$$
$$\times \left[x\left(\lambda_1^2\lambda_2^2 - \lambda_4^2\right) + y\left(\lambda_1\lambda_2^2 - \lambda_3^2\lambda_4\right) + z\left(\lambda_2 - \lambda_3\lambda_4\right)\right] + O\left(e^4\right).$$

5. Approximations and Comparisons

Having done the calculations of the objects of interest we will now approximate our results and compare the respective commutators and associators. In doing so we will have to make extensive use of the following remark.

Remark 3. Note that if $\lambda = 1 + O(e^4)$, then $\lambda^k = 1 + O(e^4)$ for any integer k. And thus a multiplication by any power of λ is essentially a multiplication by the identity plus an additional summand of order $O(e^4)$.

In order to have easy access to versions of the Remarks 1 and 2 which will prove to be useful in the following approximations we formulate two lemmata.

Lemma 1. *If $\lambda_r(\mathbf{x}, \mathbf{y})$ and $\lambda_r(\mathbf{y}, \mathbf{x})$ are of order $O(e^4)$, then*

$$\mathbf{x}/\mathbf{y} = -\mathbf{y}/\mathbf{x}$$

up to $O(e^4)$.

Proof. Using Remark 3 for $\lambda_r(\mathbf{x}, \mathbf{y})$ and $\lambda_r(\mathbf{y}, \mathbf{x})$ we get

$$\mathbf{x}/\mathbf{y} = \langle x - y, x' - y'\rangle + O\left(e^4\right)$$

and

$$\mathbf{y}/\mathbf{x} = \langle y - x, y' - x'\rangle + O\left(e^4\right),$$

and are done. □

Lemma 2. *If $\lambda_r(\mathbf{x},\mathbf{y})$ and $d(\mathbf{y},\mathbf{x})$ are of order $1+O\left(e^4\right)$, and $x-y$ is of order $O\left(e^2\right)$, then*

$$\mathbf{x}/\mathbf{y}=\mathbf{y}\backslash\mathbf{x}$$

up to $O\left(e^4\right)$.

Proof. Using Remark 3 for $\lambda_r(\mathbf{x},\mathbf{y})$ and $d(\mathbf{y},\mathbf{x})$ we get

$$\mathbf{x}/\mathbf{y}=\langle x-y,x'-y'\rangle+O\left(e^4\right)$$

and

$$\mathbf{y}\backslash\mathbf{x}=\left\langle x-y,x'-y'-e^2y'x'(x-y)\right\rangle+O\left(e^4\right).$$

Then we are done, since $e^2y'x'(x-y)$ is already of order $O\left(e^4\right)$ by the second part of the assumption. □

5.1. **The commutators.** Let us start by having a closer look at the fudge factor

$$\lambda_R=1-e^2\left(y\bar{\lambda}_1^2+x\bar{\lambda}_1\right)\left[x'\left(\lambda_1^{-1}-1\right)-y'\left(\bar{\lambda}_1^{-1}-1\right)\right]$$

of the right commutator $[\mathbf{x},\mathbf{y}]_R$. Since

- $\lambda_1^2=1+e^22yx'+O\left(e^4\right)$,
- $\lambda_1^{-1}=1-e^2yx'+O\left(e^4\right)\Rightarrow\left(\lambda_1^{-1}-1\right)=-e^2yx'+O\left(e^4\right)$,

and similarly

- $\bar{\lambda}_1^2=1+e^22xy'+O\left(e^4\right)$,
- $\left(\bar{\lambda}_1^{-1}-1\right)=-e^2xy'+O\left(e^4\right)$,

we have

$$\lambda_R=1-e^4(x+y)\left(xy'^2-yx'^2\right)+O\left(e^6\right).$$

Thus

$$\lambda_R=1+O\left(e^4\right).$$

By Remark 3 the second component of $[\mathbf{x},\mathbf{y}]_R$ now is

$$\begin{aligned}&x'\left(\lambda_1^{-1}-1\right)+y'\left(1-\bar{\lambda}_1^{-1}\right)+O\left(e^4\right)\\&=e^2\left(xy'^2-yx'^2\right)+O\left(e^4\right).\end{aligned}\tag{4}$$

Further, we have

$$\begin{aligned}&\lambda_1^2-\bar{\lambda}_1\\&=\left[1+e^22yx'+O\left(e^4\right)\right]\left[1+O\left(e^4\right)\right]-\left[1+e^2xy'\right]\\&=e^2\left(2yx'-xy'\right)+O\left(e^4\right)\end{aligned}$$

and

$$\begin{aligned}&\lambda_1-\bar{\lambda}_1^2\\&=\left(1+e^2yx'\right)-\left(1+e^22xy'\right)+O\left(e^4\right)\\&=e^2\left(yx'-2xy'\right)+O\left(e^4\right).\end{aligned}$$

Thus

$$x\left(\lambda_1^2-\bar{\lambda}_1\right)+y\left(\lambda_1-\bar{\lambda}_1^2\right)$$

$$=e^2 2xy\left[(x'-y')+y^2x'-x^2y'\right]+O\left(e^4\right). \tag{5}$$

Hence applying Remark 3 we see that

$$[\mathbf{x},\mathbf{y}]_R=e^2\left\langle 2xy\,(x'-y')+y^2x'-x^2y',xy'^2-yx'^2\right\rangle+O\left(e^4\right). \tag{6}$$

Now, we would like to compare $[\mathbf{x},\mathbf{y}]_R$ with $[\mathbf{y},\mathbf{x}]_R$ with the help of Lemma 1. All that is left to show is that the fudge factor $\bar{\lambda}_R$ of $[\mathbf{y},\mathbf{x}]_R$ is of order $1+O\left(e^4\right)$. But

$$\bar{\lambda}_R=1+e^2\left(x\lambda_1^2+y\lambda_1\right)\left[x'\left(\lambda_1^{-1}-1\right)-y'\left(\bar{\lambda}_1^{-1}-1\right)\right],$$

and by (4) we have

$$\left[x'\left(\lambda_1^{-1}-1\right)-y'\left(\bar{\lambda}_1^{-1}-1\right)\right]=O\left(e^2\right).$$

Thus

$$\bar{\lambda}_R=1+O\left(e^4\right),$$

whence

$$[\mathbf{x},\mathbf{y}]_R=-\left[\mathbf{y},\mathbf{x}\right]_R$$

up to $O\left(e^4\right)$ by Lemma 1.

Next, we will make use of Lemma 2 in order to compare the commutators in the left loop reduct with the ones we have just exhibited in the right loop reduct. First, notice that

$$[\mathbf{xy}]_1-[\mathbf{yx}]_1$$

$$=\left(x\lambda_1^2+y\lambda_1\right)-\left(y\bar{\lambda}_1^2+x\bar{\lambda}_1\right)$$

$$=O\left(e^2\right)\ \text{by (5)}. \tag{7}$$

Second, consider the fudge factor of the left commutator $[\mathbf{x},\mathbf{y}]_L$.

$$\lambda_L=1+e^2\cdot\left(y'\bar{\lambda}_1^{-1}+x'\right)\left[\left(y\bar{\lambda}_1^2+x\bar{\lambda}_1\right)-\left(x\lambda_1^2+y\lambda_1\right)\right]+O\left(e^4\right)$$

$$=1-e^2\cdot\left(y'\bar{\lambda}_1^{-1}+x'\right)\left([\mathbf{xy}]_1-[\mathbf{yx}]_1\right)+O\left(e^4\right)$$

$$=1+O\left(e^4\right)\ \text{by (7)}. \tag{8}$$

Thus the assumptions of Lemma 2 are satisfied, and we conclude that

$$[\mathbf{x},\mathbf{y}]_R=[\mathbf{x},\mathbf{y}]_L$$

up to $O\left(e^4\right)$. Similarly, we consider $[\mathbf{y},\mathbf{x}]_L$ with fudge factor

$$\bar{\lambda}_L=1+e^2\cdot\left(x'\lambda_1^{-1}+y'\right)\left[\left(x\lambda_1^2+y\lambda_1\right)-\left(y\bar{\lambda}_1^2+x\bar{\lambda}_1\right)\right]+O\left(e^4\right)$$

$$=1+e^2\cdot\left(x'\lambda_1^{-1}+y'\right)\left([\mathbf{xy}]_1-[\mathbf{yx}]_1\right)+O\left(e^4\right)$$

$$=1+O\left(e^4\right)\ \text{by (7)}. \tag{9}$$

Obviously $[\mathbf{yx}]_1 - [\mathbf{xy}]_1$ is also of order $O(e^4)$ by (7), and we can apply Lemma 2 again to see that

$$[\mathbf{y},\mathbf{x}]_R = [\mathbf{y},\mathbf{x}]_L$$

up to $O(e^4)$.

5.2. **The right associators.** It turns out that the associators are related in a very similar fashion. Let us first focus on the right loop and thus on $(\mathbf{x},\mathbf{y},\mathbf{z})_R$ and $(\mathbf{x},\mathbf{y},\mathbf{z})^*_R$. Similar to before, we will first determine the approximation of $(\mathbf{x},\mathbf{y},\mathbf{z})_R$ and then use Lemma 1 to determine $(\mathbf{x},\mathbf{y},\mathbf{z})^*_R$. Consider the fudge factor of $(\mathbf{x},\mathbf{y},\mathbf{z})_R$

$$\lambda_5 = 1 - e^2\left[x\lambda_4^2 + \left(y\lambda_3^2 + z\lambda_3\right)\lambda_4\right]\left[x'\left(\lambda_1^{-1}\lambda_2^{-1} - \lambda_4^{-1}\right) + y'\left(\lambda_2^{-1} - \lambda_3^{-1}\right)\right].$$

Since

$$\begin{aligned}
&\lambda_1^{-1}\lambda_2^{-1} - \lambda_4^{-1}\\
&= \left(1 - e^2yx' + O\left(e^4\right)\right)\left(1 - e^2z(x'+y') + O\left(e^4\right)\right) -\\
&- \left(1 - e^2x'(y+z) + O\left(e^4\right)\right)\\
&= 1 - e^2x'z - e^2y'z - e^2x'y -\\
&- 1 + e^2x'y + e^2x'z + O\left(e^4\right)\\
&= -e^2y'z + O\left(e^4\right)
\end{aligned}$$

and

$$\begin{aligned}
&\lambda_2^{-1} - \lambda_3^{-1}\\
&= \left(1 - e^2z(x'+y') + O\left(e^4\right)\right) - \left(1 - e^2zy' + O\left(e^4\right)\right)\\
&= -e^2x'z + O\left(e^4\right),
\end{aligned}$$

we have

$$\begin{aligned}
&x'\left(\lambda_1^{-1}\lambda_2^{-1} - \lambda_4^{-1}\right) + y'\left(\lambda_2^{-1} - \lambda_3^{-1}\right)\\
&= -e^2x'y'z - e^2x'y'z + O\left(e^4\right)\\
&= e^2\left(-2x'y'z\right) + O\left(e^4\right)\\
&= O(e^2).
\end{aligned}$$

Hence

$$\lambda_5 = 1 + O\left(e^4\right).$$

Using Remark 3 we now directly conclude that the *second component* of $(\mathbf{x},\mathbf{y},\mathbf{z})_R$ is

$$\begin{aligned}
&x'\left(\lambda_1^{-1}\lambda_2^{-1} - \lambda_4^{-1}\right) + y'\left(\lambda_2^{-1} - \lambda_3^{-1}\right) + O\left(e^4\right)\\
&= e^2\left(-2x'y'z\right) + O\left(e^4\right),
\end{aligned} \tag{10}$$

and the *first component* of $(\mathbf{x},\mathbf{y},\mathbf{z})_R$ is

$$x\left(\lambda_1^2\lambda_2^2-\lambda_4^2\right)+y\left(\lambda_1\lambda_2^2-\lambda_3^2\lambda_4\right)+z\left(\lambda_2-\lambda_3\lambda_4\right)+O\left(e^4\right).$$

We will reuse this formula for all the other associators later on. So let us now calculate this in detail. First of all

$$\begin{aligned}
&x\left(\lambda_1^2\lambda_2^2-\lambda_4^2\right)\\
&=x\Big[\left(1+e^2 2yx'+O\left(e^4\right)\right)\left(1+e^2\left(2zx'+2zy'\right)+O\left(e^4\right)\right)-\\
&-\left(1+e^2\left(2yx'+2zx'\right)+O\left(e^4\right)\right)\Big]\\
&=x\left[1+e^2\left(2yx'+2zx'+2zy'\right)-1-e^2\left(2yx'+2zx'\right)+O\left(e^4\right)\right]\\
&=e^2 2xzy'+O\left(e^4\right).
\end{aligned}$$

Secondly,

$$\begin{aligned}
&y\left(\lambda_1\lambda_2^2-\lambda_3^2\lambda_4\right)\\
&=y\Big[\left(1+e^2yx'\right)\left(1+e^2\left(2zx'+2zy'\right)+O\left(e^4\right)\right)-\\
&-\left(1+e^2 2zy'+O\left(e^4\right)\right)\left(1+e^2\left(yx'+zx'\right)+O\left(e^4\right)\right)\Big]\\
&=y\left[1+e^2\left(yx'+2zx'+2zy'\right)-1-e^2\left(2zy'+yx'+zx'\right)+O\left(e^4\right)\right]\\
&=e^2yzx'+O\left(e^4\right).
\end{aligned}$$

And lastly

$$\begin{aligned}
&z\left(\lambda_2-\lambda_3\lambda_4\right)\\
&=z\Big[\left(1+e^2\left(zx'+zy'\right)+O\left(e^4\right)\right)-\\
&-\left(1+e^2zy'\right)\left(1+e^2\left(yx'+zx'\right)+O\left(e^4\right)\right)\Big]\\
&=z\left[1+e^2\left(zx'+zy'\right)-1-e^2\left(zy'+yx'+zx'\right)+O\left(e^4\right)\right]\\
&=-e^2yzx'+O\left(e^4\right).
\end{aligned}$$

Thus the *first component* of $(\mathbf{x},\mathbf{y},\mathbf{z})_R$ is

$$\begin{aligned}
&\left(\lambda_1^2\lambda_2^2-\lambda_4^2\right)+y\left(\lambda_1\lambda_2^2-\lambda_3^2\lambda_4\right)+z\left(\lambda_2-\lambda_3\lambda_4\right)\\
&=e^2\left[2xzy'+yzx'-yzx'\right]+O\left(e^4\right)=e^2\left(2xzy'\right)+O\left(e^4\right).
\end{aligned}\tag{11}$$

Finally we conclude that

$$(\mathbf{x},\mathbf{y},\mathbf{z})_R=e^2\left\langle 2xzy',-2x'zy'\right\rangle+O\left(e^4\right)=e^2 2zy'\left\langle x,-x'\right\rangle+O\left(e^4\right).$$

Now, we would like to apply Lemma 1 to find out about $(\mathbf{x},\mathbf{y},\mathbf{z})_R^*$. Therefore we only need to show that

$$\lambda_5^*=1+O\left(e^4\right),$$

which is immediate, since we have seen in (10) that

$$x'\left(\lambda_1^{-1}\lambda_2^{-1}-\lambda_4^{-1}\right)+y'\left(\lambda_2^{-1}-\lambda_3^{-1}\right)=O\left(e^2\right),$$

and λ_5^* is by definition

$$1+e^2\left[(x\lambda_1^2+y\lambda_1)\lambda_2^2+z\lambda_2\right]\cdot\left[x'\left(\lambda_1^{-1}\lambda_2^{-1}-\lambda_4^{-1}\right)+y'\left(\lambda_2^{-1}-\lambda_3^{-1}\right)\right].$$

So we have

$$(\mathbf{x},\mathbf{y},\mathbf{z})_R^*=(\mathbf{x},\mathbf{y},\mathbf{z})_R$$

up to $O\left(e^4\right)$.

5.3. **The left associators.** Now, we will apply Lemma 2 to the results in the right loop reduct we just obtained to derive approximations for $(\mathbf{x},\mathbf{y},\mathbf{z})_L$ and $(\mathbf{x},\mathbf{y},\mathbf{z})_L^*$. Therefore we need to show that the assumptions of Lemma 2 are satisfied. First, notice that

$$\begin{aligned}&[(\mathbf{xy}\cdot\mathbf{z})]_1-[(\mathbf{x}\cdot\mathbf{yz})]_1\\&=x\left(\lambda_1^2\lambda_2^2-\lambda_4^2\right)+y\left(\lambda_1\lambda_2^2-\lambda_3^2\lambda_4\right)+z\left(\lambda_2-\lambda_3\lambda_4\right)\\&=O\left(e^2\right)\text{ by (11).}\end{aligned}\tag{12}$$

Second, since this factor appears in both of the fudge factors

$$\begin{aligned}d=1-e^2&\left[x'\lambda_4^{-1}+(y'\lambda_3^{-1}+z')\right]\times\\&\times\left[x\left(\lambda_1^2\lambda_2^2-\lambda_4^2\right)+y\left(\lambda_1\lambda_2^2-\lambda_3^2\lambda_4\right)+z\left(\lambda_2-\lambda_3\lambda_4\right)\right]+O\left(e^4\right)\end{aligned}$$

and

$$\begin{aligned}d^*=1+e^2&\left(x'\lambda_1^{-1}\lambda_2^{-1}+y'\lambda_2^{-1}+z'\right)\times\\&\times\left[x\left(\lambda_1^2\lambda_2^2-\lambda_4^2\right)+y\left(\lambda_1\lambda_2^2-\lambda_3^2\lambda_4\right)+z\left(\lambda_2-\lambda_3\lambda_4\right)\right]+O\left(e^4\right),\end{aligned}$$

we conclude that d and d^* are of order $1+O\left(e^4\right)$. So applying Lemma 2 to $(\mathbf{x},\mathbf{y},\mathbf{z})_R$ and $(\mathbf{x},\mathbf{y},\mathbf{z})_L$, and $(\mathbf{x},\mathbf{y},\mathbf{z})_R^*$ and $(\mathbf{x},\mathbf{y},\mathbf{z})_L^*$ respectively, we get

$$(\mathbf{x},\mathbf{y},\mathbf{z})_R=(\mathbf{x},\mathbf{y},\mathbf{z})_L\text{ and }(\mathbf{x},\mathbf{y},\mathbf{z})_R^*=(\mathbf{x},\mathbf{y},\mathbf{z})_L^*$$

up to $O\left(e^4\right)$.

6. Final Comparisons

Having done the approximations we now observe the following symmetries.

Theorem 1. *If $e^4=0$, we have*

$$-\left[\mathbf{y},\mathbf{x}\right]_L=-\left[\mathbf{y},\mathbf{x}\right]_R=\left[\mathbf{x},\mathbf{y}\right]_R=\left[\mathbf{x},\mathbf{y}\right]_L$$

and

$$-(\mathbf{x},\mathbf{y},\mathbf{z})_L^*=-(\mathbf{x},\mathbf{y},\mathbf{z})_R^*=(\mathbf{x},\mathbf{y},\mathbf{z})_R=(\mathbf{x},\mathbf{y},\mathbf{z})_L.$$

Further, let $\overleftarrow{\mathbf{x}}=\left\langle x',x\right\rangle$.

Theorem 2. *If $e^4 = 0$, we have*

$$(\mathbf{x}, \mathbf{y}, \mathbf{z})_R = (\mathbf{x}, \overleftarrow{\mathbf{z}}, \overleftarrow{\mathbf{y}})_R$$

and

$$(\overleftarrow{\mathbf{x}}, \mathbf{y}, \mathbf{z})_R = -\overleftarrow{(\mathbf{x}, \mathbf{y}, \mathbf{z})}_R.$$

Note that one may combine the identities in Theorem 1 with the symmetries given in Theorem 2 to observe even more symmetries.

7. The Special Case of $e = 2$

Let $R = \mathbb{Z}/2^{n+1}\mathbb{Z}$ (as in the motivating case, see [1]). Then $e = 2$ is certainly nilpotent in R with annihilator $E = \{0, 2^n\}$. And we get

$$(\mathbf{x}, \mathbf{y}, \mathbf{z})_R = e^3 zy' \langle x, -x' \rangle + O\left(e^4\right)$$

and

$$[\mathbf{x}, \mathbf{y}]_R = e^2 \langle x'y^2 - y'x^2, xy'^2 - yx'^2 \rangle + e^3 \langle xy\left(x' - y'\right), 0 \rangle + O\left(e^4\right).$$

Now, we will have a closer look at the Catalan loop on

$$G = \left(\left(\mathbb{Z}/2^{3+1}\mathbb{Z}\right) / \left\{0, 2^3\right\}\right)^2 = \left(\mathbb{Z}/2^3\mathbb{Z}\right)^2,$$

which turns out to be a group.

Theorem 3. $(G, \cdot, \mathbf{0})$ *is a group.*

Proof. Note that

$$(\mathbf{x}, \mathbf{y}, \mathbf{z})_R = e^3 zy' \langle x, -x' \rangle + O\left(e^4\right) = 0,$$

since $e^3 = 8 = 0$ in $\mathbb{Z}/2^3\mathbb{Z}$. Thus we have associativity, and are done, since the Catalan loop $(G, \cdot, /, \backslash, \mathbf{0})$ is a two-sided loop. □

Considering the commutator

$$[\mathbf{x}, \mathbf{y}]_R = e^2 \langle x'y^2 - y'x^2, xy'^2 - yx'^2 \rangle$$

we see that $(G, \cdot, \mathbf{0})$ is certainly *non-abelian*, since

$$[(0, 1), (1, 0)]_R = (4, 4) \neq \mathbf{0}.$$

It turns out that the commutator subgroup $[G, G]$ of G only consists of $\mathbf{0}$ and $\mathbf{4}$ as we will see in the following Proposition.

Proposition 1. $[G, G] = \{\mathbf{0}, \mathbf{4}\}$.

Proof. First, we will show that

$$[\mathbf{x}, \mathbf{y}]_R = 4 \cdot \langle x'y^2 - y'x^2, xy'^2 - yx'^2 \rangle$$

is either $\mathbf{0}$ or $\mathbf{4}$. In other words we desire to prove that

$$x'y^2 - y'x^2 \text{ and } xy'^2 - yx'^2 \tag{13}$$

are either both even or they both are odd. In the following the equalities are all to be taken modulo 2.

(1) Assume $x'y = y'x$. This is the case if and only if $x'y^2 = y'x^2$ and $xy'^2 = yx'^2$, since $x = x^2$. Thus both sums in (13) yield an even result, since a sum is even, if and only if its two summands are either both even or they are both odd.
(2) Otherwise, $x'y \neq y'x$ iff $x'y^2 \neq y'x^2$ and $xy'^2 \neq yx'^2$. Thus both sums in (13) yield an odd result.

So $[G, G]$ is generated by $\mathbf{0}$ and $\mathbf{4}$. But any multiplication involving only these two elements is componentwise addition, since the fudge factors involved are equal to 1. Hence $[G, G]$ is equal to $\{\mathbf{0}, \mathbf{4}\}$. □

Theorem 4. *The Abelianization $G/[G, G]$ is isomorphic to $\mathbb{Z}_8 \oplus \mathbb{Z}_4$.*

Proof. Firstly, note that $(0, 1)[G, G]$ is an element of order 8 in $G/[G, G]$ and generates the set

$$\langle (0, 1)[G, G] \rangle = \big\{ (0, n)[G, G] \bigm| n = 0, 1, \ldots, 7 \big\},$$

since $(0, 1)^n = (0, n)$ in G and $[G, G] = \{\mathbf{0}, \mathbf{4}\}$ by Proposition 1.
As an abelian group of order 2^5 $G/[G, G]$ is then either isomorphic to

$$\mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \text{ or } \mathbb{Z}_8 \oplus \mathbb{Z}_4.$$

Secondly, consider the quotient of $G/[G, G]$ by $\langle (0, 1)[G, G] \rangle$. Note that in this quotient the second component of a representative of any coset can be chosen to be 0, while the first component can be chosen to be between 0 and 3. Hence the quotient of $G/[G, G]$ by $\langle (0, 1)[G, G] \rangle$ consists of the elements

$$((n, 0)[G, G]) \langle (0, 1)[G, G] \rangle \text{ with } n = 0, 1, \ldots, 3.$$

Now, we can see that $((1, 0)[G, G]) \langle (0, 1)[G, G]) \rangle$ is an element of order 4 in $(G/[G, G]) \,/\, \langle (0, 1)[G, G] \rangle$, since $(1, 0)^n = (n, 0)$ in G. Hence $G/[G, G]$ is isomorphic to $\mathbb{Z}_8 \oplus \mathbb{Z}_4$. □

Finally, let us consider the center of G

$$Z(G) = \Big\{ \mathbf{x} \in G \Bigm| [\mathbf{x}, \mathbf{y}]_R = \mathbf{0} \text{ for all } \mathbf{y} \in \left(\mathbb{Z}/2^3\mathbb{Z}\right)^2 \Big\}.$$

Note that

$$[\mathbf{x}, \mathbf{y}]_R = \mathbf{0} \Leftrightarrow [\mathbf{x}, \mathbf{y}]_L = \mathbf{0} \Leftrightarrow [\mathbf{y}, \mathbf{x}]_R = \mathbf{0} \Leftrightarrow [\mathbf{y}, \mathbf{x}]_L = \mathbf{0},$$

which suggests other equivalent definitions of $Z(G)$.

Proposition 2. $Z(G) = \big\{ \mathbf{x} \in G \bigm| x \textit{ and } x' \textit{ are even} \big\}$

Proof. From the formula of $[\mathbf{x},\mathbf{y}]_R$ we see that

$$\mathbf{x} \in Z(G) \text{ iff } 4 \cdot \left\langle x'y^2 - y'x^2, xy'^2 - yx'^2 \right\rangle = \mathbf{0} \text{ for all } \mathbf{y}.$$

Or equivalently

$$\mathbf{x} \in Z(G) \text{ iff } x'y^2 - y'x^2 \text{ and } xy'^2 - yx'^2 \text{ are even for all } \mathbf{y}.$$

Using the equivalences given in the first case of the case differentiation in the proof of Proposition 1, we have

$$\mathbf{x} \in Z(G) \text{ iff } x'y = y'x \ \%2 \text{ for all } \mathbf{y}, \tag{14}$$

since $x = x^2$ %2 for all x. Choosing $\mathbf{y} = \mathbf{1}$ in (14) now shows that both components of $\mathbf{x}$ have to be equal modulo 2. Then setting $\mathbf{y} = (0, 1)$ in (14) yields that $\mathbf{x}$ is necessarily even. This condition is certainly sufficient, and we are done. □

Theorem 5. *The central quotient $G/Z(G)$ is isomorphic to the Vierergruppe.*

Proof. By Proposition 1 the central quotient is of order $(2^3 \cdot 2^3) / (2^2 \cdot 2^2) = 4$. Then $G/Z(G)$ either is the cyclic group $\mathbb{Z}_4$ or the Vierergruppe. But $G/Z(G)$ is not cyclic, since G is not abelian, and we are done. □

8. Future Work

Given the approximations for the commutators and associators, and the symmetries observed in Theorems 1 and 2, it is now of interest to find formulae using higher orders of approximation. In order to do that one may refer to the general formulae given in Section 4. Eventually we hope for a pattern to be recognized in order to characterize the structure of Catalan loops in a way similar to what was shown in Section 7 for the group case.

References

[1] L. Long and J.D.H. Smith, *Catalan Loops*, preprint, 2009.

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