

On the Equipollence of \mathcal{L}^\times and \mathcal{L}_r^\times

by

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A thesis submitted to the graduate faculty
in partial fulfillment of the requirements for the degree of
MASTER OF SCIENCE

Major: Mathematics

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2007

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ABSTRACT

In this paper the deductive systems \mathcal{L}^\times and \mathcal{L}_r^\times from [Tarski, Givant (1987)] are shown to be equipollent. Following [Tarski, Givant (1987)], a third deductive system, \mathcal{L}_s^\times , is defined. It is then noted that \mathcal{L}_s^\times is an extension of both \mathcal{L}^\times and \mathcal{L}_r^\times , thus showing that \mathcal{L}^\times and \mathcal{L}_r^\times are no stronger than \mathcal{L}_s^\times . Finally two Translation Mappings from $G : \mathcal{L}_s^\times \rightarrow \mathcal{L}^\times$ and $H : \mathcal{L}_s^\times \rightarrow \mathcal{L}_r^\times$ are defined. These mappings show that \mathcal{L}_s^\times is no stronger than either \mathcal{L}^\times or \mathcal{L}_r^\times . Therefore we conclude that, as they are both equipollent to a relative extension, \mathcal{L}^\times and \mathcal{L}_r^\times are equipollent.

CHAPTER 1. History

Set theory is at the heart of mathematics. It is generally accepted that it was Cantor's strangely named paper "On a Characteristic Property of All Real Algebraic Numbers" (see [**Cantor, Georg (1874)**]) that was the true beginning of modern set theory or, more precisely, of transfinite set theory. In this paper he showed that the real numbers were not equinumerous with the integers, thus changing how infinity, and infinite sets, could be discussed. His set theory was outlined in a six part treatise that appeared in *Mathematische Annalen* between the years 1879 and 1884. Despite objections by Kronecker, set theory was gaining popularity. In the first years of the 20th century, Lebesgue defined (analytic) measure and the Lebesgue integral in terms of Cantor's set theoretical concepts.

In 1908 Zermelo made an attempt to axiomatize set theory (see [**Zermelo, Ernst (1908)**]). This axiomatic theory did not allow the construction of the ordinal numbers. Zermelo's Axiom III of set theory invoked a concept called the "definite" property, whose operational meaning was ambiguous. Also of note, this work is also believed to be the first appearance of Cantor's Theorem that states "Every set is of lower cardinality than the set of its subsets." Around this time many others tried to axiomatize set theory. Some of the mathematicians working on set theory during this time included Fraenkel, van Neumann, Bernays, Skolem, and Gödel.

In 1922, Fraenkel and Skolem created the axiom of replacement, the axiom of regularity (or foundation), and defined this so called "definite" property as any property that could be formulated in first-order logic with some restrictions. With this definition and the two axioms mentioned above Zermelo–Fraenkel (ZF) set theory was created from Zermelo set theory. This axiomatization also had a vaguely defined concept. Specifically, the term "functions" occurred in Fraenkel's replacement axiom. The modern axiom set fixes this problem and this (modified)

axiomatization along with the axiom of choice is referred to as ZFC. It has been shown by Gödel (in Gödel's second incompleteness theorem) that the consistency of ZFC cannot be proven in ZFC.

In this paper we wish to look at three deductive systems set forth in [Tarski, Givant (1987)], namely \mathcal{L}^\times , \mathcal{L}_r^\times , and \mathcal{L}_s^\times . \mathcal{L}^\times first appeared, in a slightly different form, in [Chin, Tarski (1951)]. It has the power to express most well-known systems of set theory, but is built entirely without variables. \mathcal{L}^\times is semantically incomplete. It is shown in [Tarski, Givant (1987)] that \mathcal{L}^\times is equipollent to a fragment of first-order logic having only one binary predicate and only three variables. It is thus surprising that \mathcal{L}^\times is powerful enough to represent almost all modern models of set theory.

The predicates of \mathcal{L}^\times will be the closure under \mathbf{i} and E under $\mathbf{\odot}, +, -, \mathbf{\forall}$. The sentences of \mathcal{L}^\times will be equations between the predicates.

It is here worth noting that the axioms in \mathcal{L}^\times have the same form as an axiom set for relation algebras. Relation algebras were created by Tarski and his contemporaries in the middle of the 20th century. (For a complete overview and history of relation algebras see [Maddux, Roger D. (2006)]). The only rule of inference in \mathcal{L}^\times is the familiar rule from high school of replacing equals with equals. The formulas in \mathcal{L}^\times have the form of relation algebraic equations with one free variable E. Therefore an equation is provable in the deductive system \mathcal{L}^\times if and only if it has the form of an equation in one free variable that holds in every relation algebra. As set theory can be formalized in \mathcal{L}^\times (as shown in [Tarski, Givant (1987)], Chapter 7), \mathcal{L}^\times is undecidable.

The deductive system \mathcal{L}_r^\times is then built from \mathcal{L}^\times by removing equality, namely the = sign, from the language. When the axioms are chosen for \mathcal{L}_r^\times we start with three axioms that come from [Götlind, Erik (1947)] and [Rasiowa, H. (1949)] and can be found in [Church (1956)] (see p. 137, namely the set labeled P_G). It is stated in that text that these axioms, along with modus ponens, are enough to get all of propositional calculus, a fact that we will go into later in this paper.

We take a moment to compare \mathcal{L}_r^\times to ordinary sentential calculus \mathcal{T} . It is pointed out in

[Tarski, Givant (1987)] that \mathcal{L}_r^\times is a subformalism of \mathcal{T} . In \mathcal{T} we use a different symbol for the fixed true sentence namely \mathbf{T} instead of the notation we chose in \mathcal{L}_r^\times which is $\mathbf{\dot{1}}$. \mathcal{T} also has a fixed sentence whose truth value is undetermined (in \mathcal{L}_r^\times we used \mathbf{E}). The binary operations have a different intended meaning in \mathcal{T} then that we will suggest for their use in \mathcal{L}^\times in the beginning of Chapter 2. Namely $\mathbf{+}$ denotes disjunction, $\mathbf{\odot}$ denotes conjunction, $\mathbf{-}$ denotes negation, and $\mathbf{\smile}$ denotes something called affirmation. The affirmation symbol does not occur in many formalizations of sentential logic and is, more or less, superfluous. The only direct rule of inference is modus ponens for \Rightarrow where $A \Rightarrow B$ is an abbreviation for $A^- \mathbf{+} B$. \mathcal{T} is decidable as the true sentences are just those that are tautologies in the normal truth table evaluation.

Following the discussion on page 166-168 of [Tarski, Givant (1987)] we deduce that \mathcal{L}_r^\times is indeed a subformalism of \mathcal{T} . However \mathcal{L}_r^\times is undecidable. This formalism, \mathcal{L}_r^\times is, historically, the first undecidable subsystem of sentential logic (found in 1942). However, as it was not published until [Tarski, Givant (1987)], several other systems were published in between. (See, for example [Linial, Post (1949)], [Yntema, M.K. (1964)], and [Singletary (1964)]).

It is the purpose of this paper to fill in the proof that these two very interesting deductive systems, namely \mathcal{L}^\times and \mathcal{L}_r^\times are equipollent. We do this by introducing a third deductive system. The final deductive system, \mathcal{L}_s^\times will be built from \mathcal{L}_r^\times by adding the $=$ sign back into it, treating $=$ as a binary operation. This third system is only introduced to the ends of showing that \mathcal{L}^\times and \mathcal{L}_r^\times are equipollent. We follow the method outlined in [Tarski, Givant (1987)], showing first that \mathcal{L}_s^\times is an extension of \mathcal{L}^\times and \mathcal{L}_r^\times and then showing that there are translation mappings from \mathcal{L}_s^\times to \mathcal{L}^\times and \mathcal{L}_r^\times . These facts together will give us our desired result, that \mathcal{L}^\times and \mathcal{L}_r^\times are equipollent both in terms of expressibility and in terms of provability. Before we get to our deductive systems, we need some basic definitions and symbols.

CHAPTER 2. The deductive systems \mathcal{L}^\times , \mathcal{L}_r^\times , and \mathcal{L}_s^\times

2.1 Preliminaries

We will denote the set of all sentences of our systems by \mathcal{L}^\times , \mathcal{L}_r^\times , \mathcal{L}_s^\times by Σ^\times , Σ_r^\times , Σ_s^\times respectively. The predicates of \mathcal{L}^\times will be denoted as Π . We will use the following definition several times.

Definition 1. *The symbols $\mathring{\mathbf{0}}, \mathbf{1}, \mathbf{0}$ are defined as follows:*

$$(1) \mathring{\mathbf{0}} = \mathring{\mathbf{1}}^-$$

$$(2) \mathbf{1} = \mathring{\mathbf{1}} + \mathring{\mathbf{1}}^-$$

$$(3) \mathbf{0} = (\mathring{\mathbf{1}} + \mathring{\mathbf{1}}^-)^- \text{ or } \mathbf{0} = \mathbf{1}^-$$

We briefly mention the intended interpretations of our symbols in each of our systems. First we will assume that our systems are acting on $\mathcal{U} = \langle U, E \rangle$ where U is some collection of objects and E is a fixed binary operation. Now write the semantics of our symbols.

1. $\mathring{\mathbf{1}} = \{(x, y) | x = y, x, y \in U\}$, the identity relation.
2. $\mathring{\mathbf{0}} = \{(x, y) | x \neq y, x, y \in U\}$, the diversity relation.
3. $\mathbf{1} = \{(x, y) | x, y \in U\}$, the universal relation.
4. $\mathbf{0} = \{(x, y) | x = y \wedge x \neq y, x, y \in U\}$, the empty relation.
5. $A \odot B = \{(x, y) | \exists z (x, z) \in A \wedge (z, y) \in B\}$, relative composition (or relative multiplication).

6. $A + B = \{(x, y) | (x, y) \in A \vee (x, y) \in B\}$, absolute addition.
7. $A^- = \{(x, y) | (x, y) \notin A, (x, y) \in U \times U\}$, complementation.
8. $A^\smile = \{(x, y) | (y, x) \in A\}$, conversion.

We will not be working with the semantics of our systems, but we feel it is important to note here their interpretations to help the reader more quickly understand the document.

We are now ready to begin looking at our deductive systems.

2.2 The Deductive System \mathcal{L}^\times

We begin with \mathcal{L}^\times . The vocabulary of \mathcal{L}^\times is seven symbols, two identity symbols, $\overset{\circ}{\mathbf{1}}$ and $=$, four operators, \smile , $-$, \odot , $+$, and \mathbf{E} . \mathbf{E} is the only nonlogical constant. The set of all predicates in \mathcal{L}^\times will be denoted by $\mathbf{\Pi}$. $\mathbf{\Pi}$ is the closure of $\overset{\circ}{\mathbf{1}}$ and \mathbf{E} under $+$, $-$, \odot , and \smile . If A and B are predicates in \mathcal{L}^\times then so are $A + B$, A^- , $A \odot B$, and A^\smile . The sentences of \mathcal{L}^\times will be denoted as Σ^\times , are just the equations $A = B$ where $A, B \in \mathbf{\Pi}$.

2.2.1 Definition of X^l, X^r, X^t , and tautology

Let $X = (A = B)$ be a sentence in Σ^\times . We set:

1. X^l (the left side of X) $= A$;
2. X^r (the right side of X) $= B$;
3. $X^t = X^l$ if $X^r = \mathbf{1}$; $X^t = X^l \cdot X^r + X^{l-} \cdot X^{r-}$ otherwise;
4. X is called a tautology if $X^l = X^r$, i.e., if A coincides with B .

The logical axioms of \mathcal{L}^\times are infinite, but they can be expressed, in the usual fashion, as particular instances of a few simple axiom schemata.

2.2.2 Axioms of \mathcal{L}^\times

S is an axiom of \mathcal{L}^\times , or $S \in \Lambda^\times$, iff, for some $A, B, C \in \mathbf{\Pi}$, S coincides with one of the following equations (BI)–(BX).

$$(BI) \quad A + B = B + A$$

$$(BII) \quad A + (B + C) = (A + B) + C$$

$$(BIII) \quad (A^- + B)^- + (A^- + B^-)^- = A$$

$$(BIV) \quad A \odot (B \odot C) = (A \odot B) \odot C$$

$$(BV) \quad (A + B) \odot C = A \odot C + B \odot C$$

$$(BVI) \quad A \odot \mathbf{1} = A$$

$$(BVII) \quad A^{\sim} = A$$

$$(BVIII) \quad (A + B)^{\sim} = A^{\sim} + B^{\sim}$$

$$(BIX) \quad (A \odot B)^{\sim} = B^{\sim} \odot A^{\sim}$$

$$(BX) \quad A^{\sim} \odot (A \odot B)^- + B^- = B^-$$

Here we note that axioms (BI)-(BIII) come from [Huntington (1933a)]. It should also be noted that in [Huntington (1933b)] that Huntington corrected his original presentation of the axiom set to exclude one of the original axioms presented, namely $A + A = A$. As shown there, and aped here with the appropriate symbols from our language, this is a derivable theorem from the axiom set without it. It is also of note that Huntington outlined six different axiom sets for Boolean Algebras, focusing more on the latter three of these (which is where axioms (BI)-(BIII) originate). The other three were already written up in [Huntington (1904)]. Thus from the first three axioms we can derive many standard properties of Boolean Algebra. We look into this fact in Chapter 2.

We also want to talk about derivability in \mathcal{L}^{\times} . We will do so via the following definition.

Definition 2. Define \vdash^{\times} as follows: For any given $X \in \Sigma^{\times}$ and $\Psi \subseteq \Sigma^{\times}$, X is derivable from Ψ (in \mathcal{L}^{\times}), with notation $\Psi \vdash^{\times} X$, iff X belongs to every set $\Omega \subseteq \Sigma^{\times}$ satisfying the following conditions:

1. $\Psi \subseteq \Omega$,

2. $\Lambda^\times \subseteq \Omega$,
3. $A = A \in \Omega$ for all $A \in \mathbf{\Pi}$,
4. if $B, C, D \in \mathbf{\Pi}$ and $B = C, B = D \in \Omega$, then $C = D \in \Omega$;
5. if $B, C, D \in \mathbf{\Pi}$ and $B = C \in \Omega$, then $B \mathbf{+} D = C \mathbf{+} D, D \mathbf{+} B = D \mathbf{+} C, B^- = C^-, B \odot D = C \odot D, D \odot B = D \odot C$, and $B^\smile = C^\smile$ also belong to Ω .

Now we note some simple lemmas from this definition.

Lemma 1. $B = C, B = D \vdash^\times C = D$.

Proof. This follows directly from Definition 2 part 3. □

Lemma 2. 1. $B = C \vdash^\times B \mathbf{+} D = C \mathbf{+} D$,

$$2. B = C \vdash^\times D \mathbf{+} B = D \mathbf{+} C,$$

$$3. B = C \vdash^\times B^- = C^-,$$

$$4. B = C \vdash^\times B \odot D = C \odot D,$$

$$5. B = C \vdash^\times D \odot B = D \odot C, \text{ and}$$

$$6. B = C \vdash^\times B^\smile = C^\smile.$$

Proof. This follows directly from Definition 2 part 5. □

Lemma 3. If $\Gamma \vdash^\times X$ for every $X \in \Psi$ and $\Psi \vdash^\times Y$ then $\Gamma \vdash^\times Y$.

Proof. Let Ω be defined as it is in Definition 2. Let $X \in \Gamma$. Then X is in Ω (as $\Gamma \vdash^\times X$).

Therefore $\Gamma \subseteq \Omega$. We now note that Ω has the following properties:

1. $\Gamma \subseteq \Omega$,
2. $\Lambda^\times \subseteq \Omega$,
3. $A = A \in \Omega$ for all $A \in \mathbf{\Pi}$,

4. if $B, C, D \in \mathbf{\Pi}$ and $B = C, B = D \in \Omega$, then $C = D \in \Omega$;
5. if $B, C, D \in \mathbf{\Pi}$ and $B = C \in \Omega$, then $B + D = C + D, D + B = D + C, B^- = C^-, B \odot D = C \odot D, D \odot B = D \odot C$, and $B^\bullet = C^\bullet$ also belong to Ω .

Finally as $\Gamma \vdash^\times Y$ we have that $Y \in \Omega$ and $\Gamma \vdash^\times Y$. □

Lemma 4. $\Psi \vdash^\times X, \Psi \subseteq \Gamma$, then $\Gamma \vdash^\times X$

Proof. Assume $\Psi \vdash^\times X$. Then $X \in \Omega$ with the properties in Definition 2. Now let Ω' be any set defined by Definition 2 with Ω replaced by Ω' and Ψ replaced by Γ . Then notice that as $\Psi \subseteq \Gamma$ our Ω' has the following properties:

1. $\Psi \subseteq \Omega'$,
2. $\Lambda^\times \subseteq \Omega'$,
3. $A = A \in \Omega'$ for all $A \in \mathbf{\Pi}$,
4. if $B, C, D \in \mathbf{\Pi}$ and $B = C, B = D \in \Omega'$, then $C = D \in \Omega'$;
5. if $B, C, D \in \mathbf{\Pi}$ and $B = C \in \Omega'$, then $B + D = C + D, D + B = D + C, B^- = C^-, B \odot D = C \odot D, D \odot B = D \odot C$, and $B^\bullet = C^\bullet$ also belong to Ω' .

Therefore $X \in \Omega$ as $\Psi \vdash^\times X$. □

Lemma 5. If $X \in \Psi$ then $\Psi \vdash^\times X$

Proof. Let Ω be defined as it is in Definition 2. Then we simply note that as $X \in \Psi$ and $\Psi \subseteq \Omega$ that X is in every Ω with the properties of Definition 2. □

Lemma 6. If X is an instance of an axiom in Λ^\times then $\vdash^\times X$

Proof. Let X be an instance of an axiom in Λ^\times . Define Ω as in Definition 2 (with $\Psi = \emptyset$). Then by Definition 2 we have $X \in \Omega$. □

We will not state Lemma 6 as a reason when we want to use an axiom from our system. We will simply place the axiom name on the corresponding line and Lemma 6 will be implied.

The deductive formalism \mathcal{L}^\times is show to be equipollent to \mathcal{L}_3 , a three variable subformalism of \mathcal{L} . This proof is carried out in [Tarski, Givant (1987)] beginning on page 76. The fact that \mathcal{L}^\times is sufficient to formalize set theory systems is discussed in [Tarski, Givant (1987)] on in section 4.6. which begins on page 127.

2.2.3 Proofs of basic Boolean algebra concepts in \mathcal{L}^\times

As we need several basic lemmas from Boolean algebra in our upcoming discussion. First we want to insure that our symbol $=$ behaves like an equals sign.

Lemma 7. $\vdash^\times A = A$

Proof. Notice that this is follows from Definition 2.3. However we show here that $A = A$ in that definition can be eliminated based on the other properties.

- | | | |
|----|--|--------------|
| 1. | $\vdash^\times A^{\smile} = A$ | BVII |
| 2. | $\vdash^\times A^{\smile} = A$ | BVII |
| 3. | $A^{\smile} = A, A^{\smile} = A \vdash^\times A = A$ | Lemma 1 |
| 4. | $\vdash^\times A = A$ | Definition 2 |

□

Lemma 8. $A = B \vdash^\times B = A$

Proof.

- | | | |
|----|------------------------------------|--------------|
| 1. | $\vdash^\times A = A$ | Lemma 7 |
| 2. | $A = B, A = A \vdash^\times B = A$ | Lemma 1 |
| 3. | $A = B \vdash^\times B = A$ | Definition 2 |

□

Lemma 9. $A = B, B = C \vdash^\times A = C$

Proof.

- | | |
|---------------------------------------|------------------|
| 1. $A = B, B = C \vdash^\times B = A$ | Lemma 4, Lemma 8 |
| 2. $B = A, B = C \vdash^\times A = C$ | Lemma 1 |
| 3. $A = B, B = C \vdash^\times A = C$ | Lemma 3 |

□

We want a Lemma that states if $A = B$ and $C = D$ then we can replace any (or all) occurrences of C with D in A . We begin to prove this by making the following definition.

Definition 3. We define $\rho(A, A', C, D)$ to mean that A' can be obtained from A by replacing either no, one, or several occurrences of C in A with D . The recursive defining conditions for ρ are as follows:

- (1) $\rho(A, A, C, D)$ for all $A, C, D \in \mathbf{\Pi}$,
- (2) for any $A, C, D \in \mathbf{\Pi}$, $\rho(C, A, C, D)$ if and only if $A = C$ or $A = D$,
- (3) if $\rho(E, A, C, D)$ then $E = C$ for all $A, C, D \in \mathbf{\Pi}$,
- (4) if $\rho(\mathbf{\dot{1}}, A, C, D)$ then $\mathbf{\dot{1}} = C$ for all $A, C, D \in \mathbf{\Pi}$,
- (5) if $\rho(A + B, X, C, D)$ then there are $A', B' \in \mathbf{\Pi}$ such that $X = A' + B'$, $\rho(A, A', C, D)$ and $\rho(B, B', C, D)$
- (6) if $\rho(A \odot B, X, C, D)$ then there are $A', B' \in \mathbf{\Pi}$ such that $X = A' \odot B'$, $\rho(A, A', C, D)$ and $\rho(B, B', C, D)$
- (7) if $\rho(A^\smile, X, C, D)$ then there is some $A' \in \mathbf{\Pi}$ such that $X = A'^\smile$ and $\rho(A, A', C, D)$, and
- (8) if $\rho(A^-, X, C, D)$ then there is some $A' \in \mathbf{\Pi}$ such that $X = A'^-$ and $\rho(A, A', C, D)$.

Lemma 10. If $\rho(A, A', C, D)$ then $C = D \vdash^\times A = A'$.

Proof. Fix C and D . Define $\Pi' := \{A : \forall_{A'}(\rho(A, A', C, D) \Rightarrow C = D \vdash^\times A = A')\}$. We need to show that $A \in \Pi'$ by induction on formation of predicates. We must show the following:

- (a) $E \in \Pi'$
- (b) $\overset{\circ}{\mathbf{1}} \in \Pi'$
- (c) if $A, B \in \Pi'$ then $A + B \in \Pi'$
- (d) if $A, B \in \Pi'$ then $A \odot B \in \Pi'$
- (e) if $A \in \Pi'$ then $A^\bullet \in \Pi'$
- (f) if $A \in \Pi'$ then $A^- \in \Pi'$

Proof. of (a): Consider an arbitrary $A' \in \mathbf{\Pi}$ and assume $\rho(E, A', C, D)$. By Definition 3(3) we have $E = C$, hence $\rho(E, A', E, D)$. By Definition 3(2) either $A' = E$ or $A' = D$, so we have either $\rho(E, E, E, D)$ or $\rho(E, D, E, D)$. Therefore we need only conclude either $E = D \vdash^\times E = E$ or $E = D \vdash^\times E = D$, both of which are easily derived in \mathcal{L}^\times . \square

Proof. of (b): This proof is the same as that of (a) with $\overset{\circ}{\mathbf{1}}$ in place of E and we use Definition 3(4) instead of (3). \square

Proof. of (c): Assume $A, B \in \Pi'$. We want to show $A + B \in \mathbf{\Pi}$. Let $X \in \mathbf{\Pi}$ and assume $\rho(A + B, X, C, D)$. By Definition 3(5), there are $A', B' \in \mathbf{\Pi}$ such that $X = A' + B'$, $\rho(A, A', C, D)$, and $\rho(B, B', C, D)$. We prove at $C = D \vdash^\times A + B = X$ as follows.

- | | |
|--|-------------------------|
| 1. $C = D \vdash^\times A = A'$ | $A \in \Pi'$ |
| 2. $C = D \vdash^\times B = B'$ | $B \in \Pi'$ |
| 3. $A = A' \vdash^\times A + B = A' + B$ | Definition 3(5) |
| 4. $B' = B \vdash^\times A' + B = A' + B'$ | Definition 3(6) |
| 5. $C = D \vdash^\times A + B = A' + B$ | 1., 3., Definition 3(4) |
| 6. $C = D \vdash^\times A' + B = A' + B$ | 2., 4., Definition 3(4) |

- | | | |
|----|---------------------------------------|-----------------|
| 7. | $C = D \vdash^\times A + B = A' + B'$ | 5., 6., Lemma 9 |
| 8. | $C = D \vdash^\times A + B = X$ | $X = A' + B'$ |

□

The proofs of (c)-(f) are essentially the same as the proof of (c), with only minor changes in references and replacement of symbols with other symbols. These are left to the reader. □

Lemma 11. *If $\rho(A, A', C, D)$ and $\rho(B, B', C, D)$ then $A = B, C = D \vdash^\times A' = B'$*

Proof.

- | | | |
|----|-------------------------------------|--------------------------------|
| 1. | $C = D \vdash^\times A = A'$ | $\rho(A, A', C, D)$, Lemma 10 |
| 2. | $C = D \vdash^\times B = B'$ | $\rho(B, B', C, D)$, Lemma 10 |
| 3. | $A = B, C = D \vdash^\times A = A'$ | 1., Lemma 4 |
| 4. | $A = B, C = D \vdash^\times B = B'$ | 2., Lemma 4 |
| 5. | $A = B, C = D \vdash^\times A = B$ | Lemma 5 |
| 6. | $A = B, C = D \vdash^\times A = B'$ | 4., 5., Lemma 3, Lemma 9 |
| 7. | $A = B, C = D \vdash^\times A = B'$ | 2., 6., Lemma 1 |

□

Lemma 12. *If $\vdash^\times A = B, \vdash^\times C = D$, $\rho(A, A', C, D)$ and $\rho(B, B', C, D)$, then $\vdash^\times A' = B'$.*

Proof. This follows directly from Lemma 11. □

Next we will prove a lemma that is fairly basic, namely $\vdash^\times A + A^- = B + B^-$. In this proof we will outline all of the (tedious) steps for demonstrative purposes. We will not go into as much detail in later proofs. Here we also note that outlines of the proofs of Lemmas 13-18 can be found in [Maddux, Roger D. (1996)] and [Huntington (1933a)] or [Huntington (1933b)].

Lemma 13. $A + A^- = B + B^-$

Proof.

1. $\vdash^\times A = (A^- + B^-)^- + (A^- + B^{--})^-$ BI $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$
2. $\vdash^\times A^- = (A^{--} + B^-)^- + (A^{--} + B^{--})^-$ BI $\begin{pmatrix} A & B \\ A^- & B^- \end{pmatrix}$
3. $\vdash^\times A + A^- = A + (A^{--} + B^-)^- + (A^{--} + B^{--})^-$ 2., Lemma 12
4. $\vdash^\times A + A^- = [(A^- + B^-)^- + (A^- + B^{--})^-]$
 $\quad + [(A^{--} + B^-)^- + (A^{--} + B^{--})^-]$ 1., 3., Lemma 12
5. $\vdash^\times A^- + B^- = B^- + A^-$ BI $\begin{pmatrix} A & B \\ A^- & B^- \end{pmatrix}$
6. $\vdash^\times A + A^- = [(B^- + A^-)^- + (A^- + B^{--})^-]$
 $\quad + [(A^{--} + B^-)^- + (A^{--} + B^{--})^-]$ 4., 5., Lemma 12
7. $\vdash^\times A^- + B^{--} = B^{--} + A^-$ BI $\begin{pmatrix} A & B \\ A^- & B^{--} \end{pmatrix}$
8. $\vdash^\times A + A^- = [(B^- + A^-)^- + (B^{--} + A^-)^-]$
 $\quad + [(A^{--} + B^-)^- + (A^{--} + B^{--})^-]$ 6., 7., Lemma 12
9. $\vdash^\times A^{--} + B^- = B^- + A^{--}$ BI $\begin{pmatrix} A & B \\ A^{--} & B^- \end{pmatrix}$
10. $\vdash^\times A + A^- = [(B^- + A^-)^- + (B^{--} + A^-)^-]$
 $\quad + [(B^- + A^{--})^- + (A^{--} + B^{--})^-]$ 8., 9., Lemma 12
11. $\vdash^\times A^{--} + B^{--} = B^{--} + A^{--}$ BI $\begin{pmatrix} A & B \\ A^{--} & B^{--} \end{pmatrix}$
12. $\vdash^\times A + A^- = [(B^- + A^-)^- + (B^{--} + A^-)^-]$
 $\quad + [(B^- + A^{--})^- + (B^{--} + A^{--})^-]$ 10., 11., Lemma 12
13. $\vdash^\times A + A^- = [(B^- + A^-)^- + (B^- + A^{--})^-]$

- $$+ [(B^{--} + A^-)^- + (B^{--} + A^{--})^-] \quad \text{Lemma 9}$$
14. $\vdash^\times B = (B^- + A^-)^- + (B^- + A^{--})^- \quad \text{BIII} \begin{pmatrix} A & B \\ B & A^- \end{pmatrix}$
15. $\vdash^\times A + A^- = B + [(B^{--} + A^-)^- + (B^{--} + A^{--})^-] \quad \text{Lemma 12}$
16. $\vdash^\times B^- = (B^{--} + A^-)^- + (B^{--} + A^{--})^- \quad \text{BIII} \begin{pmatrix} A & B \\ B^- & A^- \end{pmatrix}$
17. $\vdash^\times A + A^- = B + B^- \quad \text{15., 16., Lemma 12}$

□

From this point forward we will leave out some of the details used in the last proof. Namely we will use Lemma 12 by simply quoting it as a reason instead of writing out each step, and only use the substitution notation when it is not obvious what we are substituting. We will especially avoid substitution notation in the cases when we use *BI* and *BII* as they are often used multiple times at once to rearrange terms and parenthesis. We will omit explicit references to the properties of $=$ except when necessary. Finally, when we have a proof with the lines $A = B$ and $A = C$ (usually concurrently) we will leave out the line $B = C$, quote Lemma 12 on the $A = C$ line, and the use of Lemma 9 will be implied.

The next property Boolean Algebra that we would like to have is that $\vdash^\times A = A^{--}$.

Lemma 14. $\vdash^\times A = A^{--}$

Proof.

1. $\vdash^\times A^{--} = (A^{----} + A^-)^- + (A^{----} + A^{--})^- \quad \text{BIII} \begin{pmatrix} A & B \\ A^{--} & A^- \end{pmatrix}$
2. $\vdash^\times A^{--} = (A^{----} + A^-)^- + (A^{--} + A^-)^- \quad \text{Lemma 13} \begin{pmatrix} A & B \\ A^{--} & A^- \end{pmatrix}, \text{Lemma 12}$
3. $\vdash^\times A^{--} = (A^- + A^{----})^- + (A^- + A^{--})^- \quad \text{BI, Lemma 12}$

$$\begin{array}{ll}
4. \quad \vdash^\times A = (A^- + A^{--})^- + (A^- + A^{--})^- & \text{BIII} \begin{pmatrix} A & B \\ A & A^- \end{pmatrix} \\
5. \quad \vdash^\times A^{--} = A & \text{BIII} \begin{pmatrix} A & B \\ A & A^{--} \end{pmatrix}
\end{array}$$

□

The following proof deals with the fact that $\vdash^\times A + A^- = \mathbf{1}$.

Lemma 15. $A + A^- = \mathbf{1}$

Proof. This follows directly from Lemma 13 $\begin{pmatrix} A & B \\ A & \mathbf{1} \end{pmatrix}$ and Definition 1(3). □

Now we want that $\vdash^\times A + \mathbf{1} = \mathbf{1}$.

Lemma 16. $\vdash^\times A + \mathbf{1} = \mathbf{1}$

Proof.

1. $A + \mathbf{1} = A + (A + A^-)$ Lemma 14
2. $A + \mathbf{1} = (A + A) + A^-$ BII, Lemma 10
3. $A + \mathbf{1} = (A + A) + ((A^{--} + A)^- + (A^{--} + A^-)^-)$ BIII, Lemma 10
4. $A + \mathbf{1} = (A + A) + ((A + A)^- + (A + A^-)^-)$ Lemma 26, Lemma 10
5. $A + \mathbf{1} = ((A + A) + (A + A)^-) + (A + A^-)^-$ BII
6. $A + \mathbf{1} = \mathbf{1} + \mathbf{1}^-$ Lemma 16, Lemma 12
7. $A + \mathbf{1} = \mathbf{1}$ Lemma 16, Lemma 12

□

The next lemma shows that $\vdash^\times A + \mathbf{0} = A$.

Lemma 17. $A + \mathbf{0} = A$

Proof.

- | | | |
|----|---|--------------------|
| 1. | $A + \mathbf{0} = A + \mathbf{1}^-$ | Definition 4(2) |
| 2. | $A + \mathbf{0} = A + (A + \mathbf{1})^-$ | Lemma 16 |
| 3. | $A + \mathbf{0} = ((A^- + \mathbf{1})^- + (A^- + \mathbf{1}^-)^-) + (A + \mathbf{1})^-$ | BIII, Lemma 10 |
| 4. | $A + \mathbf{0} = ((\mathbf{1} + A)^- + (\mathbf{1} + A^-)^-) + (A^- + \mathbf{1}^-)^-$ | BI, BII, Lemma 10 |
| 5. | $A + \mathbf{0} = ((\mathbf{1}^{--} + A)^- + (\mathbf{1}^{--} + A^-)^-) + (A^- + \mathbf{1}^-)^-$ | Lemma 26, Lemma 12 |
| 6. | $A + \mathbf{0} = \mathbf{1}^- + (A^- + \mathbf{1}^-)^-$ | BIII |
| 7. | $A + \mathbf{0} = (A^- + \mathbf{1})^- + (A^- + \mathbf{1}^-)^-$ | Lemma 16, Lemma 12 |
| 8. | $A + \mathbf{0} = A$ | BIII |

□

The next lemma is one that we have been leading up to. As such, it has a short proof but references many of the lemmas we have proved so far.

Lemma 18. $\vdash^\times A + A = A$

Proof.

- | | | |
|----|---|---|
| 1. | $\vdash^\times (A + A)^- = \mathbf{0} + (A + A)^-$ | BI, Lemma 17 |
| 2. | $\vdash^\times (A + A)^- = (A + A^-)^- + (A + A)^-$ | Definition 1, Lemma 15 |
| 3. | $\vdash^\times (A + A)^- = (A^{--} + A)^- + (A^{--} + A^-)^-$ | Lemma 14 |
| 4. | $\vdash^\times (A + A)^- = A^-$ | BIII $\begin{pmatrix} A & B \\ A^- & A \end{pmatrix}$ |
| 5. | $\vdash^\times (A + A)^{--} = A^{--}$ | 4., Lemma 2 |
| 6. | $\vdash^\times A + A = A$ | Lemma 14 |

□

Now that we have the basic facts from Boolean Algebra that we need, we pause to look at a rule of deduction in \mathcal{L}^\times . This rule will be very important in one of our upcoming theorems.

Lemma 19. $A = \mathbf{1} \vdash^\times B = \mathbf{1}$ iff $\vdash^\times \mathbf{1} \odot A^- \odot \mathbf{1} + B = \mathbf{1}$

This serves as a type of deduction theorem in \mathcal{L}^\times . We will use it here without proof and simply refer the reader to page 51 of [Tarski, Givant (1987)].

2.3 The formalism \mathcal{L}_r^\times

We now turn our attention to \mathcal{L}_r^\times . We will construct \mathcal{L}_r^\times by removing the equality symbol $=$ from the formalism \mathcal{L}^\times in the following way. First we note that in \mathcal{L}^\times any equation $A = B$ is logically equivalent to $A \cdot B + A^- \cdot B^- = \mathbf{1}$. So we want to replace \mathcal{L}^\times with a system whose only admitted sentences are equations of the form $C = \mathbf{1}$ with C being an arbitrary predicate. So we will delete the equality symbol $=$ and take our predicates of \mathcal{L}^\times to have a double meaning, as both a predicate and as a sentence stating that the relation designated by the predicate is the universal relation. We shall denote this reduced version of \mathcal{L}^\times by the title \mathcal{L}_r^\times . When in this formalism we shall use either $\mathbf{\Pi}$ or $\mathbf{\Sigma}_r^\times$ to denote the set of all predicate-sentences.

So how could we build such a deductive system from \mathcal{L}^\times ? Well we could take as our axiom set of \mathcal{L}^\times and use the definition in 2.1.1.3 to transform them. For example BI-BIII would look like this:

$$(BI_r) (A + B) \cdot (B + A) + (A + B)^- \cdot (B + A)^-$$

$$(BII_r) [A + (B + C)] \cdot [(A + B) + C] + [A + (B + C)]^- \cdot [(A + B) + C]^-$$

$$(BIII_r) [(A^- + B)^- + (A^- + B^-)^-] \cdot A + (A^- + B)^- + (A^- + B^-)^- \cdot A^-$$

Despite the ease of showing that these two deductive systems, with \mathcal{L}_r^\times defined this way, are equipollent in terms of expression and proof, this is a very strange and unintuitive axiomatization. It lacks elegance. Also, as noted in [Givant (2007)], the Replacement Lemma for the axioms stated in this manner “takes a rather awkward form.”

So we would like a “nicer” axiom system for \mathcal{L}_r^\times . As laid out in [Tarski, Givant (1987)], there is another system with the same premise, namely we get rid of the $=$ sign. As long as we are creating a new axiom set, why not choose a system where modus ponens is the only rule of inference?

Before we get to the axiom schemata of \mathcal{L}_r^\times we need to introduce some symbols for ease of notation. We want a symbol that replaces, in \mathcal{L}_r^\times , the operations \rightarrow in the formulation of modus ponens. We have not used the symbol \rightarrow in reference to \mathcal{L}_r^\times so we shall choose that. Formally we use the following definition.

Definition 4. *Let us define, for any $A, B \in \mathbf{II}$*

1. $(A \rightarrow B) = \mathbf{1} \odot A^- \odot \mathbf{1} + B$
2. $(A \Rightarrow B) = A^- + B$,
3. $(A \Leftrightarrow B) = (A \Rightarrow B) \cdot (B \Rightarrow A)$, and
4. $A \cdot B = (A^- + B^-)^-$.

We briefly note here that this definition will hold for \mathcal{L}_s^\times in the same way and we will not redefine it for \mathcal{L}_s^\times .

Following are the logical axioms schemata we will use for \mathcal{L}_r^\times .

2.3.1 Axioms of \mathcal{L}_r^\times

- (I_r) $(A + A) \Rightarrow A$,
- (II_r) $A \Rightarrow (A + B)$,
- (III_r) $(B \Rightarrow C) \Rightarrow [(A + B) \Rightarrow (C + A)]$,
- (IV_r) $[(A \odot B) \odot C] \Rightarrow [A \odot (B \odot C)]$,
- (V_r) $[(A + B) \odot C] \Rightarrow (A \odot C + B \odot C)$,
- (VI_r) $(A^\vee \odot B)^\vee \Rightarrow (B^\vee \odot A)$,
- (VII_r) $[A^\vee \odot (A \odot B)^-] \Rightarrow B^-$,
- (VIII_r) $(A \odot \mathbf{1}^\circ) \Rightarrow A$,
- (IX_r) $A \Rightarrow (A \odot \mathbf{1}^\circ)$,

$$(X_r) (A \Rightarrow B) \rightarrow (A \rightarrow B),$$

$$(XI_r) (A \Rightarrow B) \rightarrow [(A \odot C) \Rightarrow (B \odot C)],$$

$$(XII_r) (A^\forall \Rightarrow B) \rightarrow (A \Rightarrow B^\forall),$$

$$(XIII_r) (A \Rightarrow B^\forall) \rightarrow (A^\forall \Rightarrow B).$$

Again, we want a way to talk about derivability, so we define \vdash_r^\times .

2.3.2 Definition of \vdash_r^\times

$\Psi \vdash_r^\times X$ iff X is in every $\Omega \subseteq \Sigma_r^\times$ such that

1. $\Psi \subseteq \Omega$;
2. $\Lambda_r^\times \subseteq \Omega$;
3. if $B, C \in \Sigma_r^\times$ and $B, B \rightarrow C \in \Omega$ then $C \in \Omega$;

As stated in [Tarski, Givant (1987)] and [Givant (2007)], showing that \mathcal{L}^\times is equipollent to \mathcal{L}_r^\times under our new axiom set “presents no appreciable difficulties.” Showing that equipollence is exactly what we are setting out to do.

We want the fact that the first three of these axioms are sufficient to derive a handful of results in Propositional Calculus (that is, true in the traditional interpretation). But before taking on that endeavor, due to the similarities between \mathcal{L}_r^\times and \mathcal{L}_s^\times , we first mention the deductive system \mathcal{L}_s^\times .

2.4 The formalism \mathcal{L}_s^\times

We now build the deductive system \mathcal{L}_s^\times by taking \mathcal{L}_r^\times and reintroducing $=$ into the vocabulary of \mathcal{L}_r^\times . We will treat $=$ as a binary operator (instead of an equality symbol) and thus if $A, B \in \Sigma_s^\times$ then $A = B$ will also be in Σ_s^\times (along with the usual suspects of $A + B$, $A \odot B$, A^\forall , and A^-). Again, we will treat every meaningful expression in \mathcal{L}_s^\times as both a predicate and

a sentence that denotes the universal relation. Since $=$ can appear as arbitrarily many times in any sentence in \mathcal{L}_s^\times we will have $=$ denote the same relation as $A \Leftrightarrow B$.

The axiom schema of \mathcal{L}_s^\times will be defined by using the same axiom schema as \mathcal{L}_r^\times above but with a few additions.

2.4.1 Axioms of \mathcal{L}_s^\times

Λ_s^\times contain (I_r) – $(XIII_r)$, (II_s) – (IV_s) where $A, B, C \in \Sigma_s^\times$

(II_s) $(A = B) \Rightarrow (A \Rightarrow B)$,

(III_s) $(A = B) \Rightarrow (B \Rightarrow A)$,

(IV_s) $(A \Rightarrow B) \Rightarrow [(B \Rightarrow A) \Rightarrow (A = B)]$.

Once more we define derivability for our third deductive system.

2.4.2 Definition of \vdash_s^\times

Definition 5. $\Psi \vdash_s^\times X$ iff X is in every $\Omega \subseteq \Sigma_s^\times$ such that

1. $\Psi \subseteq \Omega$;
2. $\Lambda_s^\times \subseteq \Omega$;
3. if $B, C \in \Sigma_s^\times$ and $B, B \rightarrow C \in \Omega$ then $C \in \Omega$;

We will refer to the rule of modus ponens (for \rightarrow) several times in this paper, so we state it here as a lemma.

Lemma 20. $B, B \rightarrow C \vdash_s^\times C$

Proof. This follows directly from Definition 5. □

We will actually use this Lemma in a different form, so we restate it here.

Lemma 21. If $\vdash_s^\times B, \vdash_s^\times B \rightarrow C$, then $\vdash_s^\times C$.

Proof. This follows directly from the above lemma. □

We can derive modus ponens for \Rightarrow in \mathcal{L}_s^\times (or \mathcal{L}_r^\times) so we note that here.

Lemma 22. $A, A \Rightarrow B \vdash_s^\times B$

Proof.

1. $A, A \Rightarrow B \vdash^\times (A \Rightarrow B) \rightarrow (A \rightarrow B)$ X_r, Lemma 4
2. $A, A \Rightarrow B \vdash^\times A \rightarrow B$ 1., Lemma 21, Lemma 3
3. $A, A \Rightarrow B \vdash^\times B$ 2., Lemma 21, Lemma 3

□

Here again we will use this lemma in a different form, so we note that form here.

Lemma 23. *If $\vdash_s^\times A, \vdash_s^\times A \Rightarrow B$, then $\vdash_s^\times B$.*

Proof. This follows directly from the above lemma. □

Since these lemmas are so easily confused as they both have similar assumptions and conclusions (and, frankly since we numbered them concurrently) we want to use a different notation when referring to them. We will refer to Lemma 21 by M.P. for \rightarrow and Lemma 23 by M.P. for \Rightarrow .

2.5 Proofs of basic propositional calculus concepts in \mathcal{L}_s^\times and \mathcal{L}_r^\times

Again we will need some basic results but this time from propositional calculus. We note that Axioms I_r–III_r can be found in [Götlind, Erik (1947)]. He included a fourth axiom, specifically $A \Rightarrow A$ and posed the question as to whether or not it could be proved from the other axioms. Two years later in [Rasiowa, H. (1949)] it was shown that indeed the fourth axiom was provable from the others. These axioms were reprinted in [Church (1956)] on page 157. The following lemmas will have all capital letters (A, B, C , etc.) taken from **II**.

We start with the lemma that $\vdash_s^\times A + A^-$.

Lemma 24. $\vdash_s^\times A + A^-$

Proof.

- | | |
|--|---|
| 1. $\vdash_s^\times [(A + A) \Rightarrow A] \Rightarrow ([A^- + (A + A)] \Rightarrow (A + A^-))$ | III _r $\left(\begin{array}{ccc} A & B & C \\ A^- & A + A & A \end{array} \right)$ |
| 2. $\vdash_s^\times (A + A) \Rightarrow A$ | I _r |
| 3. $\vdash_s^\times [A^- + (A + A)] \Rightarrow (A + A^-)$ | 1., 2., M.P. for \Rightarrow |
| 4. $\vdash_s^\times A \Rightarrow (A + A)$ | I _r |
| 5. $\vdash_s^\times A^- + (A + A)$ | Definition 4(2) |
| 6. $\vdash_s^\times A + A^-$ | 3., 4., M.P. for \Rightarrow |

□

Next we want the fact that $\vdash_s^\times A \Rightarrow A^{--}$.

Lemma 25. $\vdash_s^\times A \Rightarrow A^{--}$

Proof.

- | | |
|---|---|
| 1. $\vdash_s^\times A^- + A^{--}$ | Lemma 24 $\left(\begin{array}{c} A \\ A^- \end{array} \right)$ |
| 2. $\vdash_s^\times A \Rightarrow A^{--}$ | Definition 4(2) |

□

As it should also be the case that $\vdash_s^\times A^{--} \Rightarrow A$ we prove that next.

Lemma 26. $\vdash_s^\times A^{--} \Rightarrow A$

Proof.

- | | |
|--|--|
| 1. $\vdash_s^\times A^- \Rightarrow A^{---}$ | Lemma 26 $\left(\begin{array}{c} A \\ A^- \end{array} \right)$ |
| 2. $\vdash_s^\times (A^- \Rightarrow A^{---}) \Rightarrow [(A + A^-) \Rightarrow (A^{---} + A)]$ | III _r $\left(\begin{array}{ccc} A & B & C \\ A & A & A^{--} \end{array} \right)$ |

- | | | |
|----|---|--------------------------------------|
| 3. | $\vdash_s^\times (A + A^-) \Rightarrow (A^{-^{-}} + A)$ | 1., 2., M.P. for \Rightarrow |
| 4. | $\vdash_s^\times A^{-^{-}} + A$ | 3., Lemma 24, M.P. for \Rightarrow |
| 5. | $\vdash_s^\times A^{-^{-}} \Rightarrow A$ | Definition 4(2) |

□

The next lemma we will prove tends to be one of the most difficult to derive no matter what logical axioms we start with... besides the obvious inclusion of it as an axiom.

Lemma 27. $\vdash_s^\times A \Rightarrow A$

Proof.

- | | | |
|----|--|---|
| 1. | $\vdash_s^\times A^{-^{-}} \Rightarrow A^-$ | Lemma 26 $\begin{pmatrix} A \\ A^- \end{pmatrix}$ |
| 2. | $\vdash_s^\times (A^{-^{-}} \Rightarrow A^-) \Rightarrow$
$[(A^- + A^{-^{-}}) \Rightarrow (A^- + A^-)]$ | $\text{III}_r \begin{pmatrix} A & B & C \\ A^- & A^{-^{-}} & A^- \end{pmatrix}$ |
| 3. | $\vdash_s^\times (A^- + A^{-^{-}}) \Rightarrow (A^- + A^-)$ | 1., 2. M.P. for \Rightarrow |
| 4. | $\vdash_s^\times [(A^- + A^{-^{-}}) \Rightarrow (A^- + A^-)] \Rightarrow$
$[(A^{-^{-}} + [A^- + A^{-^{-}}]) \Rightarrow$
$((A^- + A^-) + A^{-^{-})]$ | $\text{III}_r \begin{pmatrix} A & B & C \\ A^{-^{-}} & A^- + A^{-^{-}} & A^- + A^- \end{pmatrix}$ |
| 5. | $\vdash_s^\times (A^{-^{-}} + [A^- + A^{-^{-}]) \Rightarrow$
$((A^- + A^-) + A^{-^{-})$ | 3., 4. M.P. for \Rightarrow |
| 6. | $\vdash_s^\times A^- \Rightarrow (A^- + A^{-^{-})$ | $\text{II}_r \begin{pmatrix} A & B \\ A^- & A^{-^{-}} \end{pmatrix}$ |
| 7. | $\vdash_s^\times A^{-^{-}} + (A^- + A^{-^{-})$ | 6., Definition 4(2) |
| 8. | $\vdash_s^\times (A^- + A^-) + A^{-^{-}}$ | 5., 7., M.P. for \Rightarrow |

9. $\vdash_s^\times A^{- -} \Rightarrow A$ Lemma 26
10. $\vdash_s^\times (A^{- -} \Rightarrow A) \Rightarrow$
 $[[[A^- + A^-] + A^{- -}] \Rightarrow (A + [A^- + A^-])] \text{ III}_r \begin{pmatrix} A & B & C \\ A^- + A^- & A^{- -} & A \end{pmatrix}$
11. $\vdash_s^\times ([A^- + A^-] + A^{- -}) \Rightarrow (A + [A^- + A^-])$ 9., 10. M.P. for \Rightarrow
12. $\vdash_s^\times A + [A^- + A^-]$ 8., 11. M.P. for \Rightarrow
13. $\vdash_s^\times (A^- + A^-) \Rightarrow A^-$ $\text{I}_r \begin{pmatrix} A \\ A^- \end{pmatrix}$
14. $\vdash_s^\times [(A^- + A^-) \Rightarrow A^-] \Rightarrow$
 $[(A + (A^- + A^-)) \Rightarrow (A^- + A)] \text{ III}_r \begin{pmatrix} A & B & C \\ A & A^- + A^- & A^- \end{pmatrix}$
15. $\vdash_s^\times (A + (A^- + A^-)) \Rightarrow (A^- + A)$ 13., 14., M.P. for \Rightarrow
16. $\vdash_s^\times A^- + A$ 12., 15., M.P. for \Rightarrow
17. $\vdash_s^\times A \Rightarrow A$ Definition 4(2)

□

Finally we can derive the commutativity of $+$.

Lemma 28. $\vdash_s^\times A + B \Rightarrow B + A$

Proof.

1. $\vdash_s^\times A \Rightarrow A$ Lemma 27
2. $\vdash_s^\times (A \Rightarrow A) \Rightarrow [(B + A) \Rightarrow (A + B)] \text{ III}_r \begin{pmatrix} A & B & C \\ B & A & A \end{pmatrix}$
3. $\vdash_s^\times (B + A) \Rightarrow (A + B)$ 1., 2., M.P. for \Rightarrow

□

Now we want a fairly basic property of our \Rightarrow symbol.

Lemma 29. $A \Rightarrow B, B \Rightarrow C \vdash_s^\times A \Rightarrow C$

Proof.

1. $A \Rightarrow B, B \Rightarrow C \vdash_s^\times (B \Rightarrow C) \Rightarrow [(A^- + B) \Rightarrow (C + A^-)]$ $\text{III}_r \left(\begin{array}{ccc} A & B & C \\ A^- & B & C \end{array} \right)$, Lemma 4
2. $A \Rightarrow B, B \Rightarrow C \vdash_s^\times (A^- + B) \Rightarrow (C + A^-)$ 1., M.P. for \Rightarrow
3. $A \Rightarrow B, B \Rightarrow C \vdash_s^\times (A \Rightarrow B) \Rightarrow (C + A^-)$ Definition 4(2)
4. $A \Rightarrow B, B \Rightarrow C \vdash_s^\times (C + A^-)$ 3., M.P. for \Rightarrow
5. $A \Rightarrow B, B \Rightarrow C \vdash_s^\times (C + A^-) \Rightarrow (A^- + C)$ Lemma 4, Lemma 28
6. $A \Rightarrow B, B \Rightarrow C \vdash_s^\times (A^- + C)$ 4., 5., Lemma 3, M.P. for \Rightarrow
7. $A \Rightarrow B, B \Rightarrow C \vdash_s^\times A \Rightarrow C$ 6., Definition 4(2)

□

Next we would like an idea basically equivalent to “proof by cases”. Namely that if A proves C and B proves C then A or B proves C .

Lemma 30. $A \Rightarrow C, B \Rightarrow C \vdash_s^\times A + B \Rightarrow C$

Proof.

1. $A \Rightarrow C, B \Rightarrow C \vdash_s^\times (B \Rightarrow C) \Rightarrow [(A + B) \Rightarrow (C + A)]$ III_r , Lemma 4
2. $A \Rightarrow C, B \Rightarrow C \vdash_s^\times (A + B) \Rightarrow (C + A)$ 1., M.P. for \Rightarrow
3. $A \Rightarrow C, B \Rightarrow C \vdash_s^\times (A \Rightarrow C) \Rightarrow [(C + A) \Rightarrow (C + C)]$ $\text{III}_r \left(\begin{array}{ccc} A & B & C \\ C & A & C \end{array} \right)$, Lemma 4
4. $A \Rightarrow C, B \Rightarrow C \vdash_s^\times (C + A) \Rightarrow (C + C)$ 3., M.P. for \Rightarrow
5. $A \Rightarrow C, B \Rightarrow C \vdash_s^\times (A + B) \Rightarrow (C + C)$ 4., Lemma 29
8. $A \Rightarrow C, B \Rightarrow C \vdash_s^\times (C + C) \Rightarrow C$ $\text{I}_r \left(\begin{array}{c} A \\ C \end{array} \right)$, Lemma 4
9. $A \Rightarrow C, B \Rightarrow C \vdash_s^\times (A + B) \Rightarrow C$ 7., 8., Lemma 29

□

Before we get to proving the associativity of $+$ we need a small lemma.

Lemma 31. (1) $A \Rightarrow B \vdash_s^\times (C + A) \Rightarrow (B + C)$ and (2) $A \Rightarrow B \vdash_s^\times (A + C) \Rightarrow (B + C)$

Proof.

- | | |
|--|---|
| 1. $A \Rightarrow B \vdash_s^\times (A \Rightarrow B) \Rightarrow [(C + A) \Rightarrow (B + C)]$ | III _r $\begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix}$ Lemma 4 |
| 2. $A \Rightarrow B \vdash_s^\times (C + A) \Rightarrow (B + C)$ | 1., M.P. for \Rightarrow |
| 3. $A \Rightarrow B \vdash_s^\times (B + C) \Rightarrow (C + B)$ | Lemma 28 |
| 4. $A \Rightarrow B \vdash_s^\times (C + A) \Rightarrow (C + B)$ | 3., 4., Lemma 29 |
| 5. $A \Rightarrow B \vdash_s^\times (A + C) \Rightarrow (C + A)$ | Lemma 28 |

Thus we have derived (1). Now we complete the proof.

- | | |
|--|------------------|
| 6. $A \Rightarrow B \vdash_s^\times (A + C) \Rightarrow (B + C)$ | 2., 6., Lemma 29 |
|--|------------------|

□

Next we would like the associativity of $+$, but we only have enough power to prove one of the implications right now.

Lemma 32. $\vdash_s^\times [A + (B + C)] \Rightarrow [(A + B) + C]$

Proof.

- | | |
|--|--|
| 1. $\vdash_s^\times A \Rightarrow (A + B)$ | II _r |
| 2. $\vdash_s^\times (A + B) \Rightarrow [(A + B) + C]$ | II _r $\begin{pmatrix} A & B \\ A + B & C \end{pmatrix}$ |
| 3. $\vdash_s^\times A \Rightarrow [(A + B) + C]$ | 1., 2., Lemma 29 |
| 4. $\vdash_s^\times B \Rightarrow (B + A)$ | I _r $\begin{pmatrix} A & B \\ B & A \end{pmatrix}$ |

- | | | |
|----|---|---|
| 5. | $\vdash_s^\times (B + A) \Rightarrow (A + B)$ | Lemma 28 |
| 6. | $\vdash_s^\times B \Rightarrow (A + B)$ | 4., 5., Lemma 29 |
| 7. | $\vdash_s^\times B + C \Rightarrow [(A + B) + C]$ | 6., Lemma 31 $\begin{pmatrix} A & B & C \\ B & A + B & C \end{pmatrix}$ |
| 8. | $\vdash_s^\times [A + (B + C)] \Rightarrow [(A + B) + C]$ | 2., 7., Lemma 30 |

□

To get associativity from the other direction, we need the following lemma.

Lemma 33. $\vdash_s^\times [A + (B + C)] \Rightarrow [B + (A + C)]$

Proof.

- | | | |
|----|---|---|
| 1. | $\vdash_s^\times (B + C) \Rightarrow (C + B)$ | Lemma 28 |
| 2. | $\vdash_s^\times [A + (B + C)] \Rightarrow [A + (C + B)]$ | Lemma 31 $\begin{pmatrix} A & B & C \\ B + C & C + B & A \end{pmatrix}$ |
| 3. | $\vdash_s^\times [A + (C + B)] \Rightarrow [(A + C) + B]$ | Lemma 32 |
| 4. | $\vdash_s^\times [A + (B + C)] \Rightarrow [(A + C) + B]$ | 2., 3., Lemma 29 |
| 5. | $\vdash_s^\times [(A + C) + B] \Rightarrow [B + (A + C)]$ | Lemma 28 |
| 6. | $\vdash_s^\times [A + (B + C)] \Rightarrow [B + (A + C)]$ | 4., 5., Lemma 29 |

□

We can now prove the other part of the associativity of $+$.

Lemma 34. $\vdash_s^\times [(A + B) + C] \Rightarrow [A + (B + C)]$

Proof.

- | | | |
|----|---|--|
| 1. | $\vdash_s^\times (A + B) \Rightarrow [(A + B) + C]$ | $\Pi_r \begin{pmatrix} A & B \\ A + B & C \end{pmatrix}$ |
| 2. | $\vdash_s^\times [(A + B) + C] \Rightarrow [C + (A + B)]$ | Lemma 28 |

- | | | |
|-----|---|---|
| 3. | $\vdash_s^\times (A + B) \Rightarrow [C + (A + B)]$ | 1., 2., Lemma 29 |
| 4. | $\vdash_s^\times [C + (A + B)] \Rightarrow [A + (C + B)]$ | Lemma 33 |
| 5. | $\vdash_s^\times (A + B) \Rightarrow [A + (C + B)]$ | 3., 4., Lemma 29 |
| 6. | $\vdash_s^\times (C + B) \Rightarrow (B + C)$ | Lemma 28 |
| 7. | $\vdash_s^\times [A + (C + B)] \Rightarrow [A + (B + C)]$ | Lemma 31 $\begin{pmatrix} A & B & C \\ C + B & B + C & A \end{pmatrix}$ |
| 8. | $\vdash_s^\times (A + B) \Rightarrow [A + (B + C)]$ | 5., 7., Lemma 29 |
| 9. | $\vdash_s^\times C \Rightarrow [C + (A + B)]$ | $\text{II}_r \begin{pmatrix} A & B \\ C & A + B \end{pmatrix}$ |
| 10. | $\vdash_s^\times [C + (A + B)] \Rightarrow [A + (C + B)]$ | Lemma 33 $\begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix}$ |
| 11. | $\vdash_s^\times C \Rightarrow [A + (C + B)]$ | 9., 10., Lemma 29 |
| 12. | $\vdash_s^\times (C + B) \Rightarrow (B + C)$ | Lemma 28 |
| 13. | $\vdash_s^\times [A + (C + B)] \Rightarrow [A + (B + C)]$ | Lemma 31 $\begin{pmatrix} A & B & C \\ C + B & B + C & A \end{pmatrix}$ |
| 14. | $\vdash_s^\times C \Rightarrow [A + (B + C)]$ | 11., 13., Lemma 29 |
| 15. | $\vdash_s^\times [(A + B) + C] \Rightarrow [A + (B + C)]$ | 8., 14., Lemma 30 |

□

We now take a moment to notice that Axioms I_r and II_r along with Lemmas 34, 31, and 33 are those laid out as Axioms in [Russell, Whitehead (1910)]. Continuing in the fashion set forth in that book it would be possible to prove all of standard Propositional Calculus, i.e. anything that would have a value of “ T ” under normal truth table evaluations is derivable from axioms $\text{I}_r - \text{III}_r$.

Lemma 35. $\text{I}_r - \text{III}_r \vdash^\times$ the Huntington Axiom and the definition of \Leftrightarrow .

Proof. As we will only use the fact in the above paragraph twice, we simply note here the two facts we will need later. □

The axiom set from [**Russell, Whitehead (1910)**] was originally printed in [**Russell (1908)**]. In that paper Russell tried to set forth an axiom set for Propositional Calculus that would deal with several contradictions that could be found in set theory, logic, and propositional calculus, including Burali-Forti's contradiction, Richard's paradox, the Epimenides (or liar paradox), and several others. What he noticed was that all of these paradoxes seemed to have the property of self reference or, to use his word, "reflexiveness." Through a very philosophical discussion Russell slowly derives an axiom set that was not only well-founded, but functional.

CHAPTER 3. Equipollence of \mathcal{L}^\times and \mathcal{L}_r^\times

3.1 Equipollence

We need to look at the idea of two formalisms being equipollent. However we really only concern ourselves with this if one of the formalisms is an extension of the other. We do this because if two systems are equipollent to a relative extension they are equipollent to each other. Let

$$\mathcal{S}^{(1)} = \langle \Sigma^{(1)}, \vdash^{(1)} \rangle$$

be an arbitrary system and

$$\mathcal{S}^{(2)} = \langle \Sigma^{(2)}, \vdash^{(2)} \rangle$$

be an extension of it, i.e. $\Sigma^{(1)} \subseteq \Sigma^{(2)}$ and $\Psi \vdash^{(1)} X$ implies $\Psi \vdash^{(2)} X$.

3.1.1 Definitions of equipollence

Definition 6. 1. $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$ are said to be equipollent in means of expression if for every

$X \in \Sigma^{(2)}$ there is a $Y \in \Sigma^{(1)}$ such that $X \equiv^{(2)} Y$, and also for every $Y \in \Sigma^{(1)}$ there is an $X \in \Sigma^{(2)}$ such that $X \equiv^{(2)} Y$.

2. $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$ are said to be equipollent in means of proof provided, for every $\Psi \subseteq \Sigma^{(1)}$ and $X \in \Sigma^{(1)}$, we have $\Psi \vdash^{(2)} X$ iff $\Psi \vdash^{(1)} X$.

3. $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$ are simply called equipollent if they are equipollent in means of both expression and proof.

We will only deal with the second item in this definition. The fact that the deductive systems \mathcal{L}^\times , \mathcal{L}_r^\times and \mathcal{L}_s^\times are equipollent in means of expression is covered in [Tarski, Givant (1987)].

3.1.2 Translation mapping

Definition 7. G is a translation mapping from a system $\mathcal{S}^{(2)}$ to a subsystem $\mathcal{S}^{(1)}$ if

1. G is a recursive function;
2. G maps $\Sigma^{(2)}$ into $\Sigma^{(1)}$;
3. $GX = X$ for $X \in \Sigma^{(1)}$;
4. $GX \equiv^{(2)} X$ for $X \in \Sigma^{(2)}$;
5. for every $\Psi \subseteq \Sigma^{(2)}$ and $X \in \Sigma^{(2)}$, if $\Psi \vdash^{(2)} X$, then $G^*\Psi \vdash^{(1)} GX$.

Again, as we are not concerned with the semantics of our system, item 4 in the above definition will not be covered here.

3.1.3 Equipollence and translation mappings

Let $\mathcal{S}^{(1)}$ be a subsystem of $\mathcal{S}^{(2)}$. Then the following two conditions are equivalent:

1. $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$ are equipollent in means of expression and proof;
2. there is a generalized translation mapping from $\mathcal{S}^{(2)}$ to $\mathcal{S}^{(1)}$.

For a full discussion of this, see [Tarski, Givant (1987)] section 2.4.

3.1.4 Equipollence of two systems relative to a third system

When we have two systems $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$ that are both equipollent (in terms of the above sections) to a third system $\mathcal{S}^{(3)}$ we say that $\mathcal{S}^{(1)}$ is equipollent to $\mathcal{S}^{(2)}$ relative to $\mathcal{S}^{(3)}$.

3.1.5 \mathcal{L}^\times and \mathcal{L}_r^\times are subformalisms of \mathcal{L}_s^\times

We are now ready to begin looking at the equipollence of \mathcal{L}^\times and \mathcal{L}_r^\times with \mathcal{L}_s^\times . First we will establish that \mathcal{L}^\times and \mathcal{L}_r^\times are, indeed, subformalisms of \mathcal{L}_s^\times . We begin by proving the following theorem.

Theorem 1. If $\Psi \subseteq \Sigma_r^\times$ and $X \in \Sigma_r^\times$ and $\Psi \vdash_r^\times X$ then $\Psi \vdash_s^\times X$.

Proof. So let $\Psi \vdash_r^\times X$. Then X is in every $\Omega \subseteq \Sigma_r^\times$ such that $\Psi \subseteq \Omega$, $\Lambda_r^\times \subseteq \Omega$, and if $B, C \in \Sigma_r^\times$ and $B, B \rightarrow C \in \Omega$ then $C \in \Omega$. We want to show that X is in every $\Omega' \subseteq \Sigma_s^\times$ such that $\Psi \subseteq \Omega'$, $\Lambda_s^\times \subseteq \Omega'$, and if $B, C \in \Sigma_s^\times$ and $B, B \rightarrow C \in \Omega'$ then $C \in \Omega'$.

Let Ω be defined as in Definition 2. Let Ω' be defined as in Definition 5.

First we note that $\Psi \subseteq \Omega \subseteq \Omega'$. Then we note that, as $\Lambda_r^\times \subseteq \Lambda_s^\times$ and $\Lambda_s^\times \in \Omega'$ that $\Lambda_r^\times \subseteq \Omega'$. For our last condition, we let $B, B \rightarrow C \in \Omega'$, then as $B, B \rightarrow C \in \Omega$, $C \in \Omega \subseteq \Omega'$. Thus our final rule is satisfied and $\Psi \vdash_s^\times X$. \square

Therefore \mathcal{L}_r^\times is no stronger than \mathcal{L}_s^\times in terms of proof.

Theorem 2. *If $\Psi \vdash_r^\times X$ then $\Psi \vdash_s^\times X$.*

We will break the proof of this theorem into several cases. First we will show that $\vdash_s^\times \Lambda^\times$. Then we will show that $\vdash_s^\times A = A$ for all $A \in \Sigma^\times$, and if $B, C, D \in \Sigma_r^\times$ then $B = C, B = D \vdash_s^\times C = D \in \Omega$, and if $B, C, D \in \Sigma_r^\times$ then $B = C \in \Omega \vdash_s^\times B + D = C + D, D + B = D + C, B^- = C^-, B \odot D = C \odot D, D \odot B = D \odot C$, and $B^\smile = C^\smile$.

We prove this with the following Lemmas.

Starting with showing the axioms $BI - BX$ are provable from the axioms of \mathcal{L}_s^\times . In the following proofs we will do this by using axiom IV_s exclusively to get the equality. For this axiom we need $A \Rightarrow B$ and $B \Rightarrow A$ to derive $A = B$. Thus if we show the implications it would be tedious to state axiom IV_s over and over adding three lines to every proof. Thus once we have our implications we will simply quote IV_s and the two lines of implication as the reasons for our last line.

Lemma 36. $\vdash_s^\times BI$.

Proof.

- | | |
|--|---|
| 1. $\vdash_s^\times A + B \Rightarrow B + A$ | Lemma 28 |
| 2. $\vdash_s^\times B + A \Rightarrow A + B$ | Lemma 28 $\begin{pmatrix} A & B \\ B & A \end{pmatrix}$ |
| 3. $\vdash_s^\times A + B = B + A$ | 1., 2., IV_s |

□

Lemma 37. $\vdash_s^\times BII$.

Proof.

1. $\vdash_s^\times [(A + B) + C] \Rightarrow [A + (B + C)]$ Lemma 34
2. $\vdash_s^\times [A + (B + C)] \Rightarrow [(A + B) + C]$ Lemma 32
3. $\vdash_s^\times [(A + B) + C] = [A + (B + C)]$ 1., 2., IV_s

□

Lemma 38. $\vdash_s^\times BIII$.

Proof. For this case we refer to Lemma 35. We note that both implications we would need to prove that $\vdash_s^\times BIII$ are true under the standard truth table method, so they are provable from $I_r - III_r$. Thus $\vdash_s^\times BIII$. □

Lemma 39. $\vdash_s^\times BIV$.

This case involves many of the cases we have not yet proven so we will leave it to the reader to fill in this gap.

Lemma 40. $\vdash_s^\times BV$.

Proof.

1. $\vdash_s^\times [(A + B) \odot C] \Rightarrow [(A \odot C) + (B \odot C)]$ V_r
2. $\vdash_s^\times (B \Rightarrow (A + B)) \rightarrow ([B \odot C] \Rightarrow [(A + B) \odot C])$ $XI_r \begin{pmatrix} A & B & C \\ B & A + B & C \end{pmatrix}$
3. $\vdash_s^\times B \Rightarrow (B + A)$ $II_r \begin{pmatrix} A & B \\ B & A \end{pmatrix}$
4. $\vdash_s^\times (B + A) \Rightarrow (A + B)$ Lemma 28
5. $\vdash_s^\times B \Rightarrow (A + B)$ 3., 4., Lemma 29

- | | | |
|-----|---|--|
| 6. | $\vdash_s^\times [B \odot C] \Rightarrow [(A + B) \odot C]$ | 2., 5., M.P. for \rightarrow |
| 7. | $\vdash_s^\times [A \Rightarrow (A + B)] \rightarrow ([A \odot C] \Rightarrow [(A + B) \odot C])$ | XI _r $\begin{pmatrix} A & B & C \\ A & A + B & C \end{pmatrix}$ |
| 8. | $\vdash_s^\times A \Rightarrow (A + B)$ | II _r |
| 9. | $\vdash_s^\times (A \odot C) \Rightarrow [(A + B) \odot C]$ | 7., 8., M.P. for \rightarrow |
| 10. | $\vdash_s^\times [(A \odot C) + (B \odot C)] \Rightarrow [(A + B) \odot C]$ | 9., 6., Lemma 30 |
| 11. | $\vdash_s^\times (A \odot C) + (B \odot C) = (A + B) \odot C$ | 1., 10., IV _s |

□

Lemma 41. $\vdash_s^\times BVI$.

Proof.

- | | | |
|----|--|-------------------------|
| 1. | $\vdash_s^\times (A \odot \mathring{1}) \Rightarrow A$ | VIII _r |
| 2. | $\vdash_s^\times A \Rightarrow (A \odot \mathring{1})$ | IX _r |
| 3. | $\vdash_s^\times A = A \odot \mathring{1}$ | 1., 2., IV _s |

□

We do not need the next two lemmas directly, but we will refer to it several times, so we prove it here.

Lemma 42. $\vdash_s^\times A \Rightarrow A^\smile$.

Proof.

- | | | |
|----|--|--|
| 1. | $\vdash_s^\times A^\smile \Rightarrow A^\smile$ | Lemma 27 |
| 2. | $\vdash_s^\times (A^\smile \Rightarrow A^\smile) \rightarrow (A \Rightarrow A^\smile)$ | XII _r $\begin{pmatrix} A & B \\ A & A^\smile \end{pmatrix}$ |
| 3. | $\vdash_s^\times A \Rightarrow A^\smile$ | 1., 2., M.P. for \rightarrow |

□

Lemma 43. $\vdash_s^\times A^\smile \Rightarrow A$.

Proof.

- | | |
|---|--|
| 1. $\vdash_s^\times A^\smile \Rightarrow A^\smile$ | Lemma 27 |
| 2. $\vdash_s^\times (A^\smile \Rightarrow A^\smile) \rightarrow (A^\smile \Rightarrow A)$ | XIII _r $\left(\begin{array}{cc} A & B \\ A & A^\smile \end{array} \right)$ |
| 3. $\vdash_s^\times A^\smile \Rightarrow A$ | 1., 2., M.P. for \rightarrow |

□

Lemma 44. $\vdash_s^\times BVII$.

Proof. This follows directly from Lemma 43 and Lemma 42 above.

□

Again we would like a couple of simple lemmas to make our next proof easier.

Lemma 45. $\vdash_s^\times A^\smile \Rightarrow (A + B)^\smile$

Proof.

- | | |
|---|---|
| 1. $\vdash_s^\times A \Rightarrow (A + B)$ | II _r |
| 2. $\vdash_s^\times A^\smile \Rightarrow A$ | Lemma 42 |
| 3. $\vdash_s^\times A^\smile \Rightarrow (A + B)$ | 2., 1., Lemma 29 |
| 4. $\vdash_s^\times [A^\smile \Rightarrow (A + B)] \rightarrow (A^\smile \Rightarrow (A + B)^\smile)$ | XII _r $\left(\begin{array}{cc} A & B \\ A^\smile & A + B \end{array} \right)$ |
| 5. $\vdash_s^\times A^\smile \Rightarrow (A + B)^\smile$ | 3., 4., M.P. for \rightarrow |

□

Lemma 46. $\vdash_s^\times B^\smile \Rightarrow (A + B)^\smile$

Proof.

- | | | |
|----|--|--|
| 1. | $\vdash_s^\times B \Rightarrow (B + A)$ | $\text{II}_r \begin{pmatrix} A & B \\ B & A \end{pmatrix}$ |
| 2. | $\vdash_s^\times (B + A) \Rightarrow (A + B)$ | Lemma 28 |
| 3. | $\vdash_s^\times B \Rightarrow (A + B)$ | 1., 2., Lemma 29 |
| 4. | $\vdash_s^\times B^{\smile} \Rightarrow B$ | Lemma 43 |
| 5. | $\vdash_s^\times B^{\smile} \Rightarrow (A + B)$ | 4., 3., Lemma 29 |
| 6. | $\vdash_s^\times [B^{\smile} \Rightarrow (A + B)] \rightarrow (B^{\smile} \Rightarrow (A + B)^{\smile})$ | $\text{XII}_r \begin{pmatrix} A & B \\ B^{\smile} & A + B \end{pmatrix}$ |
| 7. | $\vdash_s^\times B^{\smile} \Rightarrow (A + B)^{\smile}$ | 5., 6., M.P. for \rightarrow |

□

Lemma 47. $\vdash_s^\times A^{\smile} + B^{\smile} \Rightarrow (A + B)^{\smile}$

Proof.

- | | | |
|----|--|----------|
| 1. | $\vdash_s^\times A^{\smile} \Rightarrow (A + B)^{\smile}$ | Lemma 45 |
| 2. | $\vdash_s^\times B^{\smile} \Rightarrow (A + B)^{\smile}$ | Lemma 46 |
| 3. | $\vdash_s^\times (A^{\smile} + B^{\smile}) \Rightarrow (A + B)^{\smile}$ | Lemma 30 |

□

Lemma 48. $\vdash_s^\times A \Rightarrow (A^{\smile} + B^{\smile})^{\smile}$.

Proof.

- | | | |
|----|---|---|
| 1. | $\vdash_s^\times A^{\smile} \Rightarrow (A^{\smile} + B^{\smile})$ | I_r |
| 2. | $\vdash_s^\times [A^{\smile} \Rightarrow (A^{\smile} + B^{\smile})] \rightarrow [A \Rightarrow (A^{\smile} + B^{\smile})^{\smile}]$ | $\text{XII}_r \begin{pmatrix} A & B \\ A & A^{\smile} + B^{\smile} \end{pmatrix}$ |

3. $\vdash_s^\times A \Rightarrow (A^\smile + B^\smile)^\smile$ 1., 2., M.P. for \rightarrow

□

Lemma 49. $\vdash_s^\times B \Rightarrow (A^\smile + B^\smile)^\smile$

Proof.

- | | | |
|----|---|--------------------------------|
| 1. | $\vdash_s^\times B^\smile \Rightarrow (B^\smile + A^\smile)$ | II_r |
| 2. | $\vdash_s^\times (B^\smile + A^\smile) \Rightarrow (A^\smile + B^\smile)$ | Lemma 28 |
| 3. | $\vdash_s^\times B^\smile \Rightarrow (A^\smile + B^\smile)$ | 1., 2., Lemma 29 |
| 4. | $\vdash_s^\times [B^\smile \Rightarrow (A^\smile + B^\smile)] \rightarrow [B \Rightarrow (A^\smile + B^\smile)^\smile]$ | XII_r |
| 5. | $\vdash_s^\times B \Rightarrow (A^\smile + B^\smile)^\smile$ | 3., 4., M.P. for \rightarrow |

□

Lemma 50. $\vdash_s^\times (A + B) \Rightarrow (A^\smile + B^\smile)^\smile$

Proof.

- | | | |
|----|--|----------|
| 1. | $\vdash_s^\times A \Rightarrow (A^\smile + B^\smile)^\smile$ | Lemma 48 |
| 2. | $\vdash_s^\times B \Rightarrow (A^\smile + B^\smile)^\smile$ | Lemma 49 |
| 3. | $\vdash_s^\times (A + B) \Rightarrow [(A^\smile + B^\smile)^\smile]$ | Lemma 30 |

□

Finally we are ready to show that $\vdash_s^\times \text{BVIII}$.

Lemma 51. $\vdash_s^\times \text{BVIII}$.

Proof.

- | | | |
|----|--|----------|
| 1. | $(A + B) \Rightarrow [(A^\smile + B^\smile)^\smile]$ | Lemma 50 |
|----|--|----------|

2. $[(A + B) \Rightarrow [(A^\smile + B^\smile)^\smile]] \rightarrow [(A + B)^\smile \Rightarrow (A^\smile + B^\smile)]$ XIII_r
3. $(A + B)^\smile \Rightarrow (A^\smile + B^\smile)$ 1., 2., M.P. for \rightarrow
4. $(A^\smile + B^\smile) \Rightarrow (A + B)^\smile$ Lemma 47
5. $(A + B)^\smile = (A^\smile + B^\smile)$ 3., 4., XIII_r

□

Here again we will prove two small lemmas in order to prove what we need.

Lemma 52. $(A \odot B)^\smile \Rightarrow (B^\smile \odot A^\smile)$.

Proof.

1. $\vdash_s^\times (A^\smile \odot B)^\smile \Rightarrow (B^\smile \odot A^\smile)$ VI_r $\begin{pmatrix} A & B \\ A^\smile & B \end{pmatrix}$
2. $\vdash_s^\times (A \Rightarrow A^\smile) \rightarrow [(A \odot B) \Rightarrow (A^\smile \odot B)]$ XI_r $\begin{pmatrix} A & B & C \\ A & A^\smile & B \end{pmatrix}$
3. $\vdash_s^\times A \Rightarrow A^\smile$ Lemma 42
4. $\vdash_s^\times (A \odot B) \Rightarrow (A^\smile \odot B)$ 2., 3., M.P. for \rightarrow
5. $\vdash_s^\times (A \odot B)^\smile \Rightarrow (A \odot B)$ Lemma 42
6. $\vdash_s^\times (A \odot B)^\smile \Rightarrow (A^\smile \odot B)$ 5., 4., Lemma 29
7. $\vdash_s^\times [(A \odot B)^\smile \Rightarrow (A^\smile \odot B)] \rightarrow [(A \odot B)^\smile \Rightarrow (A^\smile \odot B)^\smile]$ XII_r $\begin{pmatrix} A & B \\ (A \odot B)^\smile & A^\smile \odot B \end{pmatrix}$
8. $\vdash_s^\times (A \odot B)^\smile \Rightarrow (A^\smile \odot B)^\smile$ 6., 7., M.P. for \rightarrow
9. $\vdash_s^\times (A^\smile \odot B)^\smile \Rightarrow (B^\smile \odot A^\smile)$ VI_r $\begin{pmatrix} A & B \\ A^\smile & B \end{pmatrix}$
10. $\vdash_s^\times (A \odot B)^\smile \Rightarrow (B^\smile \odot A^\smile)$ 8., 9., Lemma 29

□

Lemma 53. $(B^\smile \odot A^\smile) \Rightarrow (A \odot B)^\smile$.

Proof.

- | | |
|--|--|
| 1. $\vdash_s^\times (B^\smile \odot A^\smile)^\smile \Rightarrow (A^\smile \odot B)$ | VI _r $\begin{pmatrix} A & B \\ B & A^\smile \end{pmatrix}$ |
| 2. $\vdash_s^\times [(B^\smile \odot A^\smile)^\smile \Rightarrow (A^\smile \odot B)] \rightarrow [(B^\smile \odot A^\smile) \Rightarrow (A^\smile \odot B)^\smile]$ | XII _r $\begin{pmatrix} A & B \\ (B^\smile \odot A^\smile) & A^\smile \odot B \end{pmatrix}$ |
| 3. $\vdash_s^\times (B^\smile \odot A^\smile) \Rightarrow (A^\smile \odot B)^\smile$ | 1., 2., M.P. for \rightarrow |
| 4. $\vdash_s^\times A^\smile \Rightarrow A$ | Lemma 42 |
| 5. $\vdash_s^\times (A^\smile \Rightarrow A) \rightarrow [(A^\smile \odot B) \Rightarrow (A \odot B)]$ | XI _r $\begin{pmatrix} A & B & C \\ A^\smile & A & B \end{pmatrix}$ |
| 6. $\vdash_s^\times (A^\smile \odot B) \Rightarrow (A \odot B)$ | 4., 5., M.P. for \rightarrow |
| 7. $\vdash_s^\times (A^\smile \odot B)^\smile \Rightarrow (A^\smile \odot B)$ | Lemma 42 |
| 8. $\vdash_s^\times (A^\smile \odot B)^\smile \Rightarrow (A \odot B)$ | 6., 7., Lemma 29 |
| 9. $\vdash_s^\times [(A^\smile \odot B)^\smile \Rightarrow (A \odot B)] \rightarrow [(A^\smile \odot B)^\smile \Rightarrow (A \odot B)^\smile]$ | XII _r $\begin{pmatrix} A & B \\ A^\smile & B \end{pmatrix}$ |
| 10. $\vdash_s^\times (A^\smile \odot B)^\smile \Rightarrow (A \odot B)^\smile$ | 8., 9., M.P. for \rightarrow |
| 11. $\vdash_s^\times (B^\smile \odot A^\smile) \Rightarrow (A \odot B)^\smile$ | 2., 10., Lemma 29 |

□

Lemma 54. $\vdash_s^\times BIX$.

Proof.

- | | |
|---|---------------------------|
| 1. $\vdash_s^\times (A \odot B)^\smile \Rightarrow (B^\smile \odot A^\smile)$ | Lemma 52 |
| 2. $\vdash_s^\times (B^\smile \odot A^\smile) \Rightarrow (A \odot B)^\smile$ | Lemma 53 |
| 3. $\vdash_s^\times B^\smile \odot A^\smile = (A \odot B)^\smile$ | 1., 2., XIII _r |

□

Lemma 55. $\vdash_s^\times BX$.

Proof.

- | | |
|--|--|
| 1. $\vdash_s^\times [A^\heartsuit \odot (A \odot B)^-] \Rightarrow B^-$ | VII _r |
| 2. $\vdash_s^\times B^- \Rightarrow B^-$ | Lemma 27 |
| 3. $\vdash_s^\times [A^\heartsuit \odot (A \odot B)^- + B^-] \Rightarrow B^-$ | 1., 2., Lemma 30 |
| 4. $\vdash_s^\times B^- \Rightarrow B^- + (A^\heartsuit \odot (A \odot B))^-$ | II _r $\left(\begin{array}{cc} A & B \\ B^- & A^- \odot (A \odot B)^\heartsuit \end{array} \right)$ |
| 5. $\vdash_s^\times (B^- + A^\heartsuit \odot (A \odot B)^-) \Rightarrow (A^\heartsuit \odot (A \odot B)^- + B^-)$ | 4., Lemma 28 |
| 6. $\vdash_s^\times B^- \Rightarrow A^\heartsuit \odot (A \odot B)^- + B^-$ | 4., 5., Lemma 29 |
| 7. $\vdash_s^\times A^\heartsuit \odot (A \odot B)^- + B^- = B^-$ | 3., 6., IV _s |

□

Lemma 56. $\vdash_s^\times A = A$ for all $A \in \mathbf{\Pi}$.

Proof.

- | | |
|--------------------------------------|-------------------------|
| 1. $\vdash_s^\times A \Rightarrow A$ | Lemma 27 |
| 2. $\vdash_s^\times A \Rightarrow A$ | 1. |
| 3. $\vdash_s^\times A = A$ | 1., 2., IV _s |

□

Lemma 57. If $B, C, D \in \mathbf{\Sigma}_r^\times$ then $B = C, B = D \vdash_s^\times C = D$.

Proof.

- | | |
|---|---|
| 1. $B = C \vdash_s^\times C \Rightarrow B$ | M.P. for \rightarrow , III _s |
| 2. $B = D \vdash_s^\times B \Rightarrow D$ | M.P. for \rightarrow , II _s |
| 3. $B = C, B = D \vdash_s^\times C \Rightarrow D$ | 1., 2., Lemma 29, Lemma 4 |
| 4. $B = D \vdash_s^\times D \Rightarrow B$ | M.P. for \rightarrow , III _s |

- | | |
|---|--|
| 5. $B = C \vdash_s^\times B \Rightarrow C$ | M.P. for \rightarrow , II_s |
| 6. $B = C, B = D \vdash_s^\times D \Rightarrow C$ | 4., 5., Lemma 29, Lemma 4 |
| 7. $B = C, B = D \vdash_s^\times C = D$ | 3., 6., IV_s |

□

Lemma 58. *If $B, C, D \in \Sigma_r^\times$ then (1) $B = C \vdash_s^\times B + D = C + D$, (2) $B = C \vdash_s^\times D + B = D + C$, (3) $B = C \vdash_s^\times B^- = C^-$, (4) $B = C \vdash_s^\times B \odot D = C \odot D$, (5) $B = C \vdash_s^\times D \odot B = D \odot C$, and (6) $B = C \vdash_s^\times B^\smile = C^\smile$.*

Proof. Below we highlight a few of the cases. The rest all proceed in a similar fashion and are thus left to the reader.

- | | |
|--|----------------|
| 1. $\vdash_s^\times B \Rightarrow C$ | II_s |
| 2. $\vdash_s^\times B + D \Rightarrow C + D$ | Lemma 31 |
| 3. $\vdash_s^\times C \Rightarrow B$ | III_s |
| 4. $\vdash_s^\times C + D \Rightarrow B + C$ | Lemma 31 |
| 5. $\vdash_s^\times B + D = C + D$ | IV_s |

We have thus proved (1) above. Now we proceed to (2).

- | | |
|--|---------------|
| 6. $\vdash_s^\times D + B \Rightarrow D + C$ | Lemma 31 |
| 7. $\vdash_s^\times D + C \Rightarrow D + B$ | Lemma 31 |
| 8. $\vdash_s^\times D + B = D + C$ | IV_s |

Thus we have proved (2). Now we proceed to (4).

- | | |
|--|---------------------------------|
| 9. $\vdash_s^\times (B \Rightarrow C) \rightarrow [(B \odot D) \Rightarrow (C \odot D)]$ | XI_r |
| 10. $\vdash_s^\times (B \odot D) \Rightarrow (C \odot D)$ | 2., 10., M.P. for \Rightarrow |

11. $\vdash_s^\times (C \Rightarrow B) \rightarrow [(C \odot D) \Rightarrow (B \odot D)]$ XI_r
12. $\vdash_s^\times (C \odot D) \Rightarrow (B \odot D)$ 4., 12., M.P. for \Rightarrow
13. $\vdash_s^\times (B \odot D) = (C \odot D)$ IV_s

Now, having proved (4), we leave the rest to the reader. □

So by Lemma 36 through Lemma 58 we have proven Theorem 2.

Thus \mathcal{L}^\times is no stronger than \mathcal{L}_s^\times in terms of proof.

3.1.6 The translation mappings from \mathcal{L}_s^\times to \mathcal{L}^\times and \mathcal{L}_r^\times

We now need the translation mappings mentioned in 3.1.2-3.1.3. Define a translation mapping $H : \Sigma_s^\times \rightarrow \Sigma_r^\times$ recursively by

$$H(\mathbf{1}) = \mathbf{1}$$

$$H(E) = E$$

$$H(A^-) = H(A)^- \text{ for } A \in \Sigma_s^\times$$

$$H(A + B) = H(A) + H(B) \text{ for } A, B \in \Sigma_s^\times$$

$$H(A \odot B) = H(A) \odot H(B) \text{ for } A, B \in \Sigma_s^\times$$

$$H(A = B) = (H(A) \Leftrightarrow H(B)) = ((H(A)^- + H(B))^- + (H(B)^- + H(A))^-)^- \text{ for } A, B \in \Sigma_s^\times$$

Define another translation mapping $G : \Sigma_s^\times \rightarrow \Sigma^\times$ by $G(A) = (H(A) = \mathbf{1})$ for every $A \in \Sigma_s^\times$.

Here, as noted in the Definition 7, we would pause to prove that $G(X) \equiv X$ and $H(X) \equiv X$ for every $X \in \Sigma_s^\times$. However, this is covered in [Tarski, Givant (1987)] so we will not prove it here.

Theorem 3. *If $\Psi \subseteq \Sigma_s^\times$, $X \in \Sigma_s^\times$, and $\Psi \vdash_s^\times X$, then*

$$H^*(\Psi) \vdash_r^\times H(X)$$

Let $\Omega = \{X : H^*(\Psi) \vdash_r^\times H(X), X \in \Sigma_s^\times\}$. Clearly, $\Psi \subseteq \Omega$.

We want to show that $\Lambda_s^\times \subseteq \Omega$, for all $B, C \in \Sigma_s^\times$, and if $B, B \rightarrow C \in \Omega$, then $C \in \Omega$. Again, we proceed by cases.

Lemma 59. $\vdash^\times H(I_r), H(II_r), H(III_r), H(IV_r), H(V_r), H(VI_r), H(VII_r), H(VIII_r), H(IX_r), H(X_r), H(XII_r),$ and $H(XIII_r)$.

Proof. If X is an instance of axioms I_r-XIII_r then $H(X)$ is simply a restatement of one of the axioms I_r-XIII_r and as $\Lambda_r^\times \subseteq \Lambda_s^\times$ our result follows immediately. \square

Lemma 60. $\vdash^\times H(II_s), H(III_s),$ and $H(IV_s)$.

Proof. Here we notice that if X is an instance of II_s-IV_s than $H(X)$ will simply be one of the following; $(H(A) \Leftrightarrow H(B)) \Rightarrow (H(A) \Rightarrow H(B))$, $(H(A) \Leftrightarrow H(B)) \Rightarrow (H(B) \Rightarrow H(A))$, or $(H(A) \Rightarrow H(B)) \Rightarrow [(H(B) \Rightarrow H(A)) \Rightarrow (H(A) \Leftrightarrow H(B))]$. These statements all follow directly from Lemma 35. \square

Lemma 61. If $B, C \in \Sigma_s^\times$ then $B, B \rightarrow C \vdash_r^\times C$.

Proof. This follows from the definitions of \vdash_r^\times and H . \square

Thus by Lemma 59 through Lemma 61 we have proved Theorem 3.

The next theorem depends a lot on the lemmas that we proved in Chapter 2. We will follow the same conventions here that we did there, as these are again proofs in \mathcal{L}^\times . Namely, we will use Lemmas 10 and 12 by quoting them and not writing out all of the steps, we will avoid substitution notation, we will not explicitly reference the properties of $=$ except when needed, and when we have $A = B$ and $A = C$ on concurrent lines we will leave out the line $B = C$ and quote Lemma 12, leaving out reference to Lemma 9.d.

Theorem 4. If $\Psi \subseteq \Sigma_s^\times$, $X \in \Sigma_s^\times$, and $\Psi \vdash_s^\times X$, then

$$G^*(\Psi) \vdash^\times G(X)$$

We need to show that $\vdash^\times \Lambda_r^\times$, $\vdash^\times A = A$ for all $A \in \Sigma_r^\times$, if $B, C, D \in \Sigma_r^\times$ then $B = C, B = D \vdash^\times C = D$, and if $B, C, D \in \Sigma_r^\times$ then $B = C \vdash^\times B + D = C + D$, $B = C \vdash^\times D + B = D + C$,

$B = C \vdash^\times B^- = C^-$, $B = C \vdash^\times B \odot D = C \odot D$, $B = C \vdash^\times D \odot B = D \odot C$, and $B = C \vdash^\times B^\bullet = C^\bullet$.

Again we proceed by cases. In the first nine of these cases we will first prove a lemma stating that the general form of the axiom, after it is mapped, will be true. We will then state a simple corollary that the mapping under H is of the same form.

Lemma 62. $\vdash^\times (A + A) \Rightarrow A$.

Proof.

1. $\vdash^\times (A + A) \Rightarrow A = (A + A)^- + A$ Lemma 7, Definition 4(2)
2. $\vdash^\times (A + A) \Rightarrow A = A^- + A$ Lemma 12, Lemma 18
3. $\vdash^\times (A + A) \Rightarrow A = \mathbf{1}$ Lemma 15

□

Lemma 63. $\vdash^\times G(I_r)$.

Proof.

1. $\vdash^\times (H(A) + H(A)) \Rightarrow H(A) = \mathbf{1}$ Lemma 62
2. $\vdash^\times H([A + A] \Rightarrow A) = \mathbf{1}$ 1., Definition 4(2)
3. $\vdash^\times G(I_r)$ 2., Definition of G

□

Lemma 64. $\vdash^\times A \Rightarrow (A + B) = \mathbf{1}$.

Proof.

1. $\vdash^\times A \Rightarrow (A + B) = A^- + (A + B)$ Lemma 7, Definition 4(2)
2. $\vdash^\times A \Rightarrow (A + B) = (A^- + A) + B$ BII
3. $\vdash^\times A \Rightarrow (A + B) = (A + A^-) + B$ BI
4. $\vdash^\times A \Rightarrow (A + B) = \mathbf{1} + B$ Lemma 15

$$5. \quad \vdash^\times A \Rightarrow (A + B) = \mathbf{1}$$

Lemma 16

□

Lemma 65. $\vdash^\times G(\text{II}_r)$.

Proof.

$$1. \quad \vdash^\times H(A) \Rightarrow (H(A) + H(B)) = \mathbf{1}$$

Lemma 64

$$2. \quad \vdash^\times H(A \Rightarrow (A + B)) = \mathbf{1}$$

Definition 4(2)

$$3. \quad \vdash^\times G(\text{II}_r)$$

2., Definition of G

□

Lemma 66. $\vdash^\times (B \Rightarrow C) \Rightarrow [(A + B) \Rightarrow (C + A)] = \mathbf{1}$.

Proof.

$$1. \quad \vdash^\times (B \Rightarrow C) \Rightarrow [(A + B) \Rightarrow (C + A)] =$$

$$(B^- + C)^- + [(A + B)^- + (C + A)] \quad \text{Definition 4(2)}$$

$$2. \quad \vdash^\times (B^- + C)^- =$$

$$[(B^- + C)^{- -} + A^-] + [(B^- + C)^{- -} + A^-]^- \quad \text{BIII} \begin{pmatrix} A & B \\ (B^- + C)^- & A \end{pmatrix}$$

$$3. \quad \vdash^\times (B^- + C)^- =$$

$$(B^- + C + A)^- + (B^- + C + A)^- \quad \text{Lemma 14}$$

$$4. \quad \vdash^\times (A + B)^- =$$

$$[(A + B)^{- -} + C]^- + [(A + B)^{- -} + C]^- \quad \text{BIII} \begin{pmatrix} A & B \\ (A + B)^- & C \end{pmatrix}$$

$$5. \quad \vdash^\times (A + B)^- =$$

$$(A + B + C)^- + (A + B + C)^- \quad \text{Lemma 14}$$

6. $\vdash^\times (B^- + C)^- + (A + B)^- =$
 $(B^- + C + A)^- + (B^- + C + A)^-$
 $+ (A + B + C)^- + (A + B + C^-)^-$ 3., 5., Lemma 2
7. $\vdash^\times (B^- + C)^- + (A + B)^- =$
 $[(A + C) + B]^- + [(A + C) + B^-]^- + (B^- + C + A^-)^-$
 $+ (A + B + C^-)^-$ BI, BII
8. $\vdash^\times (B^- + C)^- + (A + B)^- =$
 $[(A + C)^{- -} + B]^- + [(A + C)^{- -} + B^-]^-$
 $+ (B^- + C + A^-)^- + (A + B + C^-)^-$ Lemma 12, Lemma 14
9. $\vdash^\times (B^- + C)^- + (A + B)^- =$
 $(A + C)^- + (B^- + C + A^-)^- +$
 $(A + B + C^-)^-$ BIII $\begin{pmatrix} A & B \\ (A + C)^- & B \end{pmatrix}$
10. $\vdash^\times (B^- + C)^- + (A + B)^- + (C + A) =$
 $(A + C)^- + (B^- + C + A^-)^-$
 $+ (A + B + C^-)^- + (A + C)$ 9., Lemma 12
11. $\vdash^\times (B^- + C)^- + (A + B)^- + (C + A) =$
 $(A + C) + (A + C)^-$
 $+ (B^- + C + A^-)^- + (A + B + C^-)^-$ BI, BII
12. $\vdash^\times (B^- + C)^- + (A + B)^- + (C + A) =$
 $\mathbf{1} + (B^- + C + A^-)^- + (A + B + C^-)^-$ Lemma 15
13. $\vdash^\times (B^- + C)^- + (A + B)^- + (C + A) = \mathbf{1}$ Lemma 16, Lemma 12
14. $\vdash^\times (B^- + C)^- + [(A + B)^- + (C + A)] = \mathbf{1}$ BII, Lemma 12
15. $\vdash^\times (B \Rightarrow C) \Rightarrow [(A + B) \Rightarrow (C + A)] = \mathbf{1}$ Definition 4(2), Lemma 12

□

Lemma 67. $\vdash^\times G(III_r)$

Proof.

1. $\vdash^\times (H(B) \Rightarrow H(C))$
 $\Rightarrow [(H(A) + H(B)) \Rightarrow (H(C) + H(A))] = \mathbf{1}$ Lemma 7, Lemma 66
2. $\vdash^\times H([B \Rightarrow C] \Rightarrow [(A + B) \Rightarrow (C + C)]) = \mathbf{1}$ 1., Definition 4(2)
3. $\vdash^\times G(III_r)$ 2., Definition of G

□

The proof that $\vdash^\times H(IV_r), H(V_r), H(VIII_r)$, and $H(IX_r)$ all follow the same reasoning. I will outline the proof that $\vdash^\times H(IV_r)$ here.

Lemma 68. $\vdash^\times [(A \odot B) \odot C] \Rightarrow [(A \odot (B \odot C))] = \mathbf{1}$.

Proof.

1. $\vdash^\times [(A \odot B) \odot C] \Rightarrow [A \odot (B \odot C)] =$
 $[(A \odot B) \odot C]^- + [A \odot (B \odot C)]$ Lemma 7, Lemma 4(2)
2. $\vdash^\times [(A \odot B) \odot C] \Rightarrow [A \odot (B \odot C)] =$
 $[A \odot (B \odot C)]^- + [A \odot (B \odot C)]$ BIV, Lemma 12
3. $\vdash^\times [(A \odot B) \odot C] \Rightarrow [A \odot (B \odot C)] = \mathbf{1}$ Lemma 16

□

Lemma 69. $\vdash^\times G(IV_r)$.

Proof.

1. $\vdash^\times [(H(A) \odot H(B) \odot H(C)) \Rightarrow [H(A) \odot (H(B) \odot H(C))]] = \mathbf{1}$ Lemma 68

2. $\vdash^\times H([(A \odot B) \odot C] \Rightarrow [A \odot (B \odot C)]) = \mathbf{1}$ 1., Definition 4(2)
3. $\vdash^\times G(\text{IV}_r)$ 2., Definition of G

□

Lemma 70. $\vdash^\times G(V_r), G(\text{VIII}_r)$, and $G(\text{IX}_r)$.

Proof. Now we note that if we simply follow the lines of the above proof, use the definition of \Rightarrow , and quote the appropriate axiom in \mathcal{L}^\times (namely BV, BVI, and BVI again, respectively), we get the desired result. □

Lemma 71. $\vdash^\times (A^\smile \odot B)^\smile \Rightarrow (B^\smile \odot A) = \mathbf{1}$.

Proof.

1. $\vdash^\times (A^\smile \odot B)^\smile \Rightarrow (B^\smile \odot A) = (A^\smile \odot B)^\smile\text{--} + (B^\smile + A)$ Definition 4(2)
2. $\vdash^\times (A^\smile \odot B)^\smile \Rightarrow (B^\smile \odot A) = (B^\smile \odot A^\smile\text{--})\text{--} + (B^\smile \odot A)$ $BIX \begin{pmatrix} A & B \\ A^\smile & B^\smile \end{pmatrix}$
3. $\vdash^\times (A^\smile \odot B)^\smile \Rightarrow (B^\smile \odot A) = (B^\smile \odot A)\text{--} + (B^\smile \odot A)$ BVII
4. $\vdash^\times (A^\smile \odot B)^\smile \Rightarrow (B^\smile \odot A) = \mathbf{1}$ Lemma 15, Lemma 12

□

Lemma 72. $\vdash^\times G(\text{VI}_r)$.

Proof.

1. $\vdash^\times (H(A)^\smile \odot H(B))^\smile \Rightarrow (H(B)^\smile \odot H(A)) = \mathbf{1}$ Lemma 7, Lemma 71
2. $\vdash^\times H([A^\smile \odot B]^\smile \Rightarrow (B^\smile \odot A)) = \mathbf{1}$ 1., Definition 4(2)
3. $\vdash^\times G(\text{VI}_r)$ 2., Definition of G

□

Lemma 73. $\vdash^\times [A^\smile \odot (A \odot B)\text{--}] \Rightarrow B\text{--} = \mathbf{1}$.

Proof.

1. $\vdash^\times [A^\vee \odot (A \odot B)^-] \Rightarrow B^- = [A^\vee \odot (A \odot B)^-]^- + B^-$ Lemma 7, Definition 4(2)
2. $\vdash^\times [A^\vee \odot (A \odot B)^-] \Rightarrow B^- =$
 $[A^\vee \odot (A \odot B)^-]^- + (A^\vee \odot (A \odot B)^-) + B^-$ BX
3. $\vdash^\times [A^\vee \odot (A \odot B)^-] \Rightarrow B^- = \mathbf{1} + B^-$ Lemma 15
4. $\vdash^\times [A^\vee \odot (A \odot B)^-] \Rightarrow B^- = \mathbf{1}$ Lemma 16

□

Lemma 74. $\vdash^\times G(VII_r)$.

Proof.

1. $\vdash^\times [H(A)^\vee \odot (H(A) \odot H(B))^-] \Rightarrow H(B)^- = \mathbf{1}$ Lemma 7, Lemma 73
2. $\vdash^\times H([A^\vee \odot (A \odot B)^-] \Rightarrow B^-) = \mathbf{1}$ 1., Definition 4(2)
3. $\vdash^\times G(VII_r)$ 2., Definition of G

□

In the next few case we will use our Lemma 1 outlined earlier. I will outline the steps here of $G(X_r)$. Our goal is to show $G(X_r)$, so we want $H(X_r) = \mathbf{1}$. As

$$X_r = (A \Rightarrow B) \rightarrow (A \Rightarrow B)$$

we have

$$H(X_r) = \mathbf{1} \odot (H(A) \Rightarrow H(B))^- \odot \mathbf{1} + \mathbf{1} \odot (H(A)^- \odot \mathbf{1} + H(B)).$$

From our distributive laws of \odot over $+$ we can pick this apart further into

$$H(X_r) = \mathbf{1} \odot [(H(A) \Rightarrow H(B))^- + (H(A)^-) \odot \mathbf{1} + H(B)].$$

By Lemma 1 we want to show

$$(A \Rightarrow B)^- + A^- = \mathbf{1} \vdash^\times B.$$

As this will be the general argument of the next four cases we will proceed, in general, as follows. We also need some facts from relation algebras that are more involved than we are concerned with in this paper. As noted in Chapter 1, \mathcal{L}^\times has the same strength as relation algebras with one variable. We will use that fact here. We will note lines that need these facts with *R.A.*. (For more detail on this see [Givant (2007)], [Maddux, Roger D. (2006)], and section 3.2 in [Tarski, Givant (1987)].

Lemma 75. $\vdash^\times \mathbf{1} \odot (A \Rightarrow B)^- \odot \mathbf{1} + \mathbf{1} \odot A^- \odot \mathbf{1} + B = 1$.

Proof.

1. $\vdash^\times \mathbf{1} \odot (A^- + B)^- \odot \mathbf{1} + \mathbf{1} \odot A^- \odot \mathbf{1} + B =$
 $\mathbf{1} \odot [(A^- + B)^- + A^-] \odot \mathbf{1} + B$ BV
2. $\vdash^\times \mathbf{1} \odot (A^- + B)^- \odot \mathbf{1} + \mathbf{1} \odot A^- \odot \mathbf{1} + B =$
 $\mathbf{1} \odot [(A^- + B)^- + A^-]^- \odot \mathbf{1} + B$ Lemma 14
3. $\vdash^\times [(A^- + B)^- + A^-]^- = 1$ Hyp. of Lemma 1
4. $\vdash^\times [(A^- + B) \cdot A] = 1$ Definition 4
5. $\vdash^\times A^- \cdot A + B \cdot A = 1$ Distributivity of \cdot
6. $\vdash^\times B \cdot A = 1$ *R.A.*
7. $\vdash^\times B \cdot A \leq B$ *R.A.*
8. $\vdash^\times B = 1$ 6., 7.
9. $\vdash^\times \mathbf{1} \odot (A^- + B)^- \odot \mathbf{1} + \mathbf{1} \odot A^- \odot \mathbf{1} + B = 1$ Lemma 1
10. $\vdash^\times \mathbf{1} \odot (A \Rightarrow B)^- \odot \mathbf{1} + \mathbf{1} \odot A^- \odot \mathbf{1} + B = 1$ Definition 4(2)

□

Lemma 76. $\vdash^\times G(X_r)$.

Proof.

1. $\vdash^\times \mathbf{1} \odot (H(A) \Rightarrow H(B))^- \odot \mathbf{1} + \mathbf{1} \odot H(A)^- \odot \mathbf{1} + H(B) = 1$ Lemma 75

2. $\vdash^\times H([A \Rightarrow B] \rightarrow [A \rightarrow B]) = \mathbf{1}$ Definition 4(1)
3. $\vdash^\times G(X_r)$ Definition of G

□

Lemma 77. $\vdash^\times (A \odot C)^- + (B \odot C) = \mathbf{1}$.

Proof.

1. $\vdash^\times \mathbf{1} \odot (A \Rightarrow B)^- \odot \mathbf{1} + [(A \odot C) \Rightarrow (B \odot C)] =$
 $\mathbf{1} \odot (A^- + B)^- \odot \mathbf{1} + [(A \odot C)^- + (B \odot C)]$ Definition 4(2)
2. $\vdash^\times (A^- + B) = \mathbf{1}$ Hyp. of Lemma 1
3. $\vdash^\times (A^- + B) \odot C \geq (A^- + B)$ R.A.
4. $\vdash^\times (A^- \odot C) + (B \odot C) \geq \mathbf{1}$ BV, Lemma 12
5. $\vdash^\times (A \odot C)^- + (B \odot C) = \mathbf{1}$ R.A.

□

Lemma 78. $\vdash^\times G(XI_r)$.

Proof.

1. $\vdash^\times (H(A) \odot H(C))^- + (H(B) \odot H(C)) = \mathbf{1}$ Lemma 77
2. $\vdash^\times H([A \Rightarrow B] \rightarrow [(A \odot C) \Rightarrow (B \odot C)]) = \mathbf{1}$ 1., Lemma 1
3. $\vdash^\times G(XI_r)$ 2., Definition of G

□

Lemma 79. $\vdash^\times \mathbf{1} \odot (A^\smile \Rightarrow B)^\smile \odot \mathbf{1} + (A \Rightarrow B^\smile) = \mathbf{1}$.

Proof.

1. $\vdash^\times \mathbf{1} \odot (A^\smile \Rightarrow B)^- \odot \mathbf{1} + (A \Rightarrow B^\smile) =$

- | | | |
|----|--|--------------------------|
| | $1 \odot (A^{\checkmark-} + B)^{\checkmark} \odot 1 + (A^- + B^{\checkmark})$ | Lemma 7, Definition 4(2) |
| 2. | $\vdash^{\times} (A^{\checkmark-} + B) = 1$ | Hyp. of Lemma 1 |
| 3. | $\vdash^{\times} (A^{\checkmark-} + B)^{\checkmark} = 1^{\checkmark}$ | Definition 2 |
| 4. | $\vdash^{\times} A^{\checkmark-\checkmark} + B^{\checkmark} = 1$ | R.A. |
| 5. | $\vdash^{\times} A^{\checkmark\checkmark-} + B^{\checkmark} = 1$ | R.A. |
| 6. | $\vdash^{\times} A^- + B^{\checkmark} = 1$ | BVII, Lemma 12 |
| 7. | $\vdash^{\times} A \Rightarrow B^{\checkmark} = 1$ | Definition 4(2) |
| 8. | $\vdash^{\times} 1 \odot (A^{\checkmark} \Rightarrow B)^{\checkmark} \odot 1 + (A \Rightarrow B^{\checkmark}) = 1$ | Lemma 1 |

□

Lemma 80. $\vdash^{\times} G(XII_r)$.

Proof.

- | | | |
|----|--|-------------------|
| 1. | $\vdash^{\times} 1 \odot (H(A)^{\checkmark} \Rightarrow H(B))^{\checkmark} \odot 1 + (H(A) \Rightarrow H(B)^{\checkmark}) = 1$ | Lemma 79 |
| 2. | $\vdash^{\times} H(XII_r) = 1$ | Properties of H |
| 3. | $\vdash^{\times} G(XII_r)$ | Definition of G |

□

Lemma 81. $\vdash^{\times} 1 \odot (A \Rightarrow B^{\checkmark}) \odot 1 + (A^{\checkmark} \Rightarrow B) = 1$.

Proof.

- | | | |
|----|---|--------------------------|
| 1. | $\vdash^{\times} 1 \odot (A \Rightarrow B^{\checkmark}) \odot 1 + (A^{\checkmark} \Rightarrow B) =$
$1 \odot (A^- + B^{\checkmark}) \odot 1 + (A^{\checkmark-} + B)$ | Lemma 7, Definition 4(2) |
| 2. | $\vdash^{\times} A^- + B^{\checkmark} = 1$ | Hyp. of Lemma 1 |
| 3. | $\vdash^{\times} (A^- + B^{\checkmark})^{\checkmark} = 1^{\checkmark}$ | Lemma 2 |
| 4. | $\vdash^{\times} (A^- + B^{\checkmark})^{\checkmark} = 1$ | R.A. |

5. $\vdash^\times A^{\sim} + B^{\sim} = \mathbf{1}$ R.A.
6. $\vdash^\times A^{\sim-} + B = \mathbf{1}$ BVII, R.A.
7. $\vdash^\times A^{\sim} \Rightarrow B = \mathbf{1}$ Definition 4(2)
8. $\vdash^\times \mathbf{1} \odot (A \Rightarrow B^{\sim}) \odot \mathbf{1} + (A^{\sim} \Rightarrow B) = \mathbf{1}$ Lemma 1

□

Lemma 82. $\vdash^\times G(\text{XIII}_r)$.

Proof.

1. $\vdash^\times \mathbf{1} \odot (H(A) \Rightarrow H(B)^{\sim}) \odot \mathbf{1} + (H(A)^{\sim} \Rightarrow H(B)) = \mathbf{1}$ Lemma 81, Definition 4(2)
2. $\vdash^\times H(\text{XIII}_r) = \mathbf{1}$ 1., Properties of H
3. $\vdash^\times G(\text{XIII}_r)$ 2., Definition of G

□

Lemma 83. *If $A, B, C, D \in \text{Sxr}$ then (1) $\vdash^\times A = A$, (2) $B = C, B = D \vdash^\times C = D$, (3) $B = C \vdash^\times B + D = C + D$, (4) $B = C \vdash^\times D + B = D + C$, (5) $B = C \vdash^\times B^- = C^-$, (6) $B = C \vdash^\times B \odot D = C \odot D$, (7) $B = C \vdash^\times D \odot B = D \odot C$, and (8) $B = C \vdash^\times B^{\sim} = C^{\sim}$.*

Proof. Here, again, we outline some of the cases and leave the rest to the reader.

1. $\vdash^\times A = A$ Lemma 7

That completes the proof of (1). Now we look at (2).

2. $\vdash^\times C = D$ Lemma 12

Again, this completes the proof of (2).

3. $\vdash^\times B + D = C + D$ Lemma 31

This completes the proof of (3). The rest of the cases follow a similar pattern, so we leave them to the reader. □

Finally by Lemma 62 through Lemma 83 we have proved Theorem 4.

Thus by Theorems 1-4 we have shown that \mathcal{L}^\times is equipollent to \mathcal{L}_s^\times and \mathcal{L}_r^\times is equipollent to \mathcal{L}_s^\times . Finally we draw our conclusion, \mathcal{L}^\times is equipollent to \mathcal{L}_r^\times relative to \mathcal{L}_s^\times . Thus we have come to the conclusion we wanted and the proof missing from [**Tarski, Givant (1987)**] has been filled in.

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ACKNOWLEDGEMENTS

I would like to thank my committee for their patience (especially my advisor Dr. Maddux), my friends for their support, and my father Keith for getting me excited in mathematics from a very young age.