Metatheories of deductive systems

by

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ABSTRACT

A deductive system (Hilbert-style) is an algebraic closure system over the set of formulas of given propositional language. Similarly, a Gentzen system is an algebraic closure system over the set of all sequents, i.e., finite sequences of formulas, of this language. The main feature of this work is a technique that allows us to adapt the methods, previously developed in the area of algebraic logic for work with Hilbert-style deductive systems, to the case of Gentzen systems. Using the properties of the Tarski congruence, a generalization of the Leibnitz congruence, we develop an algebraic hierarchy for Gentzen systems that closely parallels the well-known algebraic hierarchy of deductive systems. This approach allows us to unify in a single framework several previously known results about algebraizable and equivalential Gentzen systems. We also obtain a characterization of weakly algebraizable Gentzen systems. The significance of Gentzen systems and related axiomatizations by Gentzen rules is due in large part to the fact that various metatheoretical properties of deductive systems can be formulated in their terms. It was observed that a number of important non-protoalgebraic deductive system that have a natural algebraic semantics also have a so-called fully adequate Gentzen system associated with them, the conjunction-disjunction fragment of the classical propositional logic being a paradigmatic example. In this work, a general criterion for the existence of a fully adequate Gentzen system for non-protoalgebraic deductive systems is obtained, and it is shown that many of the known partial results can be explained based on this general criterion. This includes such cases as the existence of fully adequate Gentzen systems for self-extensional logics with conjunction or implication, and the criterions for the existence of a fully adequate Gentzen system for protoalgebraic and weakly algebraizable logics. In another vein, it is shown that the existence of a multiterm deduction-detachment theorem in a deductive system is equivalent to the fact that, so called, axiomatic closure relations for the deductive system form a Gentzen system.

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INTRODUCTION

In very general terms we can systematize our knowledge of some area of expertise by listing all the relationships of *casuality*, i.e., statements of the form "from $\alpha_0, \alpha_2, \dots$ it follows by necessity that α" (more formally written as $\alpha_0, \alpha_2, \dots \vdash \alpha$) where $\alpha_0, \alpha_1, \dots, \alpha$ are the expressions of some language pertaining to that area. To be practical this system must at least be *finitary*, that is, we accept only the casuality relationships that depend on a finite number of *conditions*, that is, of the form $\alpha_0, \dots, \alpha_{n-1} \vdash \alpha$ where n is finite. For the same reason we want the system to be *monotone*, so if $\alpha_0, \dots, \alpha_{n-1} \vdash \alpha$ then $\beta_0, \dots, \beta_{k-1}, \alpha_0, \dots, \alpha_{n-1} \vdash \alpha$. It is also desirable that the system of knowledge be *theoretical* in nature, so each casuality relationship $\alpha_0, \dots, \alpha_{n-1} \vdash \alpha$ represents, not a single relationship, but rather a *scheme* that represents the entire family of relationships that can be obtained from it by substituting arbitrary expressions for the variables that occur in the scheme.

Generally, the linguistic expressions that the α_i represent can be quite complex: simple formulas, equations between formulas, and even entire sequents, that is formula complexes of the form $\varphi_0, \dots, \varphi_{n-1} \rhd \varphi$ where $\varphi_0, \dots, \varphi_{n-1}, \varphi$ are simple formulas. Thus the "casuality" relation can take the form of either a Hilbert-style system, an equational system, or a Gentzen system.

In this thesis we develop a general framework that allows us to study how these three different logical formalisms interact among themselves. It is based on the idea that an equational system and a Hilbert system can both be incorporated into the Gentzen formalism. An equation $\alpha = \beta$ can be presented as a pair of sequents $\{\alpha \rhd \beta, \beta \rhd \alpha\}$, while a formula α can be represented by the sequent of length one $\rhd\alpha$.

To define a Gentzen system, we start off with a language of some given functional type

$\mathcal{L}$. The elements of $\mathcal{L}$ are interpreted as logical connectives, when viewed from a logical perspective, or as operation symbols in an algebraic context. The formulas of this language form an algebra, which in fact is the absolutely free algebra with propositional variables as free generators. A Gentzen system $\mathcal{G}$ can be identified with a consequence relation $\vdash_{\mathcal{G}}$ on the set of sequents. A theory of $\mathcal{G}$ is any set of sequents closed with respect to the consequence relation. The set $\mathrm{Th}\,\mathcal{G}$ of all theories of $\mathcal{G}$ is an algebraic closure system. The largest theory of a Gentzen system is the set of all sequents.

We can impose a restriction on the shape of sequents; for instance we may require that all sequents have form $\varphi \rhd \psi$, and consider a consequence relation on the set of all sequents of this form. This is the way equational logic can be interpreted as a kind of generalized Gentzen system. Restricting to sequents of the form $\rhd\varphi$ with a single formula we get Hilbert systems. For our purposes it is convenient to refer to these as *1st-level deductive systems* and Gentzen systems that satisfy the so-called structural rules (reflexivity, exchange, and weakening) and cut (transitivity) will be called *2nd-level deductive systems*. Chapter 2 mostly contains technical results about 2nd-level systems, including their axiomatization and 2nd-level matrix semantics.

The study of 1st-level deductive systems such as the classical and intuitionistic propositional calculi, and the modal logics, has historically been supported and often led by the study of semantics, especially algebraic semantics. To accommodate this process in a broader and more abstract context the notion of the Leibnitz operator was introduced in the area of algebraic logic. Given a 1st-level deductive system $\mathcal{S}$, The Leibnitz operator of $\mathcal{S}$ is the mapping $\boldsymbol{\Omega}$ from the set $\mathrm{Th}\,\mathcal{S}$ of theories of $\mathcal{S}$ into the congruences $\mathrm{Con}\,\mathbf{Fm}_{\mathcal{L}}$ on the formula algebra where, for every $T \in \mathrm{Th}\,(\mathcal{S})$, $\boldsymbol{\Omega}\,T$ is the largest congruence *compatible* with T in the sense that if $\alpha \in T$ and $\langle \alpha, \beta \rangle \in \boldsymbol{\Omega}\,T$, then $\beta \in T$.

There is a generalization of this concept to 2nd-level systems. Using combinatorial properties of the 2nd-level Leibnitz operator (also known as Tarski operator), we classify in Chapter 3 the 2nd-level deductive systems in a hierarchy that closely parallels the well-known algebraic hierarchy of 1st-level systems. This approach allows us to unify in a single framework several previously known results about algebraizable and equivalential 2nd-level systems. We also obtain a characterization of weakly algebraizable 2nd-level deductive systems.

In the second part of the dissertation we apply the theory of 2nd-level systems developed in the first part to study metalogical properties of the 1st-level, i.e., Hilbert systems. The starting point here is the notion of a generalized model of a 1st-level deductive system. A *generalized model* for 1st-level system $\mathcal{S}$ is a pair $\langle \mathbf{A}, \mathcal{C} \rangle$ where $\mathbf{A}$ is an algebra of the appropriate language type and $\mathcal{C}$ is an algebraic closure system of $\mathcal{S}$-filters. It is *full generalized model* of $\mathcal{S}$ if $\mathcal{C}$ is the set of all $\mathcal{S}$-filters of $\mathbf{A}$ or a closely related set. A 2nd-level system $\mathcal{G}$ is *adequate* for a 1st-level system $\mathcal{S}$ if every full generalized model of $\mathcal{S}$ is a model of $\mathcal{G}$, and it is *fully adequate* for $\mathcal{S}$ if, roughly speaking, these constitute all the models of $\mathcal{G}$.

The significance for 1st-level systems of their axiomatization by Gentzen-style rules is due in large part to the fact that many important metalogical properties of these systems can be formulated in terms of rules of this kind. There are a number of important 1st-level deductive systems that have a natural algebraic semantics, but are not protoalgebraic, and hence are not amenable to the standard methods of 1st-level abstract algebraic logic. It has been observed however that many of these systems have a fully adequate 2nd-level deductive system associated with them. The conjunction-disjunction fragment of the classical propositional logic is the paradigmatic example.

Full closure relations were introduced in [14] as a tool for studying fully adequate 2nd-level systems. The finite closure relation associated with an algebraic closure system $\mathcal{C}$ is the set of all finite sequences $a_0, \ldots, a_{n-1}, a$ of elements of the universe of $\mathcal{C}$ such that a is in the closure of $\{a_0, \ldots, a_{n-1}\}$. Roughly speaking, a *full closure relation* for a 1st-level deductive system $\mathcal{S}$ is a finite closure relation associated with a *full* generalized model of $\mathcal{S}$. It was known that a 1st-level system has a fully adequate 2nd-level system if and only if the set of all full closure relations on the formula algebra is a closure system [13]. The highlight of Chapter 4 is a general criterion for the existence of a fully adequate 2nd-level systems for non-protoalgebraic 1st-level deductive system $\mathcal{S}$. More precisely, there is a fully adequate 2nd-level system for $\mathcal{S}$ if and only the full closure relations of $\mathcal{S}$ form a 2nd-level deductive system. We also show that many of the known partial results about the existence of a fully adequate Gentzen systems can be explained on the basis of this general criterion. This includes such cases as the existence of fully adequate Gentzen systems for self-extensional logics with conjunction or implication, and the known criteria for the existence of a fully adequate Gentzen system for protoalgebraic and weakly algebraizable logics.

In Chapter 5 we consider another type of finite closure relations for 1st-level systems. An *axiomatic closure relation for* $\mathcal{S}$ is a finite closure relation associated with a generalized model $\langle \mathbf{A}, \mathcal{C} \rangle$, where $\mathcal{C}$ is chosen in such way that it must include all $\mathcal{S}$-filters that contain the smallest filter in $\mathcal{C}$. It was known that the set of axiomatic closure relations is closed under finite intersections if and only if the lattice all $\mathcal{S}$-filters is distributive [13]. We show that, if we strengthen the distributivity condition, and require that the complete lattice of theories of the system is *infinitely meet-distributive over compact elements*, then the set of axiomatic closure relations is closed under finite intersections if and only if they form a closure system, and so are closed under *arbitrary* non-empty intersections. This leads to the main result of the chapter: a 1st-level system admits a deduction-detachment theorem if and only if the set of axiomatic closure relations for $\mathcal{S}$ form a Gentzen system. This is a new characterization of the deduction-detachment theorem.

Outline of the paper.

The thesis contains: abstract, acknowledgement, introduction, 5 chapters, conclusion, index and bibliography. First chapter provides the preliminary mathematical information and notation and also important facts and results from the field of algebraic logic. Some basic facts and "folklore" are given without references. In Chapter 2 we introduce the basic notion of this dissertation, a notion of a 2nd-level deductive system. The obtained elementary results have nevertheless rather technical and abstract form to suit further applications in Chapters 3,4 and 5, therefore the details of the proofs in this chapter may be skipped during the first reading. In Chapter 3 we define a 2nd-level Leibnitz (Tarski) operator. We develop an algebraic hierarchy of 2nd-level deductive systems based on properties of the 2nd-level Leibnitz operator. The results of Chapter 3 are mainly used in Chapter 4. Chapters 4 and 5 show that some important problems of logic, like the existence of deduction-detachment theorem and the existence of a fully adequate Gentzen system for Hilbert systems are in fact related to 2nd-level systems. We obtain in Chapters 4 and 5 some new results and characterizations through this relation. In Conclusion we summarize the results and sketch a plan for future work.

Main results and related works.

There are four major results in this dissertation.

I. The problem of defining an algebraic hierarchy for Gentzen systems that would be to some degree parallel in semantical and syntactical aspects to the very successful algebraic hierarchy of Hilbert systems was studied by several authors. D. Pigozzi introduced in [20] by a syntactical definition the classes of equivalential and finitely equivalential Gentzen systems and proved that they demonstrate the expected semantical behavior, similar to that of equivalential and finitely equivalential Hilbert systems. Some examples of Gentzen systems that can be called finitely algebraizable were considered in [12].

The author suggests in Def. 3.1.11 an algebraic hierarchy that is based on properties of Tarski[1] (2nd-level Leibnitz) operator. This approach has the advantage that the resulting algebraic hierarchy:

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[1] Another interesting approach based on properties of *Suszko* operator was considered in [6].

a) closely parallels (in formulation, properties and even in parts of proofs) the algebraic hierarchy for Hilbert systems based on properties Leibnitz (1st-level) operator [16, 17];

b) allows us to obtain syntactical characterizations (Thm. 3.2.3, Prop. 3.3.4, Prop 3.3.6), that are similar to and can be seen as generalizations of those for Hilbert systems.

c) the syntactical characterization for (finitely) equivalential Gentzen system coincides with that defined in [20].

The semantical properties of the hierarchy of Def. 3.1.11 in terms of closure properties of matrix classes has not been considered in this thesis.

II. The notion of the weakest structural Gentzen system adequate for a given Hilbert system was informally introduced in [14]. We denote it $\mathbf{Gcr}\,\mathcal{S}$ (Def. 2.2.9) and make a great use of it. In particular, both full closure relations (Def. 4.2.1) and axiomatic closure relations (Def. 5.1.1) are defined as special subsets of $\mathbf{Gcr}\,\mathcal{S}$. The basis for work with $\mathbf{Gcr}\,\mathcal{S}$ was laid out in [20].

III. In Chapter 4 we prove a general criterion for the existence of fully adequate Gentzen systems for Hilbert deductive systems through the existence of a graded congruence basis (Thm's. 4.3.4, 4.3.9). This criterion generalizes several known results. That includes

a) the criterion for the existence of a fully adequate Gentzen system for a protoalgebraic Hilbert system through the existence of Leibnitz generating Parameterized Graded Deduction-Detachment system (LPGDD system) [14];

as well as the following sufficient conditions [12, Thm. 4.27 and Thm. 4.45]

b) a self-extensional Hilbert system with conjunction has a fully adequate Gentzen system;

c) a self-extensional Hilbert system with implication has a fully adequate Gentzen system.

The general criterion also suggests a form of Gentzen axiomatization for fully adequate Gentzen systems (Def. 4.3.7), that can be chosen as canonical.

IV. The deduction-detachment theorem (DDT) has been extensively studied, ever since it was discovered by Tarski and Herbrand. Along with numerous results about particular deductive systems that admit the deduction-detachment theorem, a new trend become discernible in recent years: the study of the phenomenon from general positions including that of abstract algebraic logic (see [3] for references). The highlight of Chapter 5 is a new characterization

(Thm. 5.2.3) of the (multiterm) deduction-detachment theorem. It states that a Hilbert system $\mathcal{S}$ admits the DDT iff the set of all axiomatic closure relations for $\mathcal{S}$ forms a Gentzen system. This characterization, though original, has nevertheless several partially pertaining predecessors:

a) in [13] it was in fact proven that the set of axiomatic closure relations for a Hilbert system is closed under finite intersections iff the lattice of theories for this Hilbert system is distributive;

b) it was found in [9, Thm. 2.6.8] that a protoalgebraic Hilbert system admits a deduction-detachment theorem iff the lattice of theories for this Hilbert system is infinitely meet-distributive over its compact elements;

c) [14, Cor. 5.7] states that a weakly algebraizable Hilbert system has a fully adequate Gentzen system iff it admits the deduction-detachment theorem.

Special acknowledgements. I would like especially emphasize the importance of the following works in shaping the ideology of this dissertation (in historical order):

[12] J. M. Font and R. Jansana, *A general algebraic semantics for sentential logics*, Number 7 in Lecture Notes in Logic, Springer-Verlag, 1996.

[16] B. Herrmann, *Characterizing equivalential and algebraizable logics by the Leibniz operator*, Studia Logica **58** (1997), 305–323.

[20] D. Pigozzi, *Second-order algebraizable logics*, Manuscript, 1996.

[9] J. Czelakowski, *Protoalgebraic logics*, Kluwer, Dordrecht, 2001.

1. MATHEMATICAL PRELIMINARIES AND NOTATION

The contraction "iff" will be used for the phrase "if and only if", "TFAE" for "the following are equivalent", ":=" for "by definition"; "$\forall$" stands for "for all", "$\exists$" for "there exists", "&" for "and", "$\implies$" for "implies".

We employ cardinal notation for the set of natural numbers $\omega = \{0, 1, 2, \dots\}$. Thus n is a natural number iff $n \in \omega$. Also for any two natural numbers $i, n \in \omega$ we have $i < n$ iff $i \in n$. Let $\bar{a}$ be a contraction for the expression $a_0, \dots, a_{n-1}$. We will write: a sequence $\langle a_0, \dots, a_{n-1} \rangle$ as $\langle \bar{a} \rangle$ or as $\langle a_i \rangle_{i \in n}$; a non-empty sequence $\langle a_0, \dots, a_{n-1}, a \rangle$ as $a_0, \dots, a_{n-1} \triangleright a$, $\langle a_i, a \rangle_{i \in n}$, $\bar{a} \triangleright a$ or $\langle \bar{a}, a \rangle$; a set $\{a_0, \dots, a_{n-1}\}$ as $\{\bar{a}\}$. As a rule, every time we meet $\bar{a}$ in the text it can be expounded as $a_0, \dots, a_{n-1}$ for some $n \in \omega$.

The non-empty sequences are called *vectors* or *strings*. In the case $\langle \bar{a} \rangle = \langle a_i \rangle_{i \in n}$ we write $|\langle \bar{a} \rangle| = n$ and say that $\langle \bar{a} \rangle$ has *length* n. If $\{A_i\}_{i \in n}$ is a family of non-empty sets, then $\prod_{i \in n} A_i = A_0 \times \cdots \times A_{n-1} := \{\langle a_i \rangle_{i \in n} \mid a_i \in A_i\}$ is called a *cartesian product of the sets* $A_0, \dots, A_{n-1}$.

A *binary relation R between elements of A and B* is a subset $R \subseteq A \times B$. For any two binary relations $R \subseteq A \times B$, $S \subseteq B \times C$, the *composition $R \circ S$ of R and S* is defined by

$$R \circ S := \{\langle a, c \rangle \in A \times C \mid (\exists b \in B)\, aRb \,\&\, bSc\}.$$

For $\langle a, b \rangle \in R$ we often write aRb. A *function f from A into B* (written as $f : A \to B$) is a binary relation $f \subseteq A \times B$ such that 1) for every $a \in A$ there exists $b \in B$ such that $\langle a, b \rangle \in R$, 2) if $\langle a, b \rangle$, $\langle a, c \rangle \in f$, then $b = c$. We write $f(a) = b$ or, equivalently, $fa = b$ to show that $\langle a, b \rangle \in f$ and to distinguish the *argument* a of f, from the *value b of f on a*. For all $X \subseteq A$, $Y \subseteq B$, we define $fX := \{fx \mid x \in X\}$, $f^{-1}X := \{x \in A \mid (\exists y \in Y)\, fx = y\}$.

A function $f : A \to B$ is *injective* (written as $f : A \rightarrowtail B$) if $fa = fb$ implies that $a = b$;

f is *surjective* ($f : A \twoheadrightarrow B$) if $fA = B$. If $f : A \to B$ and $g : B \to C$ are two functions then the *composition of f and g* (written as gf or $f \circ g$) is a function $h : A \to C$ such that $h(a) = g(f(a))$.

We state for future reference some simple facts about images and inverse images.

Lemma 1.0.1. *Let $h : A \to B$. Then*

1. $X \subseteq h^{-1}hX$*, for every subset* $X \subseteq A$*;*
2. $hh^{-1}Y \subseteq Y$*, for every subset* $Y \subseteq B$*;*
3. $h(\bigcap_{i\in I} X_i) \subseteq \bigcap_{i\in I} hX_i$*, for every family* $\{X_i\}_{i\in I} \subseteq \mathcal{P}(A)$*;*
4. $h^{-1}(\bigcap_{i\in I} Y_i) = \bigcap_{i\in I} h^{-1}Y_i$*, for every family* $\{Y_i\}_{i\in I} \subseteq \mathcal{P}(B)$*.*

Suppose A is a set. Then $\mathcal{P}(A) := \{X \mid X \subseteq A\}$ is the *power-set* of A. We write $X \subseteq_\omega A$ if X is a finite subset of A, furthermore $\mathcal{P}_\omega(A) := \{X \mid X \subseteq_\omega A\}$. For a family of sets $\mathcal{C} \subseteq \mathcal{P}(A)$, we define $\bigcup \mathcal{C} := \bigcup_{X\in\mathcal{C}} X$, $\bigcap \mathcal{C} := \bigcap_{X\in\mathcal{C}} X$. The *$n$-th cartesian power of* of a non-empty set A is the set $A^n := \prod_{i\in n} A$ of all vectors of length n with elements from A. A^+ denotes $\bigcup_{n=1}^{\infty} A^n$, the set of all non-empty finite sequences of elements of A and $A^* := A^+ \cup \{\langle\rangle\}$ is the set of all sequences. A function $f : A^n \to A$ is called an n-ary *operation on* A. Instead of $f\langle\bar{a}\rangle$ or $f(\langle\bar{a}\rangle)$ we will write sometimes $f(\bar{a})$. A *unary operation* $f : A \to A$ is also called a *mapping* on A.

A binary relation $R \subseteq A \times A$ is *reflexive* if for all $a \in A$, aRa; *symmetric* if for all $a, b \in A$, aRb implies bRa; *transitive* if for all $a, b, c \in A$, from aRb and bRc it follows that aRc; *antisymmetric* if for all $a, b \in A$, aRb and bRa implies that $a = b$. We call $R \subseteq A \times A$ an *equivalence relation on* A if R is reflexive, symmetric and transitive; and R is a *partial order on* A if R is reflexive, transitive and antisymmetric. The set of all symmetric binary relations on A is denoted by $\operatorname{Sym} A$; the set of all equivalence relations on A by $\operatorname{Eq} A$.

If $\leqslant$ is a partial order on A and $X \subseteq A$, an element $a \in A$ such that for all $x \in X$, $x \leqslant a$ is called an *upper boundary of* X; dually, an element $a \in A$ such that for all $x \in X$, $x \geqslant a$ is called a *lower boundary of* X; $\inf X$ is the largest (if it exists) element of A among the lower boundaries of X; similarly, $\sup X$ is the smallest (if it exists) element of A among the upper boundaries of X. If inf (sup) exists for any two-element subset of A, A is called a *lower (upper) semi-lattice.* In that case, $\inf\{a, b\}$ is usually denoted by $a \wedge b$, and $\sup\{a, b\}$ as $a \vee b$, and interpreted as binary operations on A. If both $\wedge$ and $\vee$ defined for any pair of elements of A, A is called a *lattice*. If inf and sup exists for any non-empty subset of A, A is called a *complete lattice*.

For a mapping $h : A \to A$ the operator-style notation ha will be routinely used instead of function-style $h(a)$. Also any mapping h defined on A can be uniquely extended to a mapping on A^+ by the following definition:

$$h\langle a_i\rangle_{i\in n} = \langle ha_i\rangle_{i\in n}, \quad \langle a_i\rangle_{i\in n} \in A^+.$$

The latter defines a *complex* (defined on sets of elements) mapping on $\mathcal{P}(A^+)$ as follows,

$$hX = \{h\langle\bar{a}\rangle \mid \langle\bar{a}\rangle \in X\} \text{ for all } X \subseteq A^+.$$

Note that the same symbol h will be used routinely for all these mappings.

A *language type* is any non-empty set $\mathcal{L}$. The elements of $\mathcal{L}$ are called *functional symbols* in an algebraic context or *logical connectives* in a logical context. With $\mathcal{L}$ is associated an *arity* function $\rho : \mathcal{L} \to \omega$ such that ρf is the *arity* or *rank* of the functional symbol $f \in \mathcal{L}$. For each $n \in \omega$: $\mathcal{L}_n := \{f \in \mathcal{L} \mid \rho f = n\}$. An *algebra* $\mathbf{A}$ of type $\mathcal{L}$ is a pair $\langle A, \mathcal{L}^{\mathbf{A}}\rangle$, where A is a non-empty space called *universe* of $\mathbf{A}$ and $\mathcal{L}^{\mathbf{A}} = \{f^{\mathbf{A}} \mid f \in \mathcal{L}\}$ is a list of operations over the set A such that for every $f \in \mathcal{L}_n$, $f^{\mathbf{A}} : A^n \to A$. Members of $\mathcal{L}^{\mathbf{A}}$ are called *basic operations* of $\mathbf{A}$. If $\mathbf{A}$, $\mathbf{B}$ are algebras of the same type, then a mapping $h : A \to B$ is called a homomorphism of $\mathbf{A}$ into $\mathbf{B}$ (written $h : \mathbf{A} \to \mathbf{B}$), if for every $f \in \mathcal{L}_n$ and every $\langle\bar{a}\rangle \in A^n$, $hf^{\mathbf{A}}\langle\bar{a}\rangle = f^{\mathbf{B}}h\langle\bar{a}\rangle$. A homomorphism $h : \mathbf{A} \to \mathbf{A}$ is called an *endomorphism of* $\mathbf{A}$; if h is also surjective and injective, then h is an *automorphism of* $\mathbf{A}$. An equivalence relation θ on the universe A of $\mathbf{A}$ is called a *congruence relation on* $\mathbf{A}$ or a *congruence on* $\mathbf{A}$, if θ is *compatible* with the basic operations of $\mathbf{A}$, i.e., for every $f \in \mathcal{L}_n$ and all $\langle\bar{a}\rangle, \langle\bar{b}\rangle \in A^n$ if $a_i\theta b_i$ for all $i \in n$, then $f^{\mathbf{A}}(\bar{a})\theta f^{\mathbf{A}}(\bar{b})$. For every algebra A there is a pair of distinguished congruences: $0_{\mathbf{A}} := \{a \triangleright a \mid a \in A\}$ is the *trivial congruence on* $\mathbf{A}$ and $1_{\mathbf{A}} := \{a \triangleright b \mid a, b \in A\}$ is the *universal congruence on* A. The set of all congruence relations on $\mathbf{A}$ is denoted by $\operatorname{Con}\mathbf{A}$ and forms a complete lattice by inclusion with $0_{\mathbf{A}}$ as the smallest and $1_{\mathbf{A}}$ as the largest element. If for every endomorphism $h : \mathbf{A} \to \mathbf{A}$, $h\theta \subseteq \theta$, then $\theta \in \operatorname{Con}\mathbf{A}$ is called *fully invariant.* If $\theta, \eta \in \operatorname{Con}\mathbf{A}$ the join $\theta \vee \eta$ of θ and η is equal to $\bigcup_{n\in\omega}(\theta\circ\eta)^n = \theta\circ\eta \vee \theta\circ\eta\circ\theta\circ\eta \vee \theta\circ\eta\circ\theta\circ\eta\circ\theta\circ\eta \vee \ldots$.

Let $X = \{x_i\}_{i\in I}$ be a non-empty set. The set $\mathrm{Fm}_{\mathcal{L}}\, X$ of *formulas (or terms) of type* $\mathcal{L}$ *over the set of generators* X is defined recursively as follows

1. $X \subseteq \mathrm{Fm}_{\mathcal{L}}\, X$;
2. if $f \in \mathcal{L}_n$ and $\alpha_0, \ldots, \alpha_{n-1} \in \mathrm{Fm}_{\mathcal{L}}\, X$, then $\langle f, \alpha_0, \ldots, \alpha_{n-1}\rangle \in \mathrm{Fm}_{\mathcal{L}}\, X$.

Traditionally the formula $\langle f, \alpha_0, \ldots, \alpha_{n-1}\rangle$ is written as $f(\alpha_0, \ldots, \alpha_{n-1})$. Formulas will be denoted usually by small Greek letters. An element $\langle \alpha_0, \ldots, \alpha_k\rangle$ of $\mathrm{Fm}^+_{\mathcal{L}}$ is called a *sequent* and will be written usually in the form $\alpha_0, \ldots, \alpha_{k-1} \rhd \alpha_k$. We write $\alpha(p_0, \ldots, p_{n-1})$ or $\mathrm{Var}(\alpha) \subseteq \{p_0, \ldots, p_{n-1}\}$, whenever $\alpha \in \mathrm{Fm}_{\mathcal{L}}\{p_0, \ldots, p_{n-1}\}$.

We can induce the structure of an algebra on $\mathrm{Fm}_{\mathcal{L}}\, X$ by associating with each $f \in \mathcal{L}_n$ a n-ary operation $f^{\mathbf{Fm}_{\mathcal{L}} X}$ on the set $\mathrm{Fm}_{\mathcal{L}}\, X$ defined by $f^{\mathbf{Fm}_{\mathcal{L}} X}\langle\bar{\alpha}\rangle = f(\bar{\alpha})$. The superscript in this case may be omitted. This algebra $\mathbf{Fm}_{\mathcal{L}}\, X$ is called the *algebra of formulas (terms) of type* $\mathcal{L}$ *over the set of variables* X. We fix a countable set $\mathrm{Var} = \{x_0, x_1, x_2, ...\}$ of *propositional variables.* Then $\mathbf{Fm}_{\mathcal{L}}\,\mathrm{Var}$ is called *the formula algebra over (of) the language type* $\mathcal{L}$ and will be denoted $\mathbf{Fm}_{\mathcal{L}}$. The universe of $\mathbf{Fm}_{\mathcal{L}}$ is $\mathrm{Fm}_{\mathcal{L}}$.

An algebra $\mathbf{Fm}_{\mathcal{L}}\, X$ is an *absolutely free algebra over the set* X in the class of all algebras of type $\mathcal{L}$. This means that, for every algebra $\mathbf{A}$ of type $\mathcal{L}$, an arbitrary mapping $h : X \to A$ can be uniquely extended to a homomorphism $h : \mathbf{Fm}_{\mathcal{L}}\, X \to A$. In particular any homomorphism $h : \mathbf{Fm}_{\mathcal{L}}\, X \to \mathbf{A}$ is determined by the mapping $h : X \to A$. A homomorphism $h : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$ is called an *evaluation*; a homomorphism $h : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{Fm}_{\mathcal{L}}$ is called a *substitution.*

Let $\mathcal{L}$ be an arbitrary language type and suppose $\alpha \in \mathrm{Fm}_{\mathcal{L}}(\{\xi\} \cup \mathrm{Var})$. Then α defines a function $f : \mathrm{Fm}_{\mathcal{L}} \to \mathrm{Fm}_{\mathcal{L}}$, called a *unary polynomial on* $\mathbf{Fm}_{\mathcal{L}}$ or a *unary polynomial operation on* $\mathbf{Fm}_{\mathcal{L}}$, as follows $f\beta := \sigma\beta$, where $\sigma : \mathbf{Fm}_{\mathcal{L}}(\{\xi\} \cup \mathrm{Var}) \to \mathbf{Fm}_{\mathcal{L}}$ is a homomorphism such that $\sigma\xi = \beta$ and $\sigma x = x$ for all $x \in \mathrm{Var}$. Using λ-notation, we can write $f = \lambda\,\xi.\alpha$. Let

$$\mathcal{T}_{\mathcal{L}} := \{\, \lambda\,\xi.\alpha \mid \alpha \in \mathrm{Fm}_{\mathcal{L}}(\{\xi\} \cup \mathrm{Var})\}, \quad \xi \notin \mathrm{Var}\,.$$

For every algebra of type $\mathcal{L}$ we define the set $\mathcal{T}_{\mathbf{A}}$ of unary polynomials on $\mathbf{A}$

$$\mathcal{T}_{\mathbf{A}} := \{\lambda\,\xi.\alpha(\xi, \bar{a}) : A \to A \mid \lambda\,\xi.\alpha(\xi, \bar{x}) \in \mathcal{T}_{\mathcal{L}}, \langle\bar{a}\rangle \in A^{|\langle\bar{x}\rangle|}\}.$$

We will customary omit the subscript $\mathcal{L}$ whenever the language type is clear from the context. Any subset $\mathcal{J} \subseteq \mathcal{T}_{\mathbf{A}}$ defines the complex operator $\mathcal{J} : \mathcal{P}(A) \to \mathcal{P}(A)$ by $\mathcal{J}B := \{tb \mid b \in B, t \in \mathcal{J}\}$. We can also define $\mathcal{J}\mathcal{J} := \mathcal{J} \circ \mathcal{J} := \{t \circ s \mid t, s \in \mathcal{J}\}$.

A family $\mathcal{C} \subseteq \mathcal{P}(A)$ is *upward-directed* if for every pair $X, Y \in \mathcal{C}$ there is $Z \in \mathcal{C}$ such that $X, Y \subseteq Z$. A subset $\mathcal{C} \subseteq \mathcal{P}(A)$ is *algebraic* if $\bigcup \mathcal{D} \in \mathcal{C}$ for every upward-directed subfamily $\mathcal{D} \subseteq \mathcal{C}$. A family $\mathcal{C} \subseteq \mathcal{P}(A)$ is called a *closure system on (over)* A if $A \in \mathcal{C}$ and $\bigcap \mathcal{D} \in \mathcal{C}$ for every non-empty subfamily $\mathcal{D} \subseteq \mathcal{C}$. The intersection of closure systems over the same set is a closure system on that set. The intersection of algebraic closure systems over the same set is an algebraic closure system on that set. A closure system $\mathcal{C}$ over $\mathrm{Fm}_{\mathcal{L}}$ is *invariant* if for any substitution σ and any $T \in \mathcal{C}$, $\sigma^{-1}T = \{\alpha \mid \sigma\alpha \in T\} \in \mathcal{C}$, or, in other words, if $\sigma^{-1}\mathcal{C} \subseteq \mathcal{C}$ for all $\sigma : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{Fm}_{\mathcal{L}}$. The intersection of invariant closure systems over the same set $\mathrm{Fm}_{\mathcal{L}}$ is an invariant closure system on that set.

A *closure operator on* A is a mapping $\mathbf{C} : \mathcal{P}(A) \to \mathcal{P}(A)$ such that for any $X, Y \subseteq A$, $X \subseteq \mathbf{C}(X) = \mathbf{C}(\mathbf{C}(X)) \subseteq \mathbf{C}(X \cup Y)$. A set $X \in \mathcal{P}(A)$ such that $\mathbf{C}(X) = X$ is called a *closed set of* $\mathbf{C}$. A closure operator $\mathbf{C}$ is *finitary* if for any $X \subseteq A$, $\mathbf{C}(X) = \bigcup\{\mathbf{C}(Y) \mid Y \subseteq_\omega X\}$. The following relations between closure systems and closure operators are well known: 1) if $\mathbf{C}$ is a closure operator on A, then the family of its closed sets is a closure system on A; 2) if $\mathcal{C}$ ia a closure system on A, then the mapping $\mathbf{C}_{\mathcal{C}} : \mathcal{P}(A) \to \mathcal{P}(A)$ defined for each $X \subseteq A$ as $\mathbf{C}_{\mathcal{C}}X := \bigcap\{Y \in \mathcal{C} \mid X \subseteq Y\}$ is a closure operator on A; 3) $\mathcal{C}$ is algebraic iff $\mathbf{C}_{\mathcal{C}}$ is finitary. We use interchangeably the prefix and exponential notations for closure operators, thus $X^{\mathcal{C}} = \mathbf{C}_{\mathcal{C}}X$.

Every closure system $\mathcal{C}$, as a family of subsets ordered under set-inclusion, is a complete lattice. The infimum of a family $\{X_i\}_{i \in I} \subseteq \mathcal{C}$ is its intersection $\bigcap_{i \in I} X_i$, and its supremum is $\bigvee^{\mathcal{C}}_{i \in I} X_i := \mathbf{C}_{\mathcal{C}}(\bigcup_{i \in I} X_i)$; its largest element is A, and its smallest element is $\mathbf{C}_{\mathcal{C}}(\emptyset) = \bigcap \mathcal{C}$.

A *consequence relation* is a relation $\vdash_{\mathcal{S}} \subseteq \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}) \times \mathrm{Fm}_{\mathcal{L}}$ such that for all $\Gamma, \Delta \subseteq \mathrm{Fm}_{\mathcal{L}}$ and every $\alpha \in \mathrm{Fm}_{\mathcal{L}}$:

1. $\alpha \in \Gamma$ implies $\Gamma \vdash_{\mathcal{S}} \alpha$; *reflexivity*
2. $\Gamma \vdash_{\mathcal{S}} \alpha$ and $\Gamma \subseteq \Delta$ implies $\Delta \vdash_{\mathcal{S}} \alpha$; *monotonicity*
3. $\Gamma \vdash_{\mathcal{S}} \alpha$ and $\Delta \vdash_{\mathcal{S}} \beta$ for every $\beta \in \Gamma$ implies $\Delta \vdash_{\mathcal{S}} \alpha$. *transitivity*

A consequence relation $\vdash_{\mathcal{S}}$ is: 1) *finitary* if for every $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}$ from $\Gamma \vdash_{\mathcal{S}} \alpha$ it follows that $\Delta \vdash_{\mathcal{S}} \alpha$ for some finite $\Delta \subseteq \Gamma$; 2) *structural* if $\Gamma \vdash_{\mathcal{S}} \alpha$ implies $\sigma\Gamma \vdash_{\mathcal{S}} \sigma\alpha$ for all substitutions σ.

A deductive system $\mathcal{S}$ in the sense of Tarski is determined by its finitary structural consequence relation $\vdash_{\mathcal{S}}$. A *theory* for $\mathcal{S}$ is any subset T of $\mathrm{Fm}_{\mathcal{L}}$ closed under $\vdash_{\mathcal{S}}$ in the sense that if $\Gamma \vdash_{\mathcal{S}} \alpha$ and $\Gamma \subseteq T$ then $\alpha \in T$. Obviously, $\mathrm{Fm}_{\mathcal{L}}$ is a theory for any $\mathcal{S}$.

In addition, it can be shown that the set $\mathrm{Th}\,\mathcal{S}$ of all theories for $\mathcal{S}$ is closed under

1. non-empty intersections, because $\vdash_{\mathcal{S}}$ is monotone, reflexive, transitive
2. unions of upward-directed families, — finitary,
3. inverse substitutions, — structural.

where being closed under inverse substitutions means that for every $T \in \mathrm{Th}\,\mathcal{S}$ and every substitution σ, $\sigma^{-1}T := \{\alpha \mid \sigma\alpha \in T\}$ is a theory of $\mathcal{S}$ again. This leads us to the definition of a deductive system usually employed in algebraic logic.

A *deductive system (1st-level)* is a pair $\mathcal{S} = \langle \mathbf{Fm}_{\mathcal{L}}, \mathrm{Th}\,\mathcal{S} \rangle$ such that $\mathrm{Th}\,\mathcal{S} \subseteq \mathcal{P}(\mathrm{Fm}_{\mathcal{L}})$ is an algebraic invariant closure system on $\mathrm{Fm}_{\mathcal{L}}$.

The original consequence relation $\vdash_{\mathcal{S}}$ can be described in terms of a closure in $\mathrm{Th}\,\mathcal{S}$ by

$$\Gamma \vdash_{\mathcal{S}} \alpha \iff \alpha \in \Gamma^{\mathrm{Th}\,\mathcal{S}} \quad \text{where } \Gamma^{\mathrm{Th}\,\mathcal{S}} = \bigcap \{T \in \mathrm{Th}\,\mathcal{S} \mid \Gamma \subseteq T\}.$$

The following notion of a *Leibnitz congruence* played a crucial role in the development of algebraization theory for deductive systems [4, 5].

Let $\mathbf{A}$ be an algebra of type $\mathcal{L}$ and $X \subseteq A$. The *Leibnitz congruence* $\boldsymbol{\Omega}_{\mathbf{A}}X$ is defined by

$$\boldsymbol{\Omega}_{\mathbf{A}}X := \{\langle a, b\rangle \mid (\forall t \in \mathcal{T}_{\mathbf{A}})\, ta \in X \iff tb \in X\}.$$

With each deductive system $\mathcal{S}$ we associate the *Leibnitz operator* $\boldsymbol{\Omega} : \mathrm{Th}\,\mathcal{S} \to \mathrm{Con}\,\mathbf{Fm}_{\mathcal{L}}$.

Leibnitz congruence commutes with inverse surjective homomorphism, thus for every $h : \mathbf{A} \twoheadrightarrow \mathbf{B}$ and every $X \subseteq A$, $\boldsymbol{\Omega}_{\mathbf{A}}\, h^{-1}X = h^{-1}\, \boldsymbol{\Omega}_{\mathbf{B}}\, X$. The *Tarski congruence* $\widetilde{\boldsymbol{\Omega}}_{\mathbf{A}}\, \mathcal{C}$ for a family of sets $\mathcal{C} \subseteq \mathcal{P}(A)$ is defined by $\widetilde{\boldsymbol{\Omega}}_{\mathbf{A}}\, \mathcal{C} := \bigcap_{X \in \mathcal{C}} \boldsymbol{\Omega}_{\mathbf{A}}\, X$. It is easy to see that Tarski congruence also commutes with inverse surjective substitutions.

Let $\mathcal{S}$ be a 1st-level deductive system and $\mathbf{\Omega}$ be the associated Leibnitz operator. Then $\mathbf{\Omega}$ is: *monotone* if for all $T, S \in \mathrm{Th}\,\mathcal{S}$, $T \subseteq S$ implies $\mathbf{\Omega}\,T \subseteq \mathbf{\Omega}\,S$; *invariant*, if for every substitution σ and every $T \in \mathrm{Th}\,\mathcal{S}$, $\sigma^{-1}(\mathbf{\Omega}\,T) = \mathbf{\Omega}(\sigma^{-1}T)$; *continuous*, if for every upward-directed family $\mathcal{C} \subseteq \mathrm{Th}\,\mathcal{S}$, $\mathbf{\Omega}\left(\bigcup \mathcal{C}\right) = \bigcup\{\mathbf{\Omega}\,\mathcal{A} \mid \mathcal{A} \in \mathcal{C}\}$.

The following algebraizability hierarchy in terms of the properties of the associated Leibnitz operator seems has been accepted now as standard in algebraic logic [17, 9].

Let $\mathcal{S}$ be a deductive system and $\mathbf{\Omega}$ be the associated Leibnitz operator. Then $\mathcal{S}$ is

protoalgebraic,	if $\mathbf{\Omega}$ is monotone;	(PA1)
weakly algebraizable,	if $\mathbf{\Omega}$ is injective;	(WA1)
equivalential,	if $\mathbf{\Omega}$ is invariant;	(EQ1)
finitely equivalential,	if $\mathbf{\Omega}$ is invariant and continuous;	(FE1)
algebraizable,	if $\mathbf{\Omega}$ is invariant and injective;	(AL1)
finitely algebraizable,	if $\mathbf{\Omega}$ is invariant, injective and continuous.	(FA1)

There are syntactical criteria (originally definitions) for these conditions

1) [18, 4, 17] A deductive system $\mathcal{S}$ is protoalgebraic iff there is a finite set $\Delta(x, y)$ of formulas of two variables, called a *protoequivalence system*, such that

$$1.\ \vdash_{\mathcal{S}} \Delta(x, x); \quad 2.\ x, \Delta(x, y) \vdash_{\mathcal{S}} y.$$

2) [21, 7, 8, 17] A deductive system $\mathcal{S}$ is (finitely) equivalential iff there is a (finite) set $\Delta(x, y)$ of formulas of two variables, called an *equivalence system*, such that

$$1.\ \vdash_{\mathcal{S}} \Delta(x, x); \quad 2.\ x, \Delta(x, y) \vdash_{\mathcal{S}} y; \quad 3.\ \Delta(x, y) \vdash_{\mathcal{S}} \Delta(tx, ty), \text{ for every } t \in \mathcal{T}_{\mathcal{L}}.$$

3) [10, 17] A deductive system $\mathcal{S}$ is weakly algebraizable iff there is a protoequivalence system Δ and a system of equations $E(x) \subseteq \mathrm{Fm}^2_{\mathcal{L}}\{x\}$ such that

$$x \dashv\vdash_{\mathcal{S}} \bigcup\{\Delta(t\alpha(x), t\beta(x)) \mid t \in \mathcal{T}_{\mathcal{L}}, \alpha \triangleright \beta \in E\}.$$

4) [5, 17] A deductive system $\mathcal{S}$ is (finitely) algebraizable iff there is a (finite) equivalence system Δ and a system of equations $E(x) \subseteq \mathrm{Fm}^2_{\mathcal{L}}\{x\}$ such that

$$x \dashv\vdash_{\mathcal{S}} \bigcup\{\Delta(\alpha(x), \beta(x)) \mid \alpha \triangleright \beta \in E\}.$$

Suppose A and B is pair of non-empty sets. Every binary relation $R \subseteq A \times B$ induces a *Galois connection between A and B*, consisting of two mapping, called *polarities*,

$$r : \mathcal{P}(A) \to \mathcal{P}(B), \qquad s : \mathcal{P}(B) \to \mathcal{P}(A)$$

such that for every $X \subseteq A$ and $Y \subseteq B$

$$rX := \{y \in B \mid (\forall x \in X)\, xRy\}, \qquad sY := \{x \in A \mid (\forall y \in Y)\, xRy\}.$$

We list for future reference the well-known facts about Galois connections (general references for Galois connections are [2, pp.124-126], [11, pp.232-233], [15, pp.68-69])

Lemma 1.0.2. *For all $X, U \in \mathcal{P}(A)$, $Y, V \in \mathcal{P}(B)$*

1. $Y \subseteq rX \iff X \subseteq sY$;
2. $X \subseteq srX$, $Y \subseteq rsY$;
3. $X \subseteq U \implies rU \subseteq rX$, $Y \subseteq V \implies sV \subseteq sY$;
4. $sY = \bigcap_{b \in Y} s\{b\}$, $rX = \bigcap_{a \in X} r\{a\}$;
5. $s = srs$, $r = rsr$;
6. *the compositions sr and rs are closure operators on $\mathcal{P}(A)$ and $\mathcal{P}(B)$ respectively;*
7. *the images $r\mathcal{P}(A)$ of r are the fixed points of rs, so $Y = rX \iff Y = rsY$;*
8. *the images $s\mathcal{P}(B)$ of s are the fixed points of sr, so $X = sY \iff X = srX$;*
9. *the sets of fixed points are closure systems on A and B respectively; the associated complete lattices are dually isomorphic through the mappings r and s.*

2. SECOND LEVEL SYNTACTIC SYSTEMS

In this chapter we introduce a notion of a 2nd-level syntactic system, basic for this dissertation. The obtained elementary results have nevertheless rather technical and abstract form to suit further applications in Chapters 3,4 and 5, therefore the details may be skipped during the first reading.

We consider three different kinds of 2nd-level systems that can be called "deductive" in the sense of Tarski. It is important to understand the reasoning and motivation behind them.

1) The most general kind of all three is introduced in this chapter and called *syntactic systems*. For these systems a reasonable notion of associated consequence relation can be introduced. They have adequate axiomatizations by Gentzen rules, but this axiomatizations in general are not equivalent.

2) Syntactic systems that have equivalent axiomatization by Gentzen rules are called *Gentzen systems.*

3) The 2nd-level *deductive systems* are syntactic systems the theories of which admit a reasonable notion of the Leibnitz congruence, therefore allowing the Leibnitz operator and associated with former the algebraization theory. The name "deductive" is somewhat arbitrary and was chosen because the algebraic hierarchy for so-defined 2nd-level deductive systems demonstrates close parallelism with that of 1st-level deductive system.

2.1 Closure relations

Definition 2.1.1. For a set $A \neq \emptyset$ define the *compatibility relation* $\mathrel{|\!\!\sim}_A \subseteq \mathcal{P}(A) \times A^+$ as

$$X \mathrel{|\!\!\sim}_A \bar{a} \rhd a \quad \text{iff} \quad \{\bar{a}\} \subseteq X \Longrightarrow a \in X. \qquad \square$$

The binary relation $\mathrel{|\!\sim}_A$ induces a Galois connection consisting of two mappings

$$\mathbf{R}_A : \mathcal{P}(\mathcal{P}(A)) \to \mathcal{P}(A^+), \qquad \mathbf{S}_A : \mathcal{P}(A^+) \to \mathcal{P}(\mathcal{P}(A)),$$

where $\mathbf{R}$ stands for "closure Relation" and $\mathbf{S}$ for "closure System". The motivation for this terminology will become clear later. By Lemma 1.0.2, for all $\mathcal{C} \subseteq \mathcal{P}(A)$ and all $\mathcal{A} \subseteq A^+$,

$$\mathbf{S}_A\,\mathcal{A} = \{X \subseteq A \mid (\forall s \in \mathcal{A})\ X \mathrel{|\!\sim}_A s\},$$
$$\mathbf{R}_A\,\mathcal{C} = \{s \in A^+ \mid (\forall X \in \mathcal{C})\ X \mathrel{|\!\sim}_A s\}.$$

We will use as synonyms, $\mathbf{R} = \mathbf{R}_{\mathcal{L}} := \mathbf{R}_{\mathrm{Fm}_{\mathcal{L}}}$ and $\mathbf{S} = \mathbf{S}_{\mathcal{L}} := \mathbf{S}_{\mathrm{Fm}_{\mathcal{L}}}$.

Lemma 2.1.2. *If $h : A \to B$, then*

1. $X \mathrel{|\!\sim}_B hs \iff h^{-1}X \mathrel{|\!\sim}_A s$;
2. $h^{-1}\mathbf{S}_B h\mathcal{A} \subseteq \mathbf{S}_A\mathcal{A}$, *for every* $\mathcal{A} \subseteq A^+$;
3. $h\mathbf{R}_A h^{-1}\mathcal{C} \subseteq \mathbf{R}_B\mathcal{C}$, *for every* $\mathcal{C} \subseteq \mathcal{P}(B)$;

Proof. 1. Let $X \subseteq B$ and $\bar{a} \triangleright a \in A^+$. The statement follows from the implications:

$$(\Rightarrow)\quad \{\bar{a}\} \subseteq h^{-1}X \implies \{h\bar{a}\} \subseteq X \overset{2.1.1}{\implies} ha \in X \implies a \in h^{-1}X,$$
$$(\Leftarrow)\quad \{h\bar{a}\} \subseteq X \implies \{\bar{a}\} \subseteq h^{-1}X \overset{2.1.1}{\implies} a \in h^{-1}X \implies ha \in X.$$

2. For a singleton $\{s\} \subseteq A^+$

$$h^{-1}\mathbf{S}_B h\{s\} \overset{def}{=} h^{-1}\{X \subseteq B \mid X \mathrel{|\!\sim}_B hs\} = \{h^{-1}X \mid X \subseteq B, X \mathrel{|\!\sim}_B hs\}$$
$$\overset{1.}{=} \{h^{-1}X \subseteq B \mid h^{-1}X \mathrel{|\!\sim}_A s\} \subseteq \{Y \subseteq A \mid Y \mathrel{|\!\sim}_A s\} \overset{def}{=} \mathbf{S}_A\{s\}. \qquad (*)$$

Therefore, for every $\mathcal{A} \subseteq A^+$,

$$h^{-1}\mathbf{S}_B h\mathcal{A} = h^{-1}\textstyle\bigcap_{s\in\mathcal{A}} \mathbf{S}_B h\{s\} = \bigcap_{s\in\mathcal{A}} h^{-1}\mathbf{S}_B h\{s\} \overset{(*)}{\subseteq} \bigcap_{s\in\mathcal{A}} \mathbf{S}_A\{s\} = \mathbf{S}_A\mathcal{A}.$$

3. For an one-set family $\{X\} \subseteq \mathcal{P}(B)$

$$h\mathbf{R}_A h^{-1}\{X\} \overset{def}{=} h\{s \in A^+ \mid h^{-1}X \mathrel{|\!\sim}_A s\} = \{hs \mid s \in A^+, h^{-1}X \mathrel{|\!\sim}_A s\}$$
$$\overset{1.}{=} \{hs \mid X \mathrel{|\!\sim}_B hs\} \subseteq \{s \in B^+ \mid X \mathrel{|\!\sim}_B s\} \overset{def}{=} \mathbf{R}_B\{X\}. \qquad (**)$$

Therefore, for every $\mathcal{C} \subseteq \mathcal{P}(B)$,

$$h\mathbf{R}_A h^{-1}\mathcal{C} = h(\textstyle\bigcap_{X\in\mathcal{C}} \mathbf{R}_A h^{-1}\{X\}) \subseteq \bigcap_{X\in\mathcal{C}} h\mathbf{R}_A h^{-1}\{X\} \overset{(**)}{\subseteq} \bigcap_{X\in\mathcal{C}} \mathbf{R}_B\{X\} = \mathbf{R}_B\mathcal{C}. \qquad \square$$

The equivalence $X \mathrel{|\!\sim}_B hs \iff h^{-1}X \mathrel{|\!\sim}_A s$ can be seen as a generalization of properties of non-conditional, "assertive", logical propositional statements onto the class of "implicational" statements. To make it more clear, consider the following example. Let $\mathcal{S}$ be a 1st-level deductive system with theorems. Suppose $\alpha \in \mathrm{Thm}\,\mathcal{S}$. Then $\alpha \in T$, for every theory $T \in \mathrm{Th}\,\mathcal{S}$. In particular, for every substitution σ and every theory $T \in \mathrm{Th}\,\mathcal{S}$, $\mathrm{Thm}\,\mathcal{S} \subseteq \sigma^{-1}T$, since $\mathrm{Th}\,\mathcal{S}$ is closed under inverse substitutions, therefore

$$\alpha \in \sigma^{-1}T \iff \sigma^{-1}T \mathrel{|\!\!\sim}_{\mathrm{Fm}_{\mathcal{L}}} \rhd\alpha \iff T \mathrel{|\!\!\sim}_{\mathrm{Fm}_{\mathcal{L}}} \rhd\sigma\alpha \iff \sigma\alpha \in T \iff \sigma\alpha \in \mathrm{Thm}\,\mathcal{S}.$$

Lemma 2.1.3. *If $h : A \to A$, then for every $\mathcal{A} \subseteq A^+$ and every $\mathcal{C} \subseteq \mathcal{P}(A)$*

1. $h\mathcal{A} \subseteq \mathcal{A} \implies h^{-1}\mathbf{S}_A\mathcal{A} \subseteq \mathbf{S}_A\mathcal{A}$;
2. $h^{-1}\mathcal{C} \subseteq \mathcal{C} \implies h\mathbf{R}_A\mathcal{C} \subseteq \mathbf{R}_A\mathcal{C}$;
3. $h^{-1}\mathbf{S}_A\mathcal{A} \subseteq \mathbf{S}_A\mathcal{A} \iff h\mathbf{R}_A\mathbf{S}_A\mathcal{A} \subseteq \mathbf{R}_A\mathbf{S}_A\mathcal{A}$;
4. $h\mathbf{R}_A\mathcal{C} \subseteq \mathbf{R}_A\mathcal{C} \iff h^{-1}\mathbf{S}_A\mathbf{R}_A\mathcal{C} \subseteq \mathbf{S}_A\mathbf{R}_A\mathcal{C}$.

Proof.

1. $h\mathcal{A} \subseteq \mathcal{A} \implies \mathbf{S}_A\mathcal{A} \subseteq \mathbf{S}_A h\mathcal{A} \implies h^{-1}\mathbf{S}_A\mathcal{A} \subseteq h^{-1}\mathbf{S}_A h\mathcal{A} \subseteq \mathbf{S}_A\mathcal{A}$
2. $h^{-1}\mathcal{C} \subseteq \mathcal{C} \implies \mathbf{R}_A\mathcal{C} \subseteq \mathbf{R}_A h^{-1}\mathcal{A} \implies h\mathbf{R}_A\mathcal{C} \subseteq h\mathbf{R}_A h^{-1}\mathcal{A} \subseteq \mathbf{R}_A\mathcal{A}$
3. $(\Rightarrow)$ $h^{-1}\mathbf{S}_A\mathcal{A} \subseteq \mathbf{S}_A\mathcal{A} \overset{\mathit{2.}}{\implies} h\mathbf{R}_A\mathbf{S}_A\mathcal{A} \subseteq \mathbf{R}_A\mathbf{S}_A\mathcal{A}$;

   $(\Leftarrow)$ $h\mathbf{R}_A\mathbf{S}_A\mathcal{A} \subseteq \mathbf{R}_A\mathbf{S}_A\mathcal{A} \overset{\mathit{1.}}{\implies} h^{-1}\mathbf{S}_A\mathbf{R}_A\mathbf{S}_A\mathcal{A} \subseteq \mathbf{S}_A\mathbf{R}_A\mathbf{S}_A\mathcal{A} \overset{1.0.2(5)}{\implies} h^{-1}\mathbf{S}_A\mathcal{A} \subseteq \mathbf{S}_A\mathcal{A}$;
4. $(\Rightarrow)$ $h\mathbf{S}_A\mathcal{A} \subseteq \mathbf{S}_A\mathcal{A} \overset{\mathit{1.}}{\implies} h^{-1}\mathbf{S}_A\mathbf{R}_A\mathcal{C} \subseteq \mathbf{S}_A\mathbf{R}_A\mathcal{C}$;

   $(\Leftarrow)$ $h^{-1}\mathbf{S}_A\mathbf{R}_A\mathcal{C} \subseteq \mathbf{S}_A\mathbf{R}_A\mathcal{C} \overset{\mathit{2.}}{\implies} h\mathbf{R}_A\mathbf{S}_A\mathbf{R}_A\mathcal{C} \subseteq \mathbf{R}_A\mathbf{S}_A\mathbf{R}_A\mathcal{C} \overset{1.0.2(5)}{\implies} h\mathbf{R}_A\mathcal{C} \subseteq \mathbf{S}_A\mathcal{C}$. □

Lemma 2.1.4. *If $h : A \twoheadrightarrow B$, then $\mathbf{R}_A\, h^{-1}\mathcal{C} = h^{-1}\,\mathbf{R}_B\,\mathcal{C}$, for every $\mathcal{C} \subseteq \mathcal{P}(B)$.*

Proof. As before it suffices to prove the statement for $\mathcal{C} = \{X\}$.

$$\begin{aligned} \mathbf{R}_A\, h^{-1}\{X\} &= \{s \in A^+ \mid h^{-1}X \mathrel{|\!\!\sim}_A s\} \overset{2.1.2(1)}{=} \{s \in A^+ \mid X \mathrel{|\!\!\sim}_A hs\} \\ &= h^{-1}\{s \in B^+ \mid X \mathrel{|\!\!\sim}_A s\} = h^{-1}\,\mathbf{R}_B\{X\}. \end{aligned}$$ □

Definition 2.1.5. The fixed points of the operator $\mathbf{R}_A \circ \mathbf{S}_A : \mathcal{P}(A^+) \to \mathcal{P}(A^+)$ are called *finite closure relations over A*. In other words

$\mathcal{A} \subseteq A^+$ is a *finite closure relation over A* iff $\mathcal{A} = \mathbf{R}_A\mathbf{S}_A\mathcal{A}$.

Closure relations were introduced in [14] as a framework for studying metatheoretical properties of 1st-level deductive systems.

Lemma 2.1.6. *If* $\mathcal{A} \subseteq A^+$*, then* $\mathbf{S}_A\mathcal{A}$ *is an algebraic closure system on* A*.*

Proof. We have to prove that $\mathbf{S}_A\mathcal{A}$:

1) contains A; 2) is closed under non-empty intersections; 3) algebraic.

First we prove these properties for an arbitrary singleton $\{\bar{a} \rhd a\} \subseteq A^+$.

1) Obviously A is compatible with any $\bar{a} \rhd a \in A^+$.

2) Suppose $\{X_i\}_{i\in I} \subseteq \mathbf{S}_A\{\bar{a} \rhd a\}$ is non-empty family. Since

$$\{\bar{a}\} \subseteq \textstyle\bigcap_{i\in I} X_i \implies (\forall i \in I)\ \{\bar{a}\} \subseteq X_i \overset{2.1.1}{\implies} (\forall i \in I)\, a \in X_i \implies a \in \textstyle\bigcap_{i\in I} X_i,$$

therefore $\bigcap_{i\in I} X_i \in \mathbf{S}_A\{\bar{a} \rhd a\}$, and thus $\mathbf{S}_A\{\bar{a} \rhd a\}$ is a closure system on A.

3) Suppose $\mathcal{C} \subseteq \mathbf{S}_A\{\bar{a} \rhd a\}$ is a upward-directed family. If $\{\bar{a}\} \subseteq_\omega \bigcup \mathcal{C}$, then there exists $X \in \mathcal{C}$ such that $\{\bar{a}\} \subseteq X$. Since X is compatible with $\bar{a} \rhd a$, then $a \in X$, hence $a \in \bigcup \mathcal{C}$. Thus $\bigcup \mathcal{C} \in \mathbf{S}_A\{\bar{a} \rhd a\}$, therefore $\mathbf{S}_A\{\bar{a} \rhd a\}$ is algebraic.

Now let $\mathcal{A}$ be an arbitrary non-empty subset of A^+. By Lemma 1.0.2, $\mathbf{S}_A\,\mathcal{A} = \bigcap_{s\in\mathcal{A}} \mathbf{S}_A\{s\}$, therefore $\mathbf{S}_A\,\mathcal{A}$ is an algebraic closure system on A, since it is the intersection of algebraic closure systems on A. If $\mathcal{A} = \emptyset$, then $\mathbf{S}_A\,\mathcal{A} = \mathcal{P}(A)$; hence an algebraic closure system. ☐

Lemma 2.1.7. *Suppose* $\mathcal{C}$ *is a closure system on* A*. Then*

1. $\mathbf{R}_A\,\mathcal{C} = \{\bar{a} \rhd a \in A^+ \mid a \in \{\bar{a}\}^{\mathcal{C}}\}$,
2. $\{\bar{a}\}^{\mathcal{C}} = \{a \in A \mid \bar{a} \rhd a \in \mathbf{R}_A\,\mathcal{C}\}$.

Proof.

1. ($\subseteq$) $\bar{a} \rhd a \in \mathbf{R}_A\,\mathcal{C} \Longrightarrow (\forall X \in \mathcal{C})\, X \mathrel{\sim}_A \bar{a} \rhd a \implies \{\bar{a}\}^{\mathcal{C}} \mathrel{\sim}_A \bar{a} \rhd a \Longrightarrow a \in \{\bar{a}\}^{\mathcal{C}}$.

1. ($\supseteq$) Suppose $a \in \{\bar{a}\}^{\mathcal{C}}$. Then $\bar{a} \rhd a \in \mathbf{R}_A\,\mathcal{C}$, because, for every $X \in \mathcal{C}$,

$$\{\bar{a}\} \subseteq X \implies \{\bar{a}\}^{\mathcal{C}} \subseteq X \overset{2.1.1}{\implies} a \in \{\bar{a}\}^{\mathcal{C}} \subseteq X.$$

2. $\{\bar{a}\}^{\mathcal{C}} = \{a \in A \mid a \in \{\bar{a}\}^{\mathcal{C}}\} \overset{1.}{=} \{a \in A \mid \bar{a} \rhd a \in \mathbf{R}_A\,\mathcal{C}\}$. ☐

Now we can give a characterization of fixed points of the operator $\mathbf{S}_A \circ \mathbf{R}_A$.

Proposition 2.1.8 (D. Pigozzi). *Suppose* $\mathcal{C} \subseteq \mathcal{P}(A)$*. TFAE*

1. $\mathcal{C}$ *is an algebraic closure system on* A*;*

2. *$\mathcal{C}$ is a closure relation and* $\mathcal{C} = \{X \mid X = \bigcup_{\{\bar{a}\}\subseteq_\omega X}\{\bar{a}\}^{\mathcal{C}}\}$*;*

3. $\mathcal{C} = \mathbf{S}_A\mathbf{R}_A\mathcal{C}$.

Proof. ($1 \Rightarrow 2$) For every $X \in \mathcal{C}$, the equality $X = \bigcup_{\{\bar{a}\}\subseteq_\omega X}\{\bar{a}\}^{\mathcal{C}}$ follows from the inclusions:

$$(\subseteq) \quad \{\bar{a}\} \subseteq \{\bar{a}\}^{\mathcal{C}} \implies X = \bigcup\{\{\bar{a}\} \mid \{\bar{a}\} \subseteq_\omega X\} \subseteq \bigcup\{\{\bar{a}\}^{\mathcal{C}} \mid \{\bar{a}\} \subseteq_\omega X\},$$
$$(\supseteq) \quad \{\bar{a}\} \subseteq_\omega X \implies \{\bar{a}\}^{\mathcal{C}} \subseteq X^{\mathcal{C}} = X \implies \bigcup\{\{\bar{a}\}^{\mathcal{C}} \mid \{\bar{a}\} \subseteq_\omega X\} \subseteq X.$$

On the other hand, if $X = \bigcup\{\{\bar{a}\}^{\mathcal{C}} \mid \{\bar{a}\} \subseteq_\omega X\}$, then, since the family $\{\{\bar{a}\}^{\mathcal{C}} \mid \{\bar{a}\} \subseteq_\omega X\}$ is an upward-directed subfamily of $\mathcal{C}$, we get $X = \bigcup\{\{\bar{a}\}^{\mathcal{C}} \mid \{\bar{a}\} \subseteq_\omega X\} \in \mathcal{C}$, since $\mathcal{C}$ is algebraic.
($2 \Rightarrow 3$) By Lemma 1.0.2, $\mathcal{C} \subseteq \mathbf{S}_A\mathbf{R}_A\mathcal{C}$. So it suffices to show that $\mathbf{S}_A\mathbf{R}_A\mathcal{C} \subseteq \mathcal{C}$.

Suppose $X \in \mathbf{S}_A\mathbf{R}_A\mathcal{C}$ $(*)$ and let $\{\bar{a}\} \subseteq_\omega X$. Then $\{\bar{a}\}^{\mathcal{C}} \subseteq X$, because

$$a \in \{\bar{a}\}^{\mathcal{C}} \overset{2.1.7}{\iff} \bar{a} \rhd a \in \mathbf{R}_A\,\mathcal{C} \overset{(*)}{\iff} X \mathrel{|\!\!\sim}_A \bar{a} \rhd a \overset{2.1.1}{\implies} a \in X.$$

Thus $X = \bigcup_{\{\bar{a}\}\subseteq_\omega X}\{\bar{a}\}^{\mathcal{C}} \subseteq X$, therefore $X \in \mathcal{C}$, by assumption.
($3 \Leftarrow 1$) $\mathcal{C} = \mathbf{S}_A\mathbf{R}_A\mathcal{C}$ is an algebraic closure system on A by Lem. 2.1.6. □

2.2 2nd-level syntactic systems

Definition 2.2.1. We call a family $\mathcal{C} \subseteq \mathcal{P}(A)$ a *closure system in A* if

1) $\bigcup\mathcal{C} \in \mathcal{C}$; 2) $\bigcap\mathcal{D} \in \mathcal{C}$ for every non-empty subfamily $\mathcal{D} \subseteq \mathcal{C}$. □

Note the difference between a closure system *on* the set A, defined previously, and a closure system *in* the set A. If $\mathcal{C}$ is a closure system in A, then obviously $\mathcal{C}$ is a closure system on $\bigcup\mathcal{C}$. We call a closure system $\mathcal{C}$ in A *algebraic* if it is algebraic as a closure system on $\bigcup\mathcal{C}$. If $\mathcal{C}$ is a algebraic closure system in A, then it is straightforward to show that $\mathcal{D} = \mathcal{C} \cup \{A\}$ is an algebraic closure system on A. In particular, $A = \{a\}^{\mathcal{D}}$ for every $a \in A$ such that $a \notin \bigcup\mathcal{C}$.

Next is a rather technical lemma that will be useful for describing Gentzen axiomatizations of 2nd-level syntactic systems. By $X \rhd Y$ we understand the set $\{a \rhd b \mid a \in X, b \in Y\}$.

Lemma 2.2.2. *Suppose $\mathcal{C}$ is an algebraic closure system in A and let $B = \bigcup\mathcal{C}$. Then*

1. $\mathbf{R}_B\,\mathcal{C} = \mathbf{R}_A\,\mathcal{C} \cap B^+$*;*

2. $\mathcal{A} = \mathbf{R}_B\,\mathcal{C} \cup ((A\backslash B) \rhd A) \implies \mathbf{S}_A\mathcal{A} = \mathcal{C} \cup \{A\}$.

Proof. 1. For every $X \subseteq B$ and every $\bar{b} \triangleright b \in B^+$

$$\bar{b} \triangleright b \in \mathbf{R}_B\{X\} \iff (\{\bar{b}\} \subseteq X \implies b \in X) \iff b \triangleright b \in \mathbf{R}_A\{X\}.$$

Thus $\mathbf{R}_B\{X\} = \mathbf{R}_B\{X\} \cap B^+ = \mathbf{R}_A\{X\} \cap B^+$. Therefore

$$\mathbf{R}_B\,\mathcal{C} = \bigcap_{X\in\mathcal{C}} \mathbf{R}_B\{X\} = \bigcap_{X\in\mathcal{C}}(\mathbf{R}_A\{X\} \cap B^+) = (\bigcap_{X\in\mathcal{C}} \mathbf{R}_A\{X\}) \cap B^+ = \mathbf{R}_B\,\mathcal{C} \cap B^+.$$

2. $(\subseteq)$ Let $X \in \mathbf{S}_A\,\mathcal{A}$. $(*)$

a) Suppose $X \subseteq B$ and $\{\bar{b}\} \subseteq_\omega X$. Then $\{\bar{b}\}^{\mathcal{C}} \subseteq X$, because

$$b \in \{\bar{b}\}^{\mathcal{C}} \overset{2.1.7}{\Longrightarrow} \bar{b} \triangleright b \in \mathbf{R}_B\,\mathcal{C} \implies \bar{b} \triangleright b \in \mathcal{A} \overset{(*)}{\Longrightarrow} X \mathrel{|\!\!\sim}_A \bar{b} \triangleright b \overset{2.1.1}{\Longrightarrow} b \in X.$$

Thus $X = \bigcup\{\{\bar{a}\}^{\mathcal{C}} \mid \{\bar{a}\} \subseteq_\omega X\}$; hence $X \in \mathcal{C}$, since $\mathcal{C}$ is algebraic.

b) Suppose $X \not\subseteq B$ and let $a \in X \backslash B$. Then $X = A$, because

$$b \in A \implies a \triangleright b \in ((A\backslash B) \triangleright A) \subseteq \mathcal{A} \overset{(*)}{\Longrightarrow} X \mathrel{|\!\!\sim}_A a \triangleright b \overset{2.1.1}{\Longrightarrow} b \in X.$$

2. $(\supseteq)$ $\mathcal{A} = \mathbf{R}_B\mathcal{C} \cup ((A\backslash B) \triangleright A) \overset{\mathit{1.}}{\subseteq} \mathbf{R}_A\mathcal{C} \overset{1.0.2(3)}{\Longrightarrow} \mathbf{S}_A\mathbf{R}_A\mathcal{C} \subseteq \mathbf{S}_A\mathcal{A}$
$\overset{1.0.2(3)}{\Longrightarrow} \mathcal{C} \subseteq \mathbf{S}_A\mathbf{R}_A\mathcal{C} \subseteq \mathbf{S}_A\mathcal{A} \overset{2.1.6}{\Longrightarrow} \mathcal{C} \cup \{A\} \subseteq \mathbf{S}_A\,\mathcal{A}$. □

A *syntactic type* for a family $\mathcal{C} \subseteq \mathcal{P}(\mathrm{Fm}^+_{\mathcal{L}})$ is a subset $\mathbf{Type}\,\mathcal{C} \subseteq \omega$ defined as

$$\mathbf{Type}\,\mathcal{C} := \{|s| - 1 \mid s \in \bigcup\mathcal{C}\}.$$

We "downsize" the elements of $\bigcup\mathcal{C}$ by 1 for the compatibility with previous results (see [14]).

Definition 2.2.3. *A 2nd-level syntactic system* $\mathcal{R}$ is a pair $\langle \mathbf{Fm}_{\mathcal{L}}, \mathrm{Th}\,\mathcal{R}\rangle$ such that $\mathrm{Th}\,\mathcal{R} \subseteq \mathcal{P}(\mathrm{Fm}^+_{\mathcal{L}})$ is an algebraic invariant closure system in $\mathrm{Fm}^+_{\mathcal{L}}$. The elements of $\mathrm{Th}\,\mathcal{R}$ are called *theories of* $\mathcal{R}$ and $\mathrm{Thm}\,\mathcal{R} {:=} \bigcap \mathrm{Th}\,\mathcal{R}$ are *theorems of* $\mathcal{R}$. The syntactic type of $\mathcal{R}$ is $\mathbf{Type}\,\mathcal{R} := \mathbf{Type}\,\mathrm{Th}\,\mathcal{R}$. □

Suppose $\mathcal{R}$ is a 2nd-level syntactic system and let $B = \bigcup \mathrm{Th}\,\mathcal{R}$. Suppose some $\langle\bar{\alpha}\rangle \in \mathrm{Fm}^n_{\mathcal{L}} \subseteq \mathrm{Fm}^+_{\mathcal{L}}$ is contained in B. Since $\mathrm{Th}\,\mathcal{R}$ is invariant, $\sigma^{-1}B \subseteq B$ for every substitution σ. In particular, applying a substitution σ such that $\sigma\langle\bar{x}\rangle = \langle\bar{\alpha}\rangle$, we get that $\langle\bar{x}\rangle \in \sigma^{-1}\langle\bar{\alpha}\rangle \subseteq B$. Therefore any element $\langle\bar{\alpha}\rangle$ from any theory $\mathcal{A} \in \mathrm{Th}\,\mathcal{R}$ can be obtained as a substitution instance of an element of the form $\langle\bar{x}\rangle \in \mathrm{Var}^+$, that also belongs to some theory of $\mathcal{R}$ (and hence, what is really important for us, the closure $\{\langle\bar{x}\rangle\}^{\mathrm{Th}\,\mathcal{R}}$ is defined). Thus the collection

$\{\langle x_0, \ldots, x_{n-1}, x_n\rangle \mid n \in \mathbf{Type}\,\mathcal{R}\}$ can be seen as representing all possible "syntactical" forms of elements from $\bigcup \mathrm{Th}\,\mathcal{R}$. This reasoning arguably justifies the use of qualification "syntactic" in the definition of 2nd-level syntactic system.

The definition of a 2nd-level syntactic system encompasses and brings in a single context a broad family of systems that can be considered "deductive". For instance:

- Let $\mathcal{S} = \langle \mathbf{Fm}_{\mathcal{L}}, \mathrm{Th}\,\mathcal{S}\rangle$ be a 1st-level syntactic system. Define for each theory $T \in \mathrm{Th}\,\mathcal{S}$, $\rhd T := \{\rhd\alpha \mid \alpha \in T\}$. Then it is easy to check that $\mathcal{R} = \langle \mathbf{Fm}_{\mathcal{L}}, \{\rhd T\}_{T\in\mathrm{Th}\,\mathcal{S}}\rangle$ is an algebraic invariant closure system on $\rhd\mathrm{Fm}_{\mathcal{L}}$; hence $\mathcal{R}$ is a 2nd-level syntactic system with $\mathbf{Type}\,\mathcal{R} = \{0\}$. Another possible 2nd-level counterpart for $\mathcal{S}$ will be defined later in this chapter (see Definition 2.2.9).
- Consider, $\mathrm{Con}\,\mathbf{Fm}_{\mathcal{L}} \subseteq \mathcal{P}(\mathrm{Fm}^{+}_{\mathcal{L}})$. It is easy to see that $\mathcal{R}_1 = \langle \mathbf{Fm}_{\mathcal{L}}, \mathrm{Con}\,\mathbf{Fm}_{\mathcal{L}}\rangle$ is a 2nd-level syntactic system, with $\bigcup \mathrm{Con}\,\mathbf{Fm}_{\mathcal{L}} = 1_{\mathbf{Fm}_{\mathcal{L}}} = \mathrm{Fm}^2_{\mathcal{L}}$ a proper subset of $\mathrm{Fm}^{+}_{\mathcal{L}}$. Similarly, $\mathcal{R}_2 = \langle \mathbf{Fm}_{\mathcal{L}}, \mathrm{Eq}\,\mathrm{Fm}_{\mathcal{L}}\rangle$ is a 2nd-level syntactic system, where $\bigcup \mathrm{Eq}\,\mathrm{Fm}_{\mathcal{L}}$ is again $\mathrm{Fm}^2_{\mathcal{L}}$. Both $\mathbf{Type}\,\mathcal{R}_1 = \mathbf{Type}\,\mathcal{R}_2 = \{1\}$.
- Let $\mathcal{R}(k) = \langle \mathbf{Fm}_{\mathcal{L}}, \mathcal{P}(\mathrm{Fm}^k_{\mathcal{L}})\rangle$. Then $\mathcal{R}(k)$ is the weakest out of so called k-deductive systems over $\mathcal{L}$. The syntactic type of $\mathcal{R}(k)$ is $\{k-1\}$.
- Here is more non-trivial example: suppose $\mathcal{L}$ is a language type such that $\& \in \mathcal{L}_2$, so $\&$ is a binary connective. Let $B = \{\alpha\&\beta \mid \alpha, \beta \in \mathrm{Fm}_{\mathcal{L}}\} \cup \mathrm{Var}$. Then $\mathcal{R} = \langle \mathbf{Fm}_{\mathcal{L}}, \mathcal{P}(B^{+})\rangle$ is a 2nd-level syntactic system with $\mathbf{Type}\,\mathcal{R} = \omega$.
- Suppose $\mathcal{L}' \subseteq \mathcal{L}$, then $\mathcal{R} = \langle \mathbf{Fm}_{\mathcal{L}}, \mathcal{P}(\mathrm{Fm}^{+}_{\mathcal{L}'})\rangle$ is a 2nd-level syntactic system with $\mathbf{Type}\,\mathcal{R} = \omega$.

From the examples it follows that in general the syntactic type of a 2nd-level syntactic system only loosely defines the form of elements of $\mathcal{R}$.

Let $\mathcal{R} = \langle \mathbf{Fm}_{\mathcal{L}}, \mathrm{Th}\,\mathcal{R}\rangle$ be a 2nd-level syntactic system, and $X \subseteq \bigcup \mathrm{Th}\,\mathcal{R}$. Then

$$X^{\mathcal{R}} := X^{\mathrm{Th}\,\mathcal{R}} = \bigcap\{\mathcal{A} \in \mathrm{Th}\,\mathcal{R} \mid X \subseteq \mathcal{A}\}$$

is the closure of X in $\mathrm{Th}\,\mathcal{R}$. Note that the closure is not defined for all subsets of $\mathrm{Fm}^{+}_{\mathcal{L}}$.

The following lemma is a straightforward generalization of [10, Lemma 2.3].

Lemma 2.2.4. *Suppose $\mathcal{H}$ is an algebraic closure system in $\mathrm{Fm}^+_{\mathcal{L}}$. Then*

1. *if $\mathcal{H}$ is closed under inverse surjective substitutions, then $\mathcal{H} \cup \{\mathrm{Fm}^+_{\mathcal{L}}\}$ is invariant.*
2. *if $\mathbf{Type}\,\mathcal{H} \neq \omega$ or $\mathrm{Fm}^+_{\mathcal{L}} \in \mathcal{H}$, then*

$$\mathcal{H} \textit{ is invariant iff } \mathcal{H} \textit{ is closed under inverse surjective substitutions.}$$

Proof. 1) Suppose $\mathcal{H}$ is closed under inverse surjective substitutions. By Lem. 2.1.3(2), $\mathbf{R}_{\mathcal{L}}\mathcal{H}$ is closed under surjective substitutions. Fix a substitution σ. For every $s \in \mathrm{Fm}^+_{\mathcal{L}}$, there is a surjective substitution δ_s such that $\sigma s = \sigma_s s$. Therefore

$$\sigma \mathbf{R}_{\mathcal{L}}\mathcal{H} = \textstyle\bigcup_{s \in \mathbf{R}_{\mathcal{L}}\mathcal{H}} \sigma s = \bigcup_{s \in \mathbf{R}_{\mathcal{L}}\mathcal{H}} \sigma_s s \subseteq \bigcup_{s \in \mathbf{R}_{\mathcal{L}}\mathcal{H}} \sigma_s \mathbf{R}_{\mathcal{L}}\mathcal{H} \subseteq \mathbf{R}_{\mathcal{L}}\mathcal{H}.$$

Thus $\mathbf{R}_{\mathcal{L}}\mathcal{H}$ is closed under arbitrary substitutions, hence, by Lem. 2.1.3(1), $\mathbf{S}_{\mathcal{L}}\mathbf{R}_{\mathcal{L}}\mathcal{H} = \mathcal{H} \cup \{\mathrm{Fm}^+_{\mathcal{L}}\}$ is closed under arbitrary substitutions.

2) Suppose $\mathcal{H}$ is closed under inverse surjective substitutions. Then, by 1), $\mathcal{H} \cup \{\mathrm{Fm}^+_{\mathcal{L}}\}$ is closed under arbitrary substitutions. If $\{\mathrm{Fm}^+_{\mathcal{L}}\} \in \mathcal{H}$, then $\mathcal{H} = \mathcal{H} \cup \{\mathrm{Fm}^+_{\mathcal{L}}\}$, and therefore it is closed under arbitrary substitutions. If $\mathbf{Type}\,\mathcal{H} \neq \omega$, then there is no substitution σ such that $\mathrm{Fm}^+_{\mathcal{L}} = \sigma^{-1}\mathcal{A}$ for any $\mathcal{A} \in \mathcal{H}$. Therefore $\sigma\mathcal{H} \subseteq \mathcal{H}$, for every substitution σ.

The other direction is trivial: if $\mathcal{H}$ is closed under arbitrary inverse substitutions, then it is closed under inverse surjective substitutions. □

Definition 2.2.5. A 2nd-level syntactic system $\mathcal{R}$ is called *a Gentzen system*, whenever $\mathrm{Th}\,\mathcal{R}$ is a closure system *on* $\mathrm{Fm}^+_{\mathcal{L}}$.

We take a Cantor-style approach towards Gentzen rules: we view a rule not as a "rule"—description of an action, but as a list of all applications.

A *2nd-level sequent* is a string $\bar{s} \rhd s$ of sequents $\{\bar{s}, s\} \subseteq \mathrm{Fm}^+_{\mathcal{L}}$. A *Gentzen (2nd-level) rule* $\bar{s} \vdash s$ is a set of all substitution instances of the 2nd-level sequent $\bar{s} \rhd s$, i.e.,

$$\bar{s} \vdash s := \{\sigma(\bar{s} \rhd s) \mid \sigma : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{Fm}_{\mathcal{L}}\}.$$

This Gentzen rule is often also written as $\dfrac{s_0, \ldots, s_{n-1}}{s}$.

Definition 2.2.6. Let x, y, z be variables. *Standard rules* are the rules of the form

$$\begin{array}{lll}
\text{(Ax)} & \vdash \Gamma, x, \Sigma \rhd x & \text{Axioms of the 2nd-level} \\
\text{(Ex)} & \Gamma, x, y, \Sigma \rhd z \vdash \Gamma, y, x, \Sigma \rhd z & \text{Exchange} \\
\text{(W)} & \Gamma, \Sigma \rhd y \vdash \Gamma, x, \Sigma \rhd y & \text{Weakening} \\
\text{(Con)} & \Gamma, x, x, \Sigma \rhd y \vdash \Gamma, x, \Sigma \rhd y & \text{Contraction} \\
\text{(Cut)} & \Gamma, x, \Sigma \rhd y;\ \Theta \rhd x \vdash \Gamma, \Theta, \Sigma \rhd y & \text{Cut}
\end{array}$$

where Γ, Σ, Θ range over the set of finite, possibly empty, sequences of variables of $\mathrm{Fm}_{\mathcal{L}}$.

So, for instance,

$$(\mathrm{Ex}) = \Gamma, x, y, \Sigma \rhd z \vdash \Gamma, y, x, \Sigma \rhd z = \bigcup_{\langle \bar{u} \rangle, \langle \bar{v} \rangle \in \mathrm{Var}^*} \bar{u}, x, y, \bar{v} \rhd z \vdash \bar{u}, y, x, \bar{v} \rhd z.$$

We denote the collection of standard rules by (CR), i.e.,

$$(\mathrm{CR}) := (\mathrm{Cut}) \cup (\mathrm{Con}) \cup (\mathrm{Ax}) \cup (\mathrm{W}) \cup (\mathrm{Ex}).$$

A subset $\mathcal{A} \subseteq \mathrm{Fm}^{+}_{\mathcal{L}}$ is *invariant* if $\sigma \mathcal{A} \subseteq \mathcal{A}$ for every substitution σ. Note that Gentzen rules are invariant subsets of $\mathrm{Fm}^{+}_{\mathcal{L}}$.

Let $G = \{R_i\}_{i \in I}$ be a set of Gentzen rules. We define $\mathrm{Th}\, G := \mathbf{S}_{\mathcal{L}} \left(\bigcup_{i \in I} R_i \right)$.

Corollary 2.2.7.

If G is a set of Gentzen rules, then $\mathrm{Th}\, G$ *is a Gentzen system.*

Proof. By definition, $\mathrm{Th}\, G = \mathbf{S}_{\mathcal{L}}(\bigcup_{R \in G} R)$. Thus, by Lemma 2.1.6, $\mathrm{Th}\, G$ is an algebraic closure system on $\mathrm{Fm}_{\mathcal{L}}$. Also, since $\bigcup_{R \in G} R$ is an invariant set of sequents, it closed under substitutions. Therefore, by Lemma 2.1.3(1), $\mathrm{Th}\, G$ is closed under inverse substitutions. $\square$

If, for a 2nd-level syntactic system $\mathcal{G}$, $\mathrm{Th}\, \mathcal{G} = \mathrm{Th}\, G$ for some set G of Gentzen rules, then G is called an *Gentzen axiomatization* for $\mathcal{G}$. Note that every Gentzen system has an equivalent Gentzen axiomatization, as such can be chosen for example $\mathbf{R}_{\mathcal{L}}\, \mathrm{Th}\, \mathcal{G}$.

In general, there is no equivalent Gentzen axiomatization for an arbitrary 2nd-level syntactic system $\mathcal{R}$ due to the fact that $\mathrm{Fm}^{+}_{\mathcal{L}}$ may not belong to $\mathrm{Th}\, \mathcal{R}$, while $\mathrm{Fm}^{+}_{\mathcal{L}}$ is a theory of any Gentzen system. But, from Lem. 2.2.2 it follows that there is the smallest Gentzen system $\hat{\mathcal{R}}$, namely $\hat{\mathcal{R}} := \langle \mathbf{Fm}_{\mathcal{L}}, \mathrm{Th}\, \mathcal{R} \cup \{\mathrm{Fm}^{+}_{\mathcal{L}}\} \rangle$, that contains $\mathcal{R}$ and has a simple and natural interpretation. “hat” in $\hat{\mathcal{R}}$ is intended to indicate that $\hat{\mathcal{R}}$ is obtained from $\mathcal{R}$ by attaching “to the top” the largest possible Gentzen theory.

Similar to the case of 1st-level deductive systems, for every $X, Y \subseteq \bigcup \mathrm{Th}\,\mathcal{R}$ and every substitution $\sigma : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{Fm}_{\mathcal{L}}$ extended to $\mathrm{Fm}^{+}_{\mathcal{L}}$ holds

Proposition 2.2.8. *Let $\mathcal{R}$ be a 2nd-level syntactic system. Then*

1. $X \vdash_{\mathcal{R}} Y \iff X \vdash_{\hat{\mathcal{R}}} Y$;
2. $Y \subseteq X \implies X \vdash_{\mathcal{R}} Y$;
3. $X \vdash_{\mathcal{R}} Y \implies \sigma X \vdash_{\mathcal{R}} \sigma Y$;
4. $\mathcal{A} \vdash_{\mathcal{R}} X \implies X \subseteq \mathcal{A}$ *for every* $\mathcal{A} \in \mathrm{Th}\,\mathcal{R}$.

Proof. The proof is trivial. □

Some examples of Gentzen systems arise naturally in connection with 1st-level deductive systems.

Definition 2.2.9. Let $\mathcal{S}$ be a 1st-level deductive system. If $\mathcal{C} \subseteq \mathrm{Th}\,\mathcal{S}$ is an algebraic closure system on $\mathrm{Fm}_{\mathcal{L}}$, then $\mathbf{R}_A\,\mathcal{C}$ is called *a general finite closure relation for* $\mathcal{S}$ or simply *a general closure relation for* $\mathcal{S}$. The set of all general closure relations for $\mathcal{S}$ will be denoted by $\mathbf{Gcr}\,\mathcal{S}$.

For every 1st-level deductive system $\mathcal{S}$ of type there is a distinguished general closure relation $\mathbf{R}_{\mathcal{L}}\mathrm{Th}\,\mathcal{S}$, which in its turn defines a Gentzen axiomatization for some 2nd-level deductive system:

$$\vdash \mathbf{R}_{\mathcal{L}}\mathrm{Th}\,\mathcal{S} := \bigcup\{\vdash \bar{\alpha} \rhd \alpha \mid \bar{\alpha} \rhd \alpha \in \mathbf{R}_{\mathcal{L}}\mathrm{Th}\,\mathcal{S}\}.$$

Proposition 2.2.10. *For any 1st-level deductive system $\mathcal{S}$ of language type $\mathcal{L}$*

1. $\mathbf{R}_{\mathcal{L}}\mathrm{Th}\,\mathcal{S} = \{\bar{\alpha} \rhd \alpha \mid \bar{\alpha} \vdash_{\mathcal{S}} \alpha\}$,
2. $\mathbf{R}_{\mathcal{L}}\mathrm{Th}\,\mathcal{S}$ *is invariant,*
3. $\mathbf{Gcr}\,\mathcal{S} = \mathrm{Th}\,((\mathrm{CR}) \cup \vdash \mathbf{R}_{\mathcal{L}}\mathrm{Th}\,\mathcal{S})$,
4. $\mathbf{Gcr}\,\mathcal{S}$ *is a 2nd-level deductive system on* $\mathbf{Fm}_{\mathcal{L}}$,
5. $\mathbf{R}_{\mathcal{L}}\mathrm{Th}\,\mathcal{S} = \mathrm{Thm}\,(\mathbf{Gcr}\,\mathcal{S})$.

Proof.

(1) $\mathbf{R}_{\mathcal{L}}\mathrm{Th}\,\mathcal{S} \overset{def}{=} \{\bar{\alpha} \triangleright \alpha \mid \alpha \in \{\bar{\alpha}\}^{\mathrm{Th}\,\mathcal{S}}\} \overset{def}{=} \{\bar{\alpha} \triangleright \alpha \mid \bar{\alpha} \vdash_{\mathcal{S}} \alpha\}$.

(2) For any substitution $\sigma : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{Fm}_{\mathcal{L}}$

$$\bar{\alpha} \triangleright \alpha \in \mathbf{R}_{\mathcal{L}}\mathrm{Th}\,\mathcal{S} \overset{(1)}{\Longrightarrow} \bar{\alpha} \vdash_{\mathcal{S}} \alpha \implies \sigma\bar{\alpha} \vdash_{\mathcal{S}} \sigma\alpha \overset{(1)}{\Longrightarrow} \sigma\bar{\alpha} \triangleright \sigma\alpha \in \mathbf{R}_{\mathcal{L}}\mathrm{Th}\,\mathcal{S}.$$

(3) Suppose $\mathcal{A} \in \mathbf{Gcr}\,\mathcal{S}$. By definition of $\mathbf{Gcr}\,\mathcal{S}$, $\mathcal{A} = \mathbf{R}_{\mathcal{L}}\,\mathcal{C}$ for some closure system $\mathcal{C} \subseteq \mathrm{Th}\,\mathcal{S}$, hence $\mathbf{R}_{\mathcal{L}}\mathrm{Th}\,\mathcal{S} \subseteq \mathbf{R}_{\mathcal{L}}\,\mathcal{C}$. Therefore for any rule $\vdash \bar{\alpha} \triangleright \alpha$ and any substitution σ

$$\vdash \bar{\alpha} \triangleright \alpha \subseteq \vdash \mathbf{R}_{\mathcal{L}}\mathrm{Th}\,\mathcal{S} \overset{def}{\Longrightarrow} \bar{\alpha} \triangleright \alpha \in \mathbf{R}_{\mathcal{L}}\mathrm{Th}\,\mathcal{S} \overset{(2)}{\Longrightarrow} \bar{\sigma}\alpha \triangleright \sigma\alpha \in \mathbf{R}_{\mathcal{L}}\mathrm{Th}\,\mathcal{S} \subseteq \mathbf{R}_{\mathcal{L}}\,\mathcal{C}.$$

Now let $\mathcal{A} \in \mathrm{Th}\,((\mathrm{CR}) \cup \vdash \mathbf{R}_{\mathcal{L}}\mathrm{Th}\,\mathcal{S})$. Then $\mathcal{A}$ is a finite closure relation over $\mathrm{Fm}_{\mathcal{L}}^{+}$, since it is a theory of (CR). By Lemma 2.1.6, $\mathcal{A} = \mathbf{R}_{\mathcal{L}}\,\mathcal{C}$ for some algebraic closure system $\mathcal{C} \subseteq \mathcal{P}(\mathrm{Fm}_{\mathcal{L}})$. Since $\mathcal{A} \in \mathrm{Th}\,(\vdash \mathbf{R}_{\mathcal{L}}\mathrm{Th}\,\mathcal{S})$, then, for every $X \in \mathcal{C}$ and every $\bar{\alpha} \triangleright \alpha \in \mathbf{R}_{\mathcal{L}}\mathrm{Th}\,\mathcal{S} \subseteq \mathcal{A}$.

$$\{\bar{\alpha}\}^{\mathcal{C}} \subseteq X \implies \alpha \in \{\bar{\alpha}\}^{\mathcal{C}} \subseteq X^{\mathcal{C}} = X \implies X \in \mathrm{Th}\,\mathcal{S}.$$

Thus $\mathcal{C} \subseteq \mathrm{Th}\,\mathcal{S}$, therefore $\mathcal{A} = \mathbf{R}_{\mathcal{L}}\,\mathcal{C} \in \mathbf{Gcr}\,\mathcal{S}$.

(4) It follows directly from Proposition 2.2.7 and (3).

(5) By (3), any $\mathcal{A} \in \mathbf{Gcr}\,\mathcal{S}$ is closed under the rules of $\vdash \mathbf{R}_{\mathcal{L}}\mathrm{Th}\,\mathcal{S}$. Thus, for every $\mathcal{A} \in \mathbf{Gcr}\,\mathcal{S}$, we have $\mathbf{R}_{\mathcal{L}}\mathrm{Th}\,\mathcal{S} \subseteq \mathcal{A}$ and hence $\mathbf{R}_{\mathcal{L}}\mathrm{Th}\,\mathcal{S} \subseteq \mathrm{Thm}\,\mathbf{Gcr}\,\mathcal{S}$. On the other hand, since $\mathbf{R}_{\mathcal{L}}\mathrm{Th}\,\mathcal{S} \in \mathbf{Gcr}\,\mathcal{S}$, then $\mathrm{Thm}\,\mathbf{Gcr}\,\mathcal{S} \subseteq \mathbf{R}_{\mathcal{L}}\mathrm{Th}\,\mathcal{S}$. □

2.3 Matrix semantics for 2nd-level syntactic systems

A pair $\mathfrak{A} = \langle \mathbf{A}, \mathcal{A} \rangle$ is a *2nd-level matrix* if $\mathcal{A} \subseteq A^{+}$.

Definition 2.3.1. Let $\mathfrak{A} = \langle \mathbf{A}, \mathcal{A} \rangle$, $\mathfrak{B} = \langle \mathbf{B}, \mathcal{B} \rangle$ be 2nd-level matrices. We write and say that

$\mathfrak{A} \subseteq \mathfrak{B}$, if $\mathbf{A} \leqslant \mathbf{B}$ and $\mathcal{A} \subseteq \mathcal{B}$ — $\mathfrak{A}$ is a *weak submatrix* of $\mathfrak{B}$;

$\mathfrak{A} \leqslant \mathfrak{B}$, if there is $h : \mathbf{A} \rightarrowtail \mathbf{B}$ such that $\mathcal{A} = h^{-1}\mathcal{B}$ — $\mathfrak{A}$ is a *submatrix* of $\mathfrak{B}$;

$\mathfrak{A} \preccurlyeq \mathfrak{B}$, if there is $h : \mathbf{A} \twoheadrightarrow \mathbf{B}$ such that $\mathcal{A} = h^{-1}\mathcal{A}$ — $\mathfrak{B}$ is a *strong submatrix* of $\mathfrak{A}$.

Let $\mathcal{R} = \langle \mathbf{Fm}_{\mathcal{L}}, \mathrm{Th}\,\mathcal{R} \rangle$ be a 2nd-level syntactic system. We say that a 2nd-level matrix $\mathfrak{A} = \langle \mathbf{A}, \mathcal{A} \rangle$ of type $\mathcal{L}$ is a *model of* $\mathcal{R}$ if for every evaluation $h : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$, $h^{-1}\mathcal{A} \in \mathrm{Th}\,\mathcal{R}$. The class of all models for $\mathcal{R}$ is denoted by $\mathbf{Mod}\,\mathcal{R}$.

The proof of the following proposition is standard.

Proposition 2.3.2. *For every 2nd-level syntactic system* $\mathcal{R}$

$$\mathrm{Th}\,\mathcal{R} = \{\mathcal{A} \subseteq \mathrm{Fm}^{+}_{\mathcal{L}} \mid \langle \mathbf{Fm}_{\mathcal{L}}, \mathcal{A}\rangle \in \mathbf{Mod}\,\mathcal{R}\}.$$

Proof. ($\subseteq$) Suppose $\mathcal{A}$ is a theory of $\mathcal{R}$ and let $h : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{Fm}_{\mathcal{L}}$ be an evaluation, that obviously is also a substitution. Then $h^{-1}\mathcal{A} \in \mathrm{Th}\,\mathcal{R}$, since $\mathrm{Th}\,\mathcal{R}$ is closed under inverse substitutions. Therefore $\langle \mathbf{Fm}_{\mathcal{L}}, \mathcal{A}\rangle \in \mathbf{Mod}\,\mathcal{R}$.

($\supseteq$) Suppose $\mathfrak{A} = \langle \mathbf{Fm}_{\mathcal{L}}, \mathcal{A}\rangle \in \mathbf{Mod}\,\mathcal{R}$ and let $i : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{Fm}_{\mathcal{L}}$ be the identity homomorphism. Then, since $\mathfrak{A}$ is a model of $\mathcal{R}$, $\mathcal{A} = i^{-1}\mathcal{A} \in \mathbf{Mod}\,\mathcal{R}$. □

The following three lemmas were first proven in [14]. We modify the proof of two of them.

Lemma 2.3.3. *[14, Lemma 2.7]*

Let $\mathfrak{A}$ *and* $\mathfrak{B}$ *be 2nd-level matrices and* $\mathcal{R}$ *be a 2nd-level syntactic system. Then*

1. $(\mathfrak{A} \leqslant \mathfrak{B}) \implies (\mathfrak{B} \in \mathbf{Mod}\,\mathcal{R} \implies \mathfrak{A} \in \mathbf{Mod}\,\mathcal{R})$,
2. $(\mathfrak{A} \preccurlyeq \mathfrak{B}) \implies (\mathfrak{B} \in \mathbf{Mod}\,\mathcal{R} \iff \mathfrak{A} \in \mathbf{Mod}\,\mathcal{R})$.

Proof. Suppose $\mathfrak{A} \leqslant \mathfrak{B}$ and let $h : \mathbf{A} \to \mathbf{B}$ be such that $\mathcal{A} = h^{-1}\mathcal{B}$. Let $\mathfrak{B} \in \mathbf{Mod}\,\mathcal{R}$. Then $\mathfrak{A} \in \mathbf{Mod}\,\mathcal{R}$, because for every evaluation $g : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$, since $hg : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{B}$, we have that

$$g^{-1}\mathcal{A} = g^{-1}(h^{-1}\mathcal{B}) = (hg)^{-1}\mathcal{B} \in \mathrm{Th}\,\mathcal{R}.$$

Then 1) and 2) ($\Leftarrow$) follows.

2) ($\Rightarrow$) Suppose $\mathfrak{A} \preccurlyeq \mathfrak{B}$ and let $h : \mathbf{A} \twoheadrightarrow \mathbf{B}$ be such that $\mathcal{A} = h^{-1}\mathcal{B}$. Fix an evaluation $g : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{B}$. Since h is onto, then $\{h^{-1}gx \mid x \in \mathrm{Var}\}$ is a collection of non-empty sets. Choose a function $f : \mathrm{Var} \to \mathbf{A}$ such that $fx \in h^{-1}gx$. Since $\mathbf{Fm}_{\mathcal{L}}$ is absolutely free, f can be extended to a homomorphism $f : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$. Then $hf = g$, because, for every $\alpha(\bar{x}) \in \mathrm{Fm}_{\mathcal{L}}$,

$$hf\alpha(\bar{x}) = \alpha(hf\bar{x}) = \alpha(g\bar{x}) = g\alpha(\bar{x}).$$

Therefore $g^{-1}\mathcal{B} = (hf)^{-1}\mathcal{B} = f^{-1}(h^{-1}\mathcal{B}) = f^{-1}\mathcal{A} \in \mathbf{Mod}\,\mathcal{R}$. □

Lemma 2.3.4. *[14, Lemma 2.8] Let* $\mathbf{K}$ *be any upward-directed by* $\subseteq$ *family of 2nd-level matrices such that, for each* $\mathfrak{B} \in \mathbf{K}$*, there exists* $\mathfrak{A} \in \mathbf{K}$ *such that* $\mathfrak{B} \subseteq \mathfrak{A}$ *and* $\mathfrak{A} \in \mathbf{Mod}\,\mathcal{G}$*. Then* $\bigcup \mathbf{K} \in \mathbf{Mod}\,\mathcal{G}$*.*

Definition 2.3.5. Let $\mathcal{A} \subseteq A^+$. Then $\mathcal{A}$ is

1. *reflexive*, if $\bar{a} \rhd a \in \mathcal{A}$, whenever $a \in \{\bar{a}\}$;
2. *transitive*, if $a_0, \ldots, a_{k-1} \rhd a \in \mathcal{A}$ and $b_0, \ldots, b_{l-1} \rhd a_i \in \mathcal{A}$ for some $i \in k$, then $a_0, \ldots, a_{i-1}, b_0, \ldots, b_{l-1}, a_i, \ldots, a_{k-1} \rhd a \in \mathcal{A}$.
3. *regular*, if $\mathcal{A}$ is reflexive and transitive.
4. *standard*, if $\bar{a} \rhd a \in \mathcal{A}$, then $b_0, \ldots, b_{l-1} \rhd a \in \mathcal{A}$, whenever $\{b_0, \ldots, b_{l-1}\} \subseteq \{\bar{a}\}$.

Note that any regular set is standard.

Finally we can give alternative characterizations for finite closure relations.

Lemma 2.3.6. *Suppose* $\mathbf{A}$ *is an algebra and* $\mathcal{A} \subseteq A^+$*. Then TFAE*

1. *$\mathcal{A}$ is finite closure relation;*
2. *$\mathcal{A}$ is a standard subset of $\mathcal{A}^+$;*
3. *$\langle \mathbf{A}, \mathcal{A} \rangle$ is a model of the Gentzen system* (CR)*.*

Proof. The proof is straightforward. $\square$

3. ALGEBRAIZABILITY OF 2ND-LEVEL DEDUCTIVE SYSTEMS

In this chapter we develop a technique for defining an algebraic hierarchy for a special subclass of 2nd-level syntactic systems. This hierarchy closely parallels the algebraic hierarchy of 1st-level deductive systems. This striking similarity prompts a name for the subclass: 2nd-level deductive systems. However the term turns out to be somewhat arbitrary, since it relates not to deductive aspects (the closest counterpart in that case would be 2nd-level syntactic systems), but rather to semantical ones, pertaining to algebraic semantics, to be precise.

The algebraic hierarchy for 2nd-level deductive systems (as well as in the 1st-level case) is based on properties of 2nd-level Leibnitz operator, like invariancy, meet-continuity and continuity. Several important examples of 2nd-level algebraizable deductive system were considered in [12], a characterization of equivalential and finitely equivalential 2nd-level deductive systems was given in [20]. The method of this chapter is based more on an operator-style approach similar to that in [16, 17].

3.1 Leibnitz operator for 2nd-level deductive systems

Definition 3.1.1. We call a 2nd-level syntactic system a *2nd-level deductive system* if

1. every theory of $\mathcal{R}$ is a regular subset of $\mathrm{Fm}_{\mathcal{L}}^{+}$, (see Def. 2.3.5)
2. the largest theory $\bigcup \mathrm{Th}\,\mathcal{R}$ is closed under substitutions.

Note that, as a direct consequence of the second condition, $1_{\mathbf{Fm}_{\mathcal{L}}} \subseteq \mathrm{Fm}_{\mathcal{L}}^{2} \subseteq \bigcup \mathrm{Th}\,\mathcal{R}$. This allows us to take closures of symmetric sets "inside" $\mathrm{Th}\,\mathcal{R}$.

In the context of 2nd-level deductive systems, the notion of the 2nd-level Leibnitz operator has a particular simple formulation. It relates to the fact that for each regular subset $X \subseteq A^{+}$ there is the largest congruence $\theta \in \mathrm{Con}\,\mathbf{Fm}_{\mathcal{L}}$ such that $\theta \subseteq X$. This congruence we call the

Leibnitz congruence of X. We show that $\mathbf{\Omega_A} X$ is equal to the Tarski congruence of the closure system $\mathbf{S}_A X \subseteq \mathcal{P}(A)$. Later we provide a useful characterization for $\mathbf{\Omega_A} X$ through the action of $\mathcal{T}_{\mathbf{A}}$. Leibnitz congruences for theories of a 2nd-level deductive system define a 2nd-level Leibnitz operator for this system.

Definition 3.1.2. Let $\theta \in \operatorname{Con} \mathbf{A}$ and X be a regular subset of A^+. We say that θ is *compatible* with X if for every $a_0, \dots, a_{n-1} \rhd a \in X$,

$$a_0 \theta b_0, \dots, a_{n-1} \theta b_{n-1}, a \theta b \implies b_0, \dots, b_{n-1} \rhd b \in X.$$

A simple but very useful characterization for compatibility of congruences is given by

Lemma 3.1.3. *Let $\theta \in \operatorname{Con} \mathbf{A}$ and X be a regular subset of A^+. Then*

$$\theta \textit{ is compatible with } X \textit{ iff } \theta \subseteq X.$$

Proof. ($\Rightarrow$) Suppose θ is compatible with X, and let $a \rhd b \in \theta$. Since X is reflexive, $a \rhd a \in X$. Also, $a \rhd a, a \rhd b \in \theta$, therefore, by definition of compatibility, $a \rhd b \in X$.

($\Leftarrow$) Let $\theta \subseteq X$ and suppose $a_0, \dots, a_{n-1} \rhd a \in X$ and $a_0 \theta b_0, \dots, a_{n-1} \theta b_{n-1}, a \theta b$. Then, since $\theta \subseteq X$, we have $b_0 \rhd a_0, \dots, b_{n-1} \rhd a_{n-1}, a \rhd b \in X$. Therefore, by transitivity of X, from $b_0 \rhd a_0, \dots, b_{n-1} \rhd a_{n-1} \in X$ and $a_0, \dots, a_{n-1} \rhd a \in X$ it follows that $b_0, \dots, b_{n-1} \rhd a \in X$ and together with $a \rhd b \in X$, that $b_0, \dots, b_{n-1} \rhd b \in X$. □

Lemma 3.1.4.

Let $\mathbf{A}$ be an algebra and X be a regular subset of A^+. Then the set $\{\theta \in \operatorname{Con} \mathbf{A} \mid \theta \subseteq X\}$

1) is not empty, 2) is upward-directed, 3) has the largest element.

Proof. 1. Since X is reflexive, $0_{\mathbf{A}} \subseteq X$.

2. Suppose $\theta, \eta \subseteq X$. Then all relations $\theta \circ \eta, \theta \circ \eta \circ \theta, \theta \circ \eta \circ \theta \circ \eta, \dots$ contain in X, because X is transitive. Therefore $\theta \vee \eta = \theta \circ \eta \cup \theta \circ \eta \circ \theta \cup \theta \circ \eta \circ \theta \circ \eta \cup \cdots \subseteq X$.

3. Since $\{\theta \in \operatorname{Con} \mathbf{A} \mid \theta \subseteq X\}$ is upward-directed, $\bigcup \{\theta \in \operatorname{Con} \mathbf{A} \mid \theta \subseteq X\}$ is a congruence and obviously is the largest congruence in this set. □

In view of Lemma 3.1.4 the following definition is correct

Definition 3.1.5. Let $\mathbf{A}$ be an algebra. For a regular subset X of A^+ let

$$\boldsymbol{\Omega}_{\mathbf{A}} X := \bigcup\{\theta \in \operatorname{Con}\mathbf{A} \mid \theta \subseteq X\}.$$

We call $\boldsymbol{\Omega}_{\mathbf{A}} X$ the *(2nd-level) Leibnitz congruence of* X. □

We will provide a syntactical characterization for $\boldsymbol{\Omega}_{\mathbf{A}} X$, but first we need some technical results about the operator $\mathcal{T}_{\mathbf{A}} : \mathcal{P}(A) \to \mathcal{P}(A)$.

Lemma 3.1.6. *For every algebra* $\mathbf{A}$, $\mathcal{T}_{\mathbf{A}}\mathcal{T}_{\mathbf{A}} \subseteq \mathcal{T}_{\mathbf{A}}$.

Proof. We have to prove that $\mathcal{T}_{\mathbf{A}}\mathcal{T}_{\mathbf{A}} = \{s \circ t \mid s, t \in \mathcal{T}_{\mathbf{A}}\} \subseteq \mathcal{T}_{\mathbf{A}}$.

Suppose $s = \lambda\,\xi.\beta(\xi, \bar{y}) \in \mathcal{T}_{\mathbf{A}}$, $t = \lambda\,\xi.\alpha(\xi, \bar{x}) \in \mathcal{T}_{\mathbf{A}}$. Then for any $\phi \in \mathrm{Fm}_{\mathcal{L}}$

$$(st)\phi = \lambda\,\xi.\beta(\xi, \bar{y})(\lambda\,\xi.\alpha(\xi, \bar{x})\phi) = \lambda\,\xi.\beta(\xi, \bar{y})\alpha(\phi, \bar{x}) = \beta(\alpha(\phi, \bar{x}), \bar{y}) = \lambda\,\xi.\beta(\alpha(\xi, \bar{x}), \bar{y})\phi$$

where obviously $\lambda\,\xi.\beta(\alpha(\xi, \bar{x}), \bar{y}) \in \mathcal{T}_{\mathbf{A}}$. □

Every substitution σ induces a mapping $\sigma : \mathcal{T}_{\mathcal{L}} \to \mathcal{T}_{\mathcal{L}}$ given by

$$\sigma(\lambda\,\xi.\alpha(\xi, \bar{x})) = \lambda\,\xi.\alpha(\xi, \sigma\bar{x}).$$

Lemma 3.1.7. *Let* $t \in \mathcal{T}_{\mathcal{L}}$, $\phi \in \mathrm{Fm}_{\mathcal{L}}$ *and* $\sigma : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{Fm}_{\mathcal{L}}$. *Then*

1. $\sigma(t\phi) = (\sigma t)(\sigma\phi)$,
2. *if* σ *is surjective, then* $\sigma\mathcal{T}_{\mathcal{L}} = \mathcal{T}_{\mathcal{L}}$.

Proof. Let $t = \lambda\,\xi.\alpha(\xi, \bar{x})$ be an element of $\mathcal{T}_{\mathcal{L}}$.

1. $\sigma(t\phi) = \sigma(\alpha(\phi, \bar{x})) = \alpha(\sigma\phi, \sigma\bar{x}) = (\sigma t)(\sigma\phi)$.
2. Since σ is surjective, there is a sequence $\langle\bar{u}\rangle \in \sigma^{-1}\{\langle\bar{x}\rangle\}$. Therefore

$$\sigma(\lambda\,\xi.\alpha(\xi, \bar{u})) \overset{def}{=} \lambda\,\xi.\alpha(\xi, \sigma\bar{u}) = \lambda\,\xi.\alpha(\xi, \bar{x}) = t. \qquad \square$$

A simple but useful syntactic characterization of $\boldsymbol{\Omega}_{\mathbf{A}}\mathcal{A}$ was given in

Lemma 3.1.8. *[20, Basic Definability Lemma]*

Suppose $\mathcal{A}$ *is a regular subset of* A^+. *Then for every* $X \in \operatorname{Sym} A$

$$X \subseteq \boldsymbol{\Omega}_{\mathbf{A}}\mathcal{A} \iff \mathcal{T}_{\mathbf{A}} X \subseteq \mathcal{A}.$$

Proof. Consistently applying our notational conventions

$$\mathcal{T}_{\mathbf{A}}\{a \lhd\!\rhd b\} = \bigcup_{t\in\mathcal{T}_{\mathbf{A}}} t\{a \rhd b, b \rhd a\} = \bigcup_{t\in\mathcal{T}_{\mathbf{A}}} \{t(a \rhd b), t(b \rhd a)\} = \bigcup_{t\in\mathcal{T}_{\mathbf{A}}} \{ta \rhd tb, tb \rhd ta\}.$$

Let $\eta = \{a \rhd b \mid \mathcal{T}_{\mathbf{A}}\{a \lhd\!\rhd b\} \subseteq \mathcal{A}\}$. We have

(1) $\lambda\xi.\,\xi \in \mathcal{T}_{\mathbf{A}} \overset{\text{(Ax)}}{\Longrightarrow} \eta \subseteq \mathcal{A}$;

(2) $\mathcal{T}_{\mathbf{A}}\{a \rhd a\} \subseteq \mathcal{A} \implies a \rhd a \in \eta$;

(3) $a \rhd b \in \eta \iff \mathcal{T}_{\mathbf{A}}\{a \lhd\!\rhd b\} = \mathcal{T}_{\mathbf{A}}\{b \lhd\!\rhd a\} \subseteq \mathcal{A} \iff b \rhd a \in \eta$;

(4) $a \rhd b, b \rhd c \in \eta \implies (\forall t \in \mathcal{T}_{\mathbf{A}})\ ta \rhd tb, tb \rhd tc \in \mathcal{A},\ tc \rhd tb, tb \rhd ta \in \mathcal{A}$
$\overset{\text{(Cut)}}{\Longrightarrow} (\forall t \in \mathcal{T}_{\mathbf{A}})\ ta \rhd tc, tc \rhd ta \in \mathcal{A} \implies \mathcal{T}_{\mathbf{A}}\{a \lhd\!\rhd c\} \subseteq \mathcal{A} \implies a \rhd c \in \eta$;

(5) $a \rhd b \in \eta \implies \mathcal{T}_{\mathbf{A}}\{a \lhd\!\rhd b\} \subseteq \mathcal{A} \implies (\forall t \in \mathcal{T}_{\mathbf{A}})\ \mathcal{T}_{\mathbf{A}} t\{a \lhd\!\rhd b\} \subseteq \mathcal{T}_{\mathbf{A}}\{a \lhd\!\rhd b\} \subseteq \mathcal{A}$
$\implies (\forall t \in \mathcal{T}_{\mathbf{A}})\ ta \rhd tb \in \eta$;

(6) $\theta \subseteq \mathcal{A} \implies \mathcal{T}_{\mathbf{A}}\theta = \theta \subseteq \mathcal{A} \implies \theta \subseteq \eta$.

From (2)–(4) it follows that η is an equivalence relation, (5) means that η is also a congruence, from (1) it follows that $\eta \subseteq \mathcal{A}$, and (6) proves that η is maximum out of all congruences with such property. □

Although the Leibnitz congruence $\boldsymbol{\Omega}_{\mathbf{A}}$ of a subset of A (1st-level) and the Leibnitz congruence of a regular subset of A^{+} (2nd-level) bear the same name and are denoted by the same symbol, it is important to distinguish them because they are defined for radically different objects, and therefore possess inherently different properties. However, there is a connection:

Lemma 3.1.9. *For every algebra* $\mathbf{A}$ *and every closure system* $\mathcal{C}$ *in* A

$$\boldsymbol{\Omega}_{\mathbf{A}}\mathbf{R}_{A}\mathcal{C} = \widetilde{\boldsymbol{\Omega}}_{\mathbf{A}}\,\mathcal{C}.$$

Proof. It is easy to see that for every $B \subseteq A$: $\boldsymbol{\Omega}_{\mathbf{A}}B = \boldsymbol{\Omega}_{\mathbf{A}}\mathbf{R}_{A}\{B\}$. Therefore

$$\widetilde{\boldsymbol{\Omega}}_{\mathbf{A}}\,\mathcal{C} \overset{def}{=} \bigcap_{X\in\mathcal{C}} \boldsymbol{\Omega}_{\mathbf{A}}X \overset{(1)}{=} \bigcap_{X\in\mathcal{C}} \boldsymbol{\Omega}_{\mathbf{A}}\mathbf{R}_{A}\{X\} = \boldsymbol{\Omega}_{\mathbf{A}}\Big(\bigcap_{X\in\mathcal{C}} \mathbf{R}_{A}\{X\}\Big) = \boldsymbol{\Omega}_{\mathbf{A}}\mathbf{R}_{A}\mathcal{C}.$$ □

Definition 3.1.10. Suppose $\mathcal{R}$ is a 2nd-level deductive system. The associated Leibnitz operator $\boldsymbol{\Omega} : \operatorname{Th}\mathcal{R} \to \operatorname{Con}\mathbf{Fm}_{\mathcal{L}}$ is

- *invariant*, if for every substitution σ and every $\mathcal{A} \in \operatorname{Th}\mathcal{R}$, $\sigma^{-1}\,\boldsymbol{\Omega}\,\mathcal{A} = \boldsymbol{\Omega}\,\sigma^{-1}\mathcal{A}$,

- *continuous*, if for every upward-directed family $\mathcal{C} \subseteq \mathrm{Th}\,\mathcal{R}$

$$\boldsymbol{\Omega}\,(\bigcup\mathcal{C}) = \bigcup\{\boldsymbol{\Omega}\,\mathcal{A} \mid \mathcal{A} \in \mathcal{C}\}. \qquad \square$$

Definition 3.1.11. Let $\mathcal{R}$ be a 2nd-level deductive system and $\boldsymbol{\Omega} : \mathrm{Th}\,\mathcal{G} \to \mathrm{Con}\,\mathbf{Fm}_{\mathcal{L}}$ be the associated Leibnitz operator. Then $\mathcal{R}$ is

weakly algebraizable,	if $\boldsymbol{\Omega}$ is injective;	(WA2)
equivalential,	if $\boldsymbol{\Omega}$ is invariant;	(EQ2)
finitely equivalential,	if $\boldsymbol{\Omega}$ is invariant and continuous;	(FE2)
algebraizable,	if $\boldsymbol{\Omega}$ is invariant and injective;	(AL2)
finitely algebraizable,	if $\boldsymbol{\Omega}$ is invariant, injective and continuous.	(FA2)

Note almost perfect textual coincidence of this definition with that for 1st-level systems. The only difference is due to fact that a 2nd-level Leibnitz operator is always meet-continuous, therefore all 2nd-level deductive systems are trivially protoalgebraic.

Suppose $\mathcal{R}$ is a 2nd-level deductive system or a protoalgebraic 1st-level deductive system. Then $\boldsymbol{\Omega} : \mathrm{Th}\,\mathcal{R} \to \mathrm{Con}\,\mathbf{Fm}_{\mathcal{L}}$ is meet-continuous, therefore $\boldsymbol{\Omega}\,\mathrm{Th}\,\mathcal{R} := \{\boldsymbol{\Omega}\,\mathcal{A} \mid \mathcal{A} \in \mathrm{Th}\,\mathcal{R}\} \subseteq \mathrm{Con}\,\mathbf{Fm}_{\mathcal{L}}$ is a closure system. Let $\vDash_{\mathcal{R}}$ denote the consequence relation associated with the closure system $\boldsymbol{\Omega}\,\mathrm{Th}\,\mathcal{R}$, thus, for any $X, Y \subseteq \mathrm{Fm}^2_{\mathcal{L}}$,

$$X \vDash_{\mathcal{R}} Y \iff Y \subseteq \bigcap\{\boldsymbol{\Omega}\,\mathcal{A} \mid X \subseteq \boldsymbol{\Omega}\,\mathcal{A},\ \mathcal{A} \in \mathrm{Th}\,\mathcal{R}\}.$$

To avoid bulky constructions in the superscript we will use notation $(\,.\,)^{\vDash_{\mathcal{R}}}$ for the associated closure operator, so $X^{\vDash_{\mathcal{R}}} := \bigcap\{\boldsymbol{\Omega}\,\mathcal{A} \mid X \subseteq \boldsymbol{\Omega}\,\mathcal{A},\ \mathcal{A} \in \mathrm{Th}\,\mathcal{R}\}$. In particular

$$X \vDash_{\mathcal{R}} Y \iff Y \subseteq X^{\vDash_{\mathcal{R}}}.$$

According to the definition, $\vDash_{\mathcal{R}}$ is neither finitary nor structural, in general. This does not contradict the idea of a semantical consequence. However, when $\mathcal{R}$ is equivalential, $\vDash_{\mathcal{R}}$ becomes structural, and the associated consequence relation is called an *implicational system*. If $\mathcal{R}$ is finitely equivalential, then $\boldsymbol{\Omega}\,\mathrm{Th}\,\mathcal{R}$ becomes a deductive system by itself, defining the quasiequational theory of the quasivariety equivalent for $\mathcal{R}$.

3.2 Weakly algebraizable 2nd-level deductive systems

A weakly algebraizable deductive system $\mathcal{R}$ is characterized by the fact that the Leibnitz operator $\mathbf{\Omega} : \operatorname{Th}\mathcal{R} \to \operatorname{Con}\mathbf{Fm}_{\mathcal{L}}$ is injective, or, equivalently, there is an 1-1 correspondence between theories of $\mathcal{R}$ and the set $\mathbf{\Omega}\operatorname{Th}\mathcal{R}$ of Leibnitz congruences for $\mathcal{R}$. It alone is enough to obtain a characterization through the existence of a graded congruence basis for $\mathcal{R}$. We start with a technical lemma.

Lemma 3.2.1. *Suppose* $\mathcal{H} \subseteq \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}^{+})$ *is a closure system in* $\mathrm{Fm}_{\mathcal{L}}^{+}$ *of regular sets such that* $\mathbf{\Omega} : \mathcal{H} \to \operatorname{Con}\mathbf{Fm}_{\mathcal{L}}$ *is injective. Then, for every* $\mathcal{A} \in \mathcal{H}$*,*

1. $\mathbf{\Omega}\,(\mathbf{\Omega}\,\mathcal{A})^{\mathcal{H}} = \mathbf{\Omega}\,\mathcal{A}$*,*
2. $\mathcal{A} = (\mathbf{\Omega}\,\mathcal{A})^{\mathcal{H}}$*.*

Proof. (1) Let $\mathcal{A} \in \mathcal{H}$. Then

$$(\subseteq) \quad \mathbf{\Omega}\,\mathcal{A} \overset{def}{\subseteq} \mathcal{A} \implies (\mathbf{\Omega}\,\mathcal{A})^{\mathcal{H}} \subseteq \mathcal{A}^{\mathcal{H}} = \mathcal{A} \implies \mathbf{\Omega}\,(\mathbf{\Omega}\,\mathcal{A})^{\mathcal{H}} \subseteq \mathbf{\Omega}\,\mathcal{A},$$

$$(\supseteq) \quad \mathbf{\Omega}\,\mathcal{A} \subseteq (\mathbf{\Omega}\,\mathcal{A})^{\mathcal{H}} \implies \mathbf{\Omega}\,\mathcal{A} \subseteq \mathbf{\Omega}\,(\mathbf{\Omega}\,\mathcal{A})^{\mathcal{H}}.$$

(2) From (1), since $\mathbf{\Omega}$ is injective, it follows that $\mathcal{A} = (\mathbf{\Omega}\,\mathcal{A})^{\mathcal{H}}$. □

Suppose $N \subseteq \omega$ and $\mathcal{E} = \bigcup_{n\in N} \varepsilon_n$ is a set such that for each $n \in N$

$$\varepsilon_n(\bar{x}, x) = \{\alpha_i(\bar{x}, x) \triangleleft\!\triangleright \beta_i(\bar{x}, x)\}_{i\in I} \subseteq_{\omega} \mathrm{Fm}_{\mathcal{L}}^{2}, \quad \langle \bar{x} \rangle = \langle x_0, \ldots, x_{n-1} \rangle$$

is a finite symmetric set of ordered pairs of formulas depending on a set $\{\bar{x}, x\}$ of distinct variables. Let $\mathcal{H} \subseteq \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}^{+})$ be a closure system in $\mathrm{Fm}_{\mathcal{L}}^{+}$ such that all $\mathcal{A} \in \mathcal{H}$ are regular sets.

Definition 3.2.2. $\mathcal{E}$ is *a graded congruence basis for* $\mathcal{H}$ if $N = \mathbf{Type}\,\mathcal{H}$ and for all $\mathcal{A} \in \mathcal{H}$

$$\bar{\alpha} \triangleright \alpha \in \mathcal{A} \iff \varepsilon_{|\bar{\alpha}|}(\bar{\alpha}, \alpha) \subseteq \mathbf{\Omega}\,\mathcal{A}. \qquad \square$$

A graded congruence basis $\mathcal{E}$ for $\mathcal{H}$ defines an operator $\mathcal{E} : \mathcal{H} \to \operatorname{Sym}\mathrm{Fm}_{\mathcal{L}}$ by

$$\mathcal{E}\mathcal{A} = \bigcup\{\varepsilon_{|\bar{\alpha}|}(\bar{\alpha}, \alpha) \mid \bar{\alpha} \triangleright \alpha \in \mathcal{A}\}, \text{ for every } \mathcal{A} \in \mathcal{H}.$$

We will see that the existence of a graded congruence basis is equivalent to a 2nd-level deductive system to be weakly algebraizable.

Theorem 3.2.3. *Let $\mathcal{R}$ be a 2nd-level deductive system. Then*

$\mathcal{R}$ is weakly algebraizable iff it has a graded congruence basis.

Proof. ($\Rightarrow$). Fix $n \in \mathbf{Type}\,\mathcal{R}$, and let $\bar{x} \rhd x$, $|\bar{x}| = n$, be a vector of distinct variables. By Lemma 3.2.1(2),

$$\{\bar{x} \rhd x\}^{\mathcal{R}} = (\mathbf{\Omega}\{\bar{x} \rhd x\}^{\mathcal{R}})^{\mathcal{R}}.$$

Since $\mathrm{Th}\,\mathcal{G}$ is algebraic, there exists a finite set $X \subseteq_\omega \mathbf{\Omega}\{\bar{x} \rhd x\}^{\mathcal{R}}$, which can also be chosen symmetric, such that $\{\bar{x} \rhd x\}^{\mathcal{R}} = X^{\mathcal{R}}$. Also

$$X \subseteq \mathbf{\Omega}\{\bar{x} \rhd x\}^{\mathcal{R}} \implies X \subseteq \mathcal{T}X \subseteq \mathbf{\Omega}\,\{\bar{x} \rhd x\}^{\mathcal{R}} \implies \{\bar{x} \rhd x\}^{\mathcal{R}} = X^{\mathcal{R}} \subseteq (\mathcal{T}X)^{\mathcal{R}} \subseteq \{\bar{x} \rhd x\}^{\mathcal{R}}.$$

Being finite, X depends on a finite set of variables $\mathrm{Var}\,X$. Therefore there exists a surjective substitution $\sigma : \mathrm{Fm}_{\mathcal{L}} \twoheadrightarrow \mathrm{Fm}_{\mathcal{L}}$, such that $\sigma(\bar{x} \rhd x) = \bar{x} \rhd x$ and $\sigma(\mathrm{Var}\,X) \subseteq \{\bar{x}, x\}$. Denote $\varepsilon_n(\bar{x}, x) = \sigma X$. Then

$$\{\bar{x} \rhd x\}^{\mathcal{R}} = \{\sigma(\bar{x} \rhd x)\}^{\mathcal{R}} = (\sigma X)^{\mathcal{R}} = (\varepsilon_n(\bar{x}, x))^{\mathcal{R}},$$

$$\{\bar{x} \rhd x\}^{\mathcal{R}} = \{\sigma(\bar{x} \rhd x)\}^{\mathcal{R}} = (\sigma(\mathcal{T}X))^{\mathcal{R}} = ((\sigma\mathcal{T})(\sigma X))^{\mathcal{R}} = (\mathcal{T}\varepsilon_n(\bar{x}, x))^{\mathcal{R}}.$$

Lemma 3.2.4. *For every $\bar{\alpha} \rhd \alpha \in \mathrm{Fm}^+_{\mathcal{L}}$*

1. $\{\bar{\alpha} \rhd \alpha\}^{\mathcal{R}} = (\varepsilon_{|\bar{\alpha}|}(\bar{\alpha}, \alpha))^{\mathcal{R}}$,
2. $\varepsilon_{|\bar{\alpha}|}(\bar{\alpha}, \alpha) \subseteq \mathbf{\Omega}\{\bar{\alpha} \rhd \alpha\}^{\mathcal{R}}$.

Proof. Let σ be a surjective substitution, such that $\sigma(\bar{x} \rhd x) = \bar{\alpha} \rhd \alpha$. Then

(1) $\{\bar{\alpha} \rhd \alpha\}^{\mathcal{R}} = \{\sigma(\bar{x} \rhd x)\}^{\mathcal{R}} = (\sigma(\varepsilon_n(\bar{x}, x)))^{\mathcal{R}} = (\varepsilon_n(\bar{\alpha}, \alpha))^{\mathcal{R}}$.

(2) $\{\bar{\alpha} \rhd \alpha\}^{\mathcal{R}} = \{\sigma(\bar{x} \rhd x)\}^{\mathcal{R}}$

$$= (\sigma(\mathcal{T}\varepsilon_n(\bar{x}, x)))^{\mathcal{R}} = ((\sigma\mathcal{T})(\sigma\varepsilon_n(\bar{x}, x)))^{\mathcal{R}} = (\mathcal{T}\varepsilon_n(\bar{\alpha}, \alpha))^{\mathcal{R}}.$$

So, by Lemma 3.1.8, $\varepsilon_n(\bar{\alpha}, \alpha) \subseteq \mathbf{\Omega}\{\bar{\alpha} \rhd \alpha\}^{\mathcal{R}}$. □

Thus, for every $\mathcal{A} \in \mathrm{Th}\,\mathcal{G}$,

$$\bar{\alpha} \rhd \alpha \in \mathcal{A} \implies \{\bar{\alpha} \rhd \alpha\}^{\mathcal{R}} \subseteq \mathcal{A}^{\mathcal{R}} = \mathcal{A} \implies \mathbf{\Omega}\{\bar{\alpha} \rhd \alpha\}^{\mathcal{R}} \subseteq \mathbf{\Omega}\,\mathcal{A} \overset{3.2.4(2)}{\implies} \varepsilon_{|\bar{\alpha}|}(\bar{\alpha}, \alpha) \subseteq \mathbf{\Omega}\,\mathcal{A},$$

$$\varepsilon_{|\bar{\alpha}|}(\bar{\alpha}, \alpha) \subseteq \mathbf{\Omega}\,\mathcal{A} \implies \{\bar{\alpha} \rhd \alpha\}^{\mathcal{R}} \overset{3.2.4(1)}{=} (\varepsilon_{|\bar{\alpha}|}(\bar{\alpha}, \alpha))^{\mathcal{R}} \subseteq (\mathbf{\Omega}\,\mathcal{A})^{\mathcal{R}} \overset{3.2.1}{=} \mathcal{A} \implies \bar{\alpha} \rhd \alpha \in \mathcal{A}.$$

Therefore $\mathcal{E} := \bigcup_{n \in \mathbf{Type}\,\mathcal{R}} \varepsilon_n$ is a graded congruence basis for $\mathcal{R}$.

($\Leftarrow$) Suppose $\mathcal{A}, \mathcal{B} \in \operatorname{Th}\mathcal{G}$ and $\boldsymbol{\Omega}\,\mathcal{A} = \boldsymbol{\Omega}\,\mathcal{B}$. Then

$$\bar{\alpha} \rhd \alpha \in \mathcal{A} \iff \varepsilon_{|\bar{\alpha}|}(\bar{\alpha}, \alpha) \subseteq \boldsymbol{\Omega}\,\mathcal{A} \iff \varepsilon_{|\bar{\alpha}|}(\bar{\alpha}, \alpha) \subseteq \boldsymbol{\Omega}\,\mathcal{B} \iff \bar{\alpha} \rhd \alpha \in \mathcal{B}.$$

Thus $\boldsymbol{\Omega} : \operatorname{Th}\mathcal{G} \to \operatorname{Con}\mathbf{Fm}_{\mathcal{L}}$ is injective; hence $\operatorname{Th}\mathcal{G}$ is weakly algebraizable. □

The following proposition shows that an operator associated with a graded congruence basis provides a faithful interpretation of $\vdash_{\mathcal{R}}$ in $\vDash_{\mathcal{R}}$.

Proposition 3.2.5. *If $\mathcal{E}$ is a graded congruence basis for a 2nd-level deductive system $\mathcal{R}$ then*

1. $\boldsymbol{\Omega}\,(U^{\mathcal{R}}) = (\mathcal{E}U)^{\vDash_{\mathcal{R}}}$,
2. $U \vdash_{\mathcal{R}} V \iff \mathcal{E}U \vDash_{\mathcal{R}} \mathcal{E}V$,

where U, V are arbitrary subsets of $\mathrm{Fm}^{+}_{\mathcal{L}}$.

Proof. By definition of a graded congruence basis for $\mathcal{R}$, for every $\mathcal{A} \in \operatorname{Th}\mathcal{G}$,

$$\bar{\alpha} \rhd \alpha \in \mathcal{A} \iff \mathcal{E}\{\bar{\alpha} \rhd \alpha\} \subseteq \boldsymbol{\Omega}\,\mathcal{A}. \qquad (*)$$

(1) The statement follows from the inclusions

$$(\subseteq) \quad (\mathcal{E}U \subseteq \boldsymbol{\Omega}\,\mathcal{A} \overset{(*)}{\Longrightarrow} U \subseteq \mathcal{A} \Longrightarrow U^{\mathcal{R}} \subseteq \mathcal{A} \Longrightarrow \boldsymbol{\Omega}(U^{\mathcal{R}}) \subseteq \boldsymbol{\Omega}\,\mathcal{A})$$

$$\Longrightarrow \boldsymbol{\Omega}\,(U^{\mathcal{R}}) \subseteq \bigcap\{\boldsymbol{\Omega}\,\mathcal{A} \mid \mathcal{A} \in \operatorname{Th}\mathcal{G},\ \mathcal{E}U \subseteq \boldsymbol{\Omega}\,\mathcal{A}\} \overset{def}{\iff} (\mathcal{E}U)^{\vDash_{\mathcal{R}}}.$$

$$(\supseteq) \quad U \subseteq U^{\mathcal{R}} \overset{(*)}{\Longrightarrow} \mathcal{E}U \subseteq \boldsymbol{\Omega}\,(U^{\mathcal{R}}) \Longrightarrow (\mathcal{E}U)^{\vDash_{\mathcal{R}}} \subseteq \boldsymbol{\Omega}\,(U^{\mathcal{R}}).$$

(2) $U \vdash_{\mathcal{R}} \bar{\alpha} \rhd \alpha \overset{def}{=} \bar{\alpha} \rhd \alpha \in U^{\mathcal{R}} \overset{(*)}{\iff} \mathcal{E}\{\bar{\alpha} \rhd \alpha\} \subseteq \boldsymbol{\Omega}\,(U^{\mathcal{R}}) \overset{(1)}{=} (\mathcal{E}U)^{\vDash_{\mathcal{R}}}$. □

3.3 Equivalential 2nd-level deductive systems

Let $\mathcal{A}$ be a theory of a 2nd-level deductive system $\mathcal{R}$. Recall the definition, that $\mathcal{A}$ is invariant if it is closed under substitutions, i.e., $\sigma\mathcal{A} \subseteq \mathcal{A}$ for every substitution σ. If $\mathcal{A}$ is invariant and $\mathcal{R}$ is equivalential, then for every substitution σ:

$$\sigma\mathcal{A} \subseteq \mathcal{A} \Longrightarrow \mathcal{A} \subseteq \sigma^{-1}\sigma\mathcal{A} \subseteq \sigma^{-1}\mathcal{A} \Longrightarrow \boldsymbol{\Omega}\,\mathcal{A} \subseteq \boldsymbol{\Omega}\,\sigma^{-1}\mathcal{A} \overset{\text{(EQ2)}}{=} \sigma^{-1}\,\boldsymbol{\Omega}\,\mathcal{A}$$

$$\Longrightarrow \sigma\,\boldsymbol{\Omega}\,\mathcal{A} \subseteq \sigma\sigma^{-1}\,\boldsymbol{\Omega}\,\mathcal{A} \subseteq \boldsymbol{\Omega}\,\mathcal{A}.$$

Thus, if $\mathcal{R}$ is equivalential, then the Leibnitz congruence of every invariant theory of $\mathcal{R}$ is a fully invariant congruence. This fact results in a close connection between consequence relations $\vdash_{\mathcal{R}}$ and $\vDash_{\mathcal{R}}$. First we prove a general result about arbitrary 2nd-level deductive systems.

Lemma 3.3.1. *Let $\mathcal{R}$ be a 2nd-level deductive system. Then for all $X, Y \in \operatorname{Sym}\operatorname{Fm}_{\mathcal{L}}$*

1. $X \dashv\models_{\mathcal{R}} \mathcal{T}X$,
2. $X \dashv\models_{\mathcal{R}} \boldsymbol{\Omega}(\mathcal{T}X)^{\mathcal{R}}$,
3. $X \models_{\mathcal{R}} Y \iff \mathcal{T}X \vdash_{\mathcal{R}} \mathcal{T}Y$.

Proof. (1) Since $X \subseteq \mathcal{T}X$ yields $X^{\models_{\mathcal{R}}} \subseteq (\mathcal{T}X)^{\models_{\mathcal{R}}}$, it suffices to show only the other inclusion. For every $\mathcal{A} \in \operatorname{Th}\mathcal{G}$, $X \subseteq \boldsymbol{\Omega}\,\mathcal{A}$ implies $\mathcal{T}X \subseteq \boldsymbol{\Omega}\,\mathcal{A}$. Therefore

$$\mathcal{T}X \subseteq \bigcap\{\boldsymbol{\Omega}\,\mathcal{A} \mid X \subseteq \boldsymbol{\Omega}\,\mathcal{A},\, \mathcal{A} \in \operatorname{Th}\mathcal{G}\} \overset{def}{=} X^{\models_{\mathcal{R}}} \overset{(mon)}{\Longrightarrow} (\mathcal{T}X)^{\models_{\mathcal{R}}} \subseteq X^{\models_{\mathcal{R}}}.$$

(2) The statement follows from the inclusions

$(\subseteq)$ $\mathcal{T}X \subseteq (\mathcal{T}X)^{\mathcal{R}} \overset{3.1.8}{\iff} X \subseteq \boldsymbol{\Omega}\,(\mathcal{T}X)^{\mathcal{R}} \implies X^{\models_{\mathcal{R}}} \subseteq \boldsymbol{\Omega}\,(\mathcal{T}X)^{\mathcal{R}}$.

$(\supseteq)$ $\forall \mathcal{A} \in \operatorname{Th}\mathcal{G}$: $X \subseteq \boldsymbol{\Omega}\,\mathcal{A} \overset{3.1.8}{\iff} \mathcal{T}X \subseteq \mathcal{A} \Longrightarrow (\mathcal{T}X)^{\mathcal{R}} \subseteq \mathcal{A} \Longrightarrow \boldsymbol{\Omega}\,(\mathcal{T}X)^{\mathcal{R}} \subseteq \boldsymbol{\Omega}\,\mathcal{A}$.

Therefore $\boldsymbol{\Omega}\,(\mathcal{T}X)^{\mathcal{R}} \subseteq \bigcap\{\boldsymbol{\Omega}\mathcal{A} \mid X \subseteq \boldsymbol{\Omega}\,\mathcal{A},\, \mathcal{A} \in \operatorname{Th}\mathcal{G}\} \overset{def}{=} X^{\models_{\mathcal{R}}}$.

(3) $X \models_{\mathcal{R}} Y \overset{def}{\iff} Y \subseteq X^{\models_{\mathcal{R}}} \overset{2.}{=} \boldsymbol{\Omega}\,(\mathcal{T}X)^{\mathcal{R}} \overset{3.1.8}{\iff} \mathcal{T}Y \subseteq (\mathcal{T}X)^{\mathcal{R}} \overset{def}{\iff} \mathcal{T}X \vdash_{\mathcal{R}} \mathcal{T}Y$. □

For an arbitrary 2nd-level deductive system, according to Lemma 3.3.1(3), the operator $\mathcal{T} : \operatorname{Sym}\operatorname{Fm}_{\mathcal{L}} \to \operatorname{Sym}\operatorname{Fm}_{\mathcal{L}}$ provides a faithful interpretation of $\models_{\mathcal{R}}$ in $\vdash_{\mathcal{R}}$. If $\mathcal{R}$ is equivalential, $\mathcal{T}$ can be replaced by a weaker operator.

Definition 3.3.2. Let $\mathcal{M}_{\mathcal{L}} := \{t \in \mathcal{T}_{\mathcal{L}} \mid t = \lambda\,\xi.\,\alpha(\xi, x, y)\}$ be the set of unary polynomials over $\operatorname{Fm}_{\mathcal{L}}$ that depend only on variables x and y. Then for every subset $\mathcal{J} \subseteq \mathcal{M}_{\mathcal{L}}$ the definition

$$\tilde{\mathcal{J}}(\alpha \rhd \beta) := \{\, \phi(\alpha, \alpha, \beta) \lhd\!\rhd \phi(\beta, \alpha, \beta) \mid \lambda\,\xi.\,\phi(\xi, x, y) \in \mathcal{J}\},$$

gives rise to a complex operator $\tilde{\mathcal{J}} : \mathcal{P}(\operatorname{Fm}^2_{\mathcal{L}}) \to \operatorname{Sym}\operatorname{Fm}_{\mathcal{L}}$.

The index $\mathcal{L}$ in $\mathcal{M}_{\mathcal{L}}$ will be routinely omitted. □

For every $\mathcal{J} \subseteq \mathcal{M}$ we can define the operator $\tilde{\mathcal{J}}^{-1} : \mathcal{P}(\operatorname{Fm}^{+}_{\mathcal{L}}) \to \operatorname{Fm}^2_{\mathcal{L}}$ as follows

$$\tilde{\mathcal{J}}^{-1}\mathcal{A} = \{\alpha \lhd\!\rhd \beta \mid \tilde{\mathcal{J}}\{\alpha \lhd\!\rhd \beta\} \subseteq \mathcal{A}\}.$$

Lemma 3.3.3. *The following holds*

1. *$\mathcal{M}_{\mathcal{L}} = \sigma\mathcal{T}_{\mathcal{L}}$, for every substitution σ such that $\sigma(x \rhd y) = x \rhd y$ and $\sigma\operatorname{Var} \subseteq \{x, y\}$;*
2. *for every substitution σ and every $\mathcal{J} \subseteq \mathcal{M}_{\mathcal{L}}$, $\sigma\tilde{\mathcal{J}} = \tilde{\mathcal{J}}\sigma$;*

3. for every substitution σ and every $\mathcal{J} \subseteq \mathcal{M}_{\mathcal{L}}$, $\sigma^{-1}\tilde{\mathcal{J}}^{-1} = \tilde{\mathcal{J}}^{-1}\sigma^{-1}$.

4. if $\mathcal{J}$ is finite, then $\tilde{\mathcal{J}}^{-1} : \mathcal{P}(\mathrm{Fm}^{+}_{\mathcal{L}}) \to \mathrm{Fm}^{2}_{\mathcal{L}}$ is continuous.

Proof. 1) Let σ be a substitution such that $\sigma(x \rhd y) = x \rhd y$ and $\sigma\,\mathrm{Var} \subseteq \{x, y\}$. Trivially, for every $t \in \mathcal{T}_{\mathcal{L}}$, $\sigma t \in \mathcal{M}_{\mathcal{L}}$. Also, if $t \in \mathcal{M}_{\mathcal{L}}$ then $t = \sigma t \in \sigma\mathcal{T}_{\mathcal{L}}$.

2) For every $\alpha \rhd \beta \in \mathrm{Fm}^{+}_{\mathcal{L}}$

$$\begin{aligned}\sigma\tilde{\mathcal{J}}\{\alpha \rhd \beta\} &= \sigma\{\, \phi(\alpha,\alpha,\beta) \lhd\rhd \phi(\beta,\alpha,\beta) \mid \lambda\,\xi.\,\phi(\xi,x,y) \in \mathcal{J}\} \\ &= \{\, \phi(\sigma\alpha,\sigma\alpha,\sigma\beta) \lhd\rhd \phi(\sigma\beta,\sigma\alpha,\sigma\beta) \mid \lambda\,\xi.\,\phi(\xi,x,y) \in \mathcal{J}\} = \tilde{\mathcal{J}}\{\sigma\alpha \rhd \sigma\beta\} = \tilde{\mathcal{J}}\sigma\{\alpha \rhd \beta\},\end{aligned}$$

therefore, for every $\mathcal{A} \subseteq \mathrm{Fm}^{+}_{\mathcal{L}}$,

$$\sigma\tilde{\mathcal{J}}\mathcal{A} = \bigcup_{s \in \mathcal{A}} \sigma\tilde{\mathcal{J}}\{s\} = \bigcup_{s \in \mathcal{A}} \tilde{\mathcal{J}}\sigma\{s\} = \tilde{\mathcal{J}}\sigma\mathcal{A}.$$

3) For every $X \in \mathrm{Sym}\,\mathrm{Fm}_{\mathcal{L}}$

$$\begin{aligned}\alpha \rhd \beta \in \sigma^{-1}\tilde{\mathcal{J}}^{-1}X &\iff \sigma\alpha \rhd \sigma\beta \in \tilde{\mathcal{J}}^{-1}X \iff \tilde{\mathcal{J}}(\sigma\alpha \rhd \sigma\beta) \subseteq X \\ &\iff \{\, \phi(\sigma\alpha,\sigma\alpha,\sigma\beta) \lhd\rhd \phi(\sigma\beta,\sigma\alpha,\sigma\beta) \mid \lambda\,\xi.\,\phi(\xi,x,y) \in \mathcal{J}\} \subseteq X \\ &\iff \sigma\{\, \phi(\alpha,\alpha,\beta) \lhd\rhd \phi(\beta,\alpha,\beta) \mid \lambda\,\xi.\,\phi(\xi,x,y) \in \mathcal{J}\} \subseteq X \\ &\iff \{\, \phi(\alpha,\alpha,\beta) \lhd\rhd \phi(\beta,\alpha,\beta) \mid \lambda\,\xi.\,\phi(\xi,x,y) \in \mathcal{J}\} \subseteq \sigma^{-1}X \\ &\iff \tilde{\mathcal{J}}\{\alpha \rhd \beta\} \subseteq \sigma^{-1}X \iff \alpha \rhd \beta \in \tilde{\mathcal{J}}^{-1}\sigma^{-1}X.\end{aligned}$$

4) Suppose $\{\mathcal{A}_i\}_{i \in I} \subseteq \mathcal{P}(\mathrm{Fm}^{+}_{\mathcal{L}})$ is an upward-directed family. We have to show that

$$\tilde{\mathcal{J}}^{-1}(\textstyle\bigcup_{i \in I} \mathcal{A}_i) = \textstyle\bigcup_{i \in I} \tilde{\mathcal{J}}^{-1}\mathcal{A}_i.$$

The direction $\supseteq$ is trivial, because $\tilde{\mathcal{J}}^{-1}$ is monotone. For the other direction, pick any $\alpha \rhd \beta \in \tilde{\mathcal{J}}^{-1}(\bigcup_{i \in I} \mathcal{A}_i)$. Then, by definition, $\tilde{\mathcal{J}}\{\alpha \lhd\rhd \beta\} \subseteq_{\omega} \bigcup_{i \in I} \mathcal{A}_i$. Therefore, since $\{\mathcal{A}_i\}_{i \in I}$ is upward-directed, there exists $\mathcal{A} \in \{\mathcal{A}_i\}_{i \in I}$ such that $\tilde{\mathcal{J}}\{\alpha \lhd\rhd \beta\} \subseteq \mathcal{A}$. Thus $\alpha \rhd \beta \in \tilde{\mathcal{J}}^{-1}\mathcal{A} \subseteq \tilde{\mathcal{J}}^{-1}(\bigcup_{i \in I} \mathcal{A}_i)$. □

In equivalential deductive systems $\tilde{\mathcal{J}}$ replaces the operator $\mathcal{T}$. and the fact that $\tilde{\mathcal{J}}$ commutes with substitutions plays the crucial role in the characterization of equivalential deductive systems by Leibnitz operator. Note that in the case of $\mathcal{T}$, $\sigma(\mathcal{T}X) = (\sigma\mathcal{T})(\sigma X) \subseteq \mathcal{T}(\sigma X)$.

The following proposition is an analog of [17, Theorem 4.5].

Proposition 3.3.4. *Let $\mathcal{R}$ be a 2nd-level deductive system $\mathcal{R}$. TFAE*

1. *$\mathcal{R}$ is equivalential;*
2. *$\sigma\,\boldsymbol{\Omega}\,\mathcal{A} \subseteq \boldsymbol{\Omega}(\sigma\mathcal{A})^{\mathcal{R}}$, for all substitutions σ and theories $\mathcal{A} \in \mathrm{Th}\,\mathcal{G}$;*
3. *$\mathcal{T}X \dashv\vdash_{\mathcal{R}} \tilde{\mathcal{M}}X$, for every $X \in \mathrm{Sym}\,\mathrm{Fm}_{\mathcal{L}}$.*

Proof. ($1 \Rightarrow 2$) For every theory $\mathcal{A} \in \mathrm{Th}\,\mathcal{G}$,

$$\mathcal{A} \overset{1.0.1}{\subseteq} \sigma^{-1}\sigma\mathcal{A} = \sigma^{-1}(\sigma\mathcal{A}) \subseteq \sigma^{-1}(\sigma\mathcal{A})^{\mathcal{R}} \overset{\text{mon}}{\Longrightarrow} \boldsymbol{\Omega}\,\mathcal{A} \subseteq \boldsymbol{\Omega}\,\sigma^{-1}(\sigma\mathcal{A})^{\mathcal{R}} \overset{\mathit{1.}}{=} \sigma^{-1}\,\boldsymbol{\Omega}(\sigma\mathcal{A})^{\mathcal{R}}$$
$$\Longrightarrow \sigma\,\boldsymbol{\Omega}\,\mathcal{A} \subseteq \sigma\sigma^{-1}\,\boldsymbol{\Omega}(\sigma\mathcal{A})^{\mathcal{R}} \overset{1.0.1}{\subseteq} \boldsymbol{\Omega}(\sigma\mathcal{A})^{\mathcal{R}}.$$

($2 \Rightarrow 3$) Note, that for every $A \subseteq \bigcup \mathrm{Th}\,\mathcal{G}$ and every substitution σ,

$$C_{\mathcal{R}}\,\sigma\,C_{\mathcal{R}}A = C_{\mathcal{R}}\,\sigma A \implies \boldsymbol{\Omega}\,C_{\mathcal{R}}\,\sigma\,C_{\mathcal{R}}A = \boldsymbol{\Omega}\,C_{\mathcal{R}}\,\sigma A. \quad (*)$$

Let σ be a substitution such that $\sigma(x \rhd y) = x \rhd y$ and $\sigma\,\mathrm{Var} = \{x, y\}$. Then

$$\sigma\,\boldsymbol{\Omega}\,C_{\mathcal{R}}(\mathcal{T}\{x \lhd\!\rhd y\}) \overset{\mathit{2.}}{\subseteq} \boldsymbol{\Omega}\,C_{\mathcal{R}}\sigma C_{\mathcal{R}}(\mathcal{T}\{x \lhd\!\rhd y\})$$
$$\overset{(*)}{=} \boldsymbol{\Omega}\,C_{\mathcal{R}}(\sigma\mathcal{T}\{x \lhd\!\rhd y\}) \overset{3.1.7(2)}{=} \boldsymbol{\Omega}\,C_{\mathcal{R}}((\sigma\mathcal{T})\,\sigma\{x \lhd\!\rhd y\}) \overset{3.3.3}{=} \boldsymbol{\Omega}\,C_{\mathcal{R}}(\tilde{\mathcal{M}}\{x \lhd\!\rhd y\}).$$

Therefore

$$\mathcal{T}\{x \lhd\!\rhd y\} \subseteq (\mathcal{T}\{x \lhd\!\rhd y\})^{\mathcal{R}} \overset{3.1.8}{\iff} \{x \lhd\!\rhd y\} \subseteq \boldsymbol{\Omega}\,(\mathcal{T}\{x \lhd\!\rhd y\})^{\mathcal{R}}$$
$$\implies \{x \lhd\!\rhd y\} = \sigma\{x \lhd\!\rhd y\} \subseteq \sigma\,\boldsymbol{\Omega}\,(\mathcal{T}\{x \lhd\!\rhd y\})^{\mathcal{R}} \subseteq \boldsymbol{\Omega}\,(\tilde{\mathcal{M}}\{x \lhd\!\rhd y\})^{\mathcal{R}}$$
$$\overset{3.1.8}{\iff} \mathcal{T}\{x \lhd\!\rhd y\} \subseteq (\tilde{\mathcal{M}}\{x \lhd\!\rhd y\})^{\mathcal{R}} \implies (\mathcal{T}\{x \lhd\!\rhd y\})^{\mathcal{R}} \subseteq (\tilde{\mathcal{M}}\{x \lhd\!\rhd y\})^{\mathcal{R}}.$$

So, since trivially $\tilde{\mathcal{M}}\{x \lhd\!\rhd y\} \subseteq \mathcal{T}\{x \lhd\!\rhd y\}$, we have

$$(\tilde{\mathcal{M}}\{x \lhd\!\rhd y\})^{\mathcal{R}} = (\mathcal{T}\{x \lhd\!\rhd y\})^{\mathcal{R}}.$$

Consider any $\alpha \rhd \beta \in \mathrm{Fm}^2_{\mathcal{L}}$ and let σ be a surjective substitution such that $\sigma(x \rhd y) = \alpha \rhd \beta$. Then

$$(\tilde{\mathcal{M}}\{x \lhd\!\rhd y\})^{\mathcal{R}} = (\mathcal{T}\{x \lhd\!\rhd y\})^{\mathcal{R}} \implies (\sigma\tilde{\mathcal{M}}\{x \lhd\!\rhd y\})^{\mathcal{R}} = (\sigma\mathcal{T}\{x \lhd\!\rhd y\})^{\mathcal{R}}$$
$$\implies (\tilde{\mathcal{M}}\sigma\{x \lhd\!\rhd y\})^{\mathcal{R}} = ((\sigma\mathcal{T})\sigma\{x \lhd\!\rhd y\})^{\mathcal{R}} \overset{3.1.7}{\Longrightarrow} (\tilde{\mathcal{M}}\{\alpha \lhd\!\rhd \beta\})^{\mathcal{R}} = (\mathcal{T}\{\alpha \lhd\!\rhd \beta\})^{\mathcal{R}}.$$

($3 \Rightarrow 1$) Consider the operator $\tilde{\mathcal{M}}^{-1} : \mathrm{Th}\,\mathcal{G} \to \mathrm{Sym}\,\mathrm{Fm}_{\mathcal{L}}$ defined as follows

$$\tilde{\mathcal{M}}^{-1}\mathcal{A} = \{\alpha \lhd\!\rhd \beta \mid \tilde{\mathcal{M}}\{\alpha \lhd\!\rhd \beta\} \subseteq \mathcal{A}\}.$$

Then, for every $\mathcal{A} \in \mathrm{Th}\,\mathcal{G}$,

$$\tilde{\mathcal{M}}^{-1}\mathcal{A} = \{\alpha \lhd\!\rhd \beta \mid \tilde{\mathcal{M}}\{\alpha \lhd\!\rhd \beta\} \subseteq \mathcal{A}\} \overset{\mathit{3.}}{=} \{\alpha \lhd\!\rhd \beta \mid \mathcal{T}\{\alpha \lhd\!\rhd \beta\} \subseteq \mathcal{A}\} \overset{3.1.8}{=} \boldsymbol{\Omega}\mathcal{A}. \quad (**)$$

Therefore, for every substitution σ,

$$\sigma^{-1}\boldsymbol{\Omega}\mathcal{A} \overset{(**)}{=} \sigma^{-1}\tilde{\mathcal{M}}^{-1}\mathcal{A} \overset{3.3.3}{=} \tilde{\mathcal{M}}^{-1}\sigma^{-1}\mathcal{A} \overset{(**)}{=} \boldsymbol{\Omega}\sigma^{-1}\mathcal{A}. \qquad \square$$

Corollary 3.3.5. *Let $\mathcal{R}$ be a 2nd-level finitely equivalential deductive system. Then*

1. *for all $X, Y \in \operatorname{Sym}\operatorname{Fm}_{\mathcal{L}}$: $X \vDash_{\mathcal{R}} Y \iff \tilde{\mathcal{M}}X \vdash_{\mathcal{R}} \tilde{\mathcal{M}}Y$,*
2. *$\vDash_{\mathcal{R}}$ is structural.*

Proof. 1. $X \vDash_{\mathcal{R}} Y \overset{3.3.1}{\iff} \mathcal{T}X \vdash_{\mathcal{R}} \mathcal{T}Y \overset{3.3.4(3)}{\iff} \tilde{\mathcal{M}}X \vdash_{\mathcal{R}} \tilde{\mathcal{M}}Y$.

2. For every substitution σ

$$\begin{aligned} X \vDash_{\mathcal{R}} Y &\overset{1.}{\Longrightarrow} \tilde{\mathcal{M}}X \vdash_{\mathcal{R}} \tilde{\mathcal{M}}Y \Longrightarrow \sigma(\tilde{\mathcal{M}}X) \vdash_{\mathcal{R}} \sigma(\tilde{\mathcal{M}}Y) \\ &\overset{3.3.3(2)}{\Longrightarrow} \tilde{\mathcal{M}}(\sigma X) \vdash_{\mathcal{R}} \tilde{\mathcal{M}}(\sigma Y) \overset{1.}{\Longrightarrow} \sigma X \vDash_{\mathcal{R}} \sigma Y. \qquad \square \end{aligned}$$

The finitely equivalential 2nd-level deductive systems also demonstrate properties similar to their 1st-level counterparts. The following proposition is an analog of [17, Theorem 4.6].

Proposition 3.3.6. *Let $\mathcal{R}$ be a 2nd-level deductive system $\mathcal{R}$. TFAE*

1. *$\mathcal{R}$ is finitely equivalential;*
2. *there is $\mathcal{J} \subseteq_{\omega} \mathcal{M}$ such that for every $X \in \operatorname{Sym}\operatorname{Fm}_{\mathcal{L}}$: $\mathcal{T}X \dashv\vdash_{\mathcal{R}} \tilde{\mathcal{J}}X$;*

Proof. ($1 \Rightarrow 2$) Let $X \in \operatorname{Sym}\operatorname{Fm}_{\mathcal{L}}$ be a finite symmetric set. The family $\{(\tilde{\mathcal{K}}X)^{\mathcal{R}} \mid \mathcal{K} \subseteq_{\omega} \mathcal{M}\}$ is upward-directed and $\bigcup\{(\tilde{\mathcal{K}}X)^{\mathcal{R}} \mid \mathcal{K} \subseteq_{\omega} \mathcal{M}\} = (\tilde{\mathcal{M}}X)^{\mathcal{R}}$. Therefore

$$\boldsymbol{\Omega}\,(\mathcal{T}X)^{\mathcal{R}} \overset{3.3.4(3)}{=} \boldsymbol{\Omega}\,(\tilde{\mathcal{M}}X)^{\mathcal{R}} = \boldsymbol{\Omega}\,(\bigcup\{(\tilde{\mathcal{K}}X)^{\mathcal{R}} \mid \mathcal{K} \subseteq_{\omega} \mathcal{M}\}) \overset{1.}{=} \bigcup\{\,\boldsymbol{\Omega}(\tilde{\mathcal{K}}X)^{\mathcal{R}} \mid \mathcal{K} \subseteq_{\omega} \mathcal{M}\}. \quad (*)$$

$$\text{Then} \quad X \overset{3.3.1(2)}{\dashv\vDash_{\mathcal{R}}} \boldsymbol{\Omega}\,(\mathcal{T}X)^{\mathcal{R}} \overset{3.1.8}{\Longrightarrow} X \subseteq \boldsymbol{\Omega}\,(\mathcal{T}X)^{\mathcal{R}} \overset{(*)}{=} \bigcup\{\,\boldsymbol{\Omega}(\tilde{\mathcal{K}}X)^{\mathcal{R}} \mid \mathcal{K} \subseteq_{\omega} \mathcal{M}\}.$$

Thus, there exists $\mathcal{J} \subseteq_{\omega} \mathcal{M}$ such that $X \subseteq \boldsymbol{\Omega}\,(\tilde{\mathcal{J}}X)^{\mathcal{R}}$. Then

$$X \subseteq \boldsymbol{\Omega}\,(\tilde{\mathcal{J}}X)^{\mathcal{R}} \overset{3.1.8}{\iff} \mathcal{T}X \subseteq (\tilde{\mathcal{J}}X)^{\mathcal{R}}.$$

Therefore, for an arbitrary $Y \in \operatorname{Sym}\operatorname{Fm}_{\mathcal{L}}$,

$$\mathcal{T}Y = \bigcup\{\mathcal{T}X \mid X \in \operatorname{Sym}\operatorname{Fm}_{\mathcal{L}}, X \subseteq_{\omega} Y\} \subseteq \bigcup\{(\tilde{\mathcal{J}}X)^{\mathcal{R}} \mid X \in \operatorname{Sym}\operatorname{Fm}_{\mathcal{L}}, X \subseteq_{\omega} Y\} = (\tilde{\mathcal{J}}Y)^{\mathcal{R}}$$

Thus $\tilde{\mathcal{J}}Y \vdash_{\mathcal{R}} \mathcal{T}Y$. The other direction is trivial, because $\tilde{\mathcal{J}}Y \subseteq \mathcal{T}Y$.

($2 \Rightarrow 1$) As well as before, for every $\mathcal{A} \in \operatorname{Th}\mathcal{G}$,

$$\tilde{\mathcal{J}}^{-1}\mathcal{A} = \{\alpha \lhd\!\rhd \beta \mid \tilde{\mathcal{J}}\{\alpha \lhd\!\rhd \beta\} \subseteq \mathcal{A}\} \overset{2.}{=} \{\alpha \lhd\!\rhd \beta \mid \mathcal{T}\{\alpha \lhd\!\rhd \beta\} \subseteq \mathcal{A}\} \overset{3.1.8}{=} \boldsymbol{\Omega}\mathcal{A}. \quad (**)$$

Then $\mathcal{R}$ is equivalential, because for every substitution σ,

$$\sigma^{-1}\boldsymbol{\Omega}\mathcal{A} \overset{(**)}{=} \sigma^{-1}\tilde{\mathcal{J}}^{-1}\mathcal{A} \overset{3.3.3}{=} \tilde{\mathcal{J}}^{-1}\sigma^{-1}\mathcal{A} \overset{(**)}{=} \boldsymbol{\Omega}\,\sigma^{-1}\mathcal{A}.$$

Also, $\boldsymbol{\Omega}$ is continuous, because for every upward-directed family $\{\mathcal{A}_i\}_{i\in I}$of theories of $\mathcal{R}$.

$$\boldsymbol{\Omega}(\textstyle\bigcup_{i\in I}\mathcal{A}_i) \overset{(**)}{=} \tilde{\mathcal{J}}^{-1}(\textstyle\bigcup_{i\in I}\mathcal{A}_i) \overset{3.3.3}{=} \textstyle\bigcup_{i\in I}\tilde{\mathcal{J}}^{-1}\mathcal{A}_i \overset{(**)}{=} \textstyle\bigcup_{i\in I}\boldsymbol{\Omega}\,\mathcal{A}_i. \qquad \square$$

Similarly to the 1st-level case holds

Corollary 3.3.7. *Let $\mathcal{R}$ be a 2nd-level finitely equivalential deductive system. Then*

1. *there is $\mathcal{J} \subseteq_\omega \mathcal{M}$ such that for all $X, Y \in \operatorname{Sym}\operatorname{Fm}_{\mathcal{L}}$: $X \vDash_{\mathcal{S}} Y \iff \tilde{\mathcal{J}}X \vdash_{\mathcal{R}} \tilde{\mathcal{J}}Y$,*
2. *$\vDash_{\mathcal{R}}$ is finitary.*

Proof. 1. $X \vDash_{\mathcal{R}} Y \overset{3.3.1}{\iff} \mathcal{T}Y \vdash_{\mathcal{R}} \mathcal{T}X \overset{3.3.6(2)}{\iff} \tilde{\mathcal{J}}Y \vdash_{\mathcal{R}} \tilde{\mathcal{J}}Y$.

2. Suppose $X, Y \subseteq \operatorname{Sym}\operatorname{Fm}_{\mathcal{L}}$ such that $X \vDash_{\mathcal{R}} Y$ and Y is finite. Then $\tilde{\mathcal{J}}X \vdash_{\mathcal{R}} \tilde{\mathcal{J}}Y$, and since $\vdash_{\mathcal{R}}$ is finitary there is a finite subset $Z \subseteq_\omega X$, which can be chosen also symmetric, such that $\tilde{\mathcal{J}}Z \vdash_{\mathcal{R}} \tilde{\mathcal{J}}Y$, and therefore $Z \vDash_{\mathcal{R}} Y$. $\square$

The properties of an algebraizable (finitely algebraizable) logics are just mechanical combinations of those for weakly algebraizable and equivalential (finitely equivalential) ones. We put the summary in tables.

1) Let $X, Y \subseteq \operatorname{Fm}^2_{\mathcal{L}}$,

2) (PA) means protoalgebraic, (EQ) stands for equivalential, (FE) is finitely equivalential,

3) Δ is a protoequivalence system, that exists for every protoalgebraic deductive system,

4) $\Delta\tilde{\mathcal{K}}\{x \triangleright y\}$, $\mathcal{K} \subseteq \mathcal{M}$ is an equivalence system, that exists for every equivalential deductive

system; $\mathcal{K}$ can be chosen finite if the system is finitely equivalential.

	1st level	2nd level
PA	$X \vDash_{\mathcal{S}} Y \iff (\Delta\mathcal{T})X \vdash_{\mathcal{S}} (\Delta\mathcal{T})Y$	$X \vDash_{\mathcal{R}} Y \iff \mathcal{T}X \vdash_{\mathcal{R}} \mathcal{T}X$
EQ	$X \vDash_{\mathcal{S}} Y \iff (\Delta\hat{\mathcal{M}})X \vdash_{\mathcal{S}} (\Delta\hat{\mathcal{M}})Y$	$X \vDash_{\mathcal{R}} Y \iff \hat{\mathcal{M}}X \vdash_{\mathcal{R}} \hat{\mathcal{M}}Y$
FE	$X \vDash_{\mathcal{S}} Y \iff (\Delta\hat{\mathcal{K}})X \vdash_{\mathcal{S}} (\Delta\hat{\mathcal{K}})Y$	$X \vDash_{\mathcal{R}} Y \iff \hat{\mathcal{J}}X \vdash_{\mathcal{R}} \hat{\mathcal{J}}Y$

Note that $\Delta t(\alpha \rhd \beta) = \Delta(t\alpha, t\beta)$ in infix notation looks like $t\alpha\Delta t\beta$ which bears striking syntactical similarity to $t\alpha \rhd t\beta$ of the 2-level case. Thus by using this notation we can give a precise meaning to the informal observation first made apparently by D. Pigozzi, that the role of Δ in the 2nd-level case is played by $\rhd$.

The lineage of weakly algebraizable systems looks like following.

1. $S, T \subseteq \mathrm{Fm}_{\mathcal{L}}$, $X, Y \subseteq \mathrm{Fm}^2_{\mathcal{L}}$, $U, V \subseteq \bigcup \mathrm{Th}\,\mathcal{G}$,
2. $E(x) \subseteq \mathrm{Fm}^2_{\mathcal{L}}\{x\}$ be a system of equations that exists for every weakly algebraizable 1st-level deductive system [10]. Let for every $A \subseteq \mathrm{Fm}_{\mathcal{L}}$, $EA = \bigcup\{E(\alpha) \mid \alpha \in A\}$. It is possible to compose E with Δ, so for instance $(E\Delta)\,\theta = E(\Delta\theta) = \{\gamma(\delta(\alpha,\beta)) \rhd \varepsilon(\delta(\alpha,\beta)) \mid \gamma \rhd \varepsilon \in E, \delta \in \Delta, \alpha \rhd \beta \in \theta\}$.
3. Let $\mathcal{E} = \bigcup_{n \in \mathbf{Type}\,\mathcal{R}} \varepsilon_n$ be a graded congruence basis for a weakly algebraizable 2nd-level deductive system $\mathcal{R}$, that exists by Theorem 3.2.3. As before we associate with $\mathcal{E}$ an operator $\mathcal{E} : \mathcal{P}(\mathrm{Fm}^+_{\mathcal{L}}) \to \mathcal{P}(\mathrm{Fm}^2_{\mathcal{L}})$ by $\mathcal{E}U = \bigcup\{\varepsilon_{|\bar{\alpha}|}(\bar{\alpha}, \alpha) \mid \bar{\alpha} \rhd \alpha \in U\}$.

	1st level	2nd level
WA	$S \vdash_{\mathcal{S}} T \iff ES \models_{\mathcal{S}} ET$	$U \vdash_{\mathcal{R}} V \iff \mathcal{E}U \models_{\mathcal{R}} \mathrm{E}V$
	$X \mathrel{\dashv\models}_{\mathcal{S}} (E\Delta\mathcal{T})X$	$X \mathrel{\dashv\models}_{\mathcal{R}} (\mathcal{E}\mathcal{T})X$
AL	$S \vdash_{\mathcal{S}} T \iff ES \models_{\mathcal{S}} ET$	$U \vdash_{\mathcal{R}} V \iff \mathcal{E}U \models_{\mathcal{R}} \mathcal{E}V$
	$X \mathrel{\dashv\models}_{\mathcal{S}} (E\Delta\hat{\mathcal{M}})X$	$X \mathrel{\dashv\models}_{\mathcal{R}} (\mathcal{E}\hat{\mathcal{M}})X$
FA	$S \vdash_{\mathcal{S}} T \iff ES \models_{\mathcal{S}} ET$	$U \vdash_{\mathcal{R}} V \iff \mathcal{E}U \models_{\mathcal{R}} \mathcal{E}V$
	$X \mathrel{\dashv\models}_{\mathcal{S}} (E\Delta\hat{\mathcal{K}})X$	$X \mathrel{\dashv\models}_{\mathcal{R}} (\mathcal{E}\hat{\mathcal{J}})X$

Notes to Chapter 3.

It was shown in [20], that a 2nd-level deductive system $\mathcal{R}$ is equivalential iff the class $\mathbf{Mod}\,\mathcal{R}$ of 2nd-level models for $\mathcal{R}$ is closed under submatrices, and $\mathcal{R}$ is finitely equivalential iff it is equivalential and $\mathbf{Mod}\,\mathcal{R}$ is closed under filtered products. Equivalential 2nd-level deductive systems were defined in [20] through an operator essentially equivalent to $\tilde{\mathcal{M}}$.

4. FULL CLOSURE RELATIONS

A full closure relation for a deductive system $\mathcal{S}$ can be defined as a general closure relation generated by a Leibnitz congruence. Full closure relations were studied in algebraic logic primarily in connection with deduction-detachment theorem [14] and fully adequate Gentzen systems. The former arise in a number cases when non-protoalgebraic 1st-level deductive system has a natural algebraic semantics [12], the paradigmatic example here is being the conjunction-disjunction fragment of a classical propositional logic [19].

In this chapter, we will prove the criterion for the existence of a fully adequate Gentzen system for a 1-st level deductive system, through the existence of, so called, *a graded congruence basis*. The proof exploits the fact that a deductive system $\mathcal{S}$ has a fully adequate Gentzen system if and only if its full closure relations form a 2nd-level deductive system. This 2nd-level deductive system $\mathbf{Fcr}\,\mathcal{S}$ which corresponds to a fully adequate Gentzen system will be shown is always weakly algebraizable and, according to Proposition 3.2.5, the graded congruence basis represents a faithful translation of the deductive theory of $\mathcal{G}$ into the equational theory of the class $\mathrm{Alg}^*\,\mathcal{S}$.

4.1 Strong Galois connections

Suppose $\mathfrak{X} \subseteq \mathcal{P}(A)$ is an algebraic family and $\mathcal{C}$ is a closure system in A such that for every $Y \in \mathcal{C}$ the set $\{X \in \mathfrak{X} \mid X \subseteq Y\}$ is non-empty and upward-directed. We will describe the properties of the special Galois connection, that we will call *strong*, determined by the relation $\subseteq$ between elements of $\mathfrak{X}$ and $\mathcal{C}$. For every $Y \in \mathcal{C}$, we can uniquely define $\mathcal{O}Y = \bigcup\{X \in \mathfrak{X} \mid X \subseteq Y\} \in \mathfrak{X}$, therefore defining an operator $\mathcal{O} : \mathcal{C} \to \mathfrak{X}$. Thus $\mathcal{O}Y$ is the largest element of $\mathfrak{X}$ contained in Y.

Let C be the closure operator associated with $\mathcal{C}$. Here is some easy facts about the relations between $\mathcal{O}$ and C:

Lemma 4.1.1.

1. $\mathcal{O}Y \subseteq Y$;
2. $\mathcal{O}$ *is monotone;*
3. *if* $\mathfrak{X}$ *is a closure system in* A, *then* $\mathcal{O}$ *is meet-continuous;*
4. $\mathcal{O}C\mathcal{O} = \mathcal{O}$;
5. $\mathcal{O}$ *is injective on* $\{C\mathcal{O}Y \mid Y \in \mathcal{C}\}$;
6. $C\mathcal{O}Y = \bigcap\{Z \in \mathcal{C} \mid \mathcal{O}Y \subseteq Z\}$.

Proof. The proof is trivial. □

Not so trivial properties we formulate in the following lemmas:

Lemma 4.1.2. *Let* $\mathcal{C}$ *be an algebraic closure system in* A, $C : \mathcal{P}(A) \to \mathcal{C}$ *be a closure operator associated with* $\mathcal{C}$, *and* $\mathcal{O} : \mathcal{C} \to \mathcal{P}(A)$ *be an operator such that*

a) $\mathcal{O}C\mathcal{O} = \mathcal{O}$,
b) $\mathcal{O}$ *is monotone.*
c) $\mathcal{O}X \subseteq X$, *for every* $X \in \mathcal{C}$,

Then $\{C\mathcal{O}X \mid X \in \mathcal{C}\}$ *is algebraic.*

Proof. Denote $\mathcal{D} = \{C\mathcal{O}X \mid X \in \mathcal{C}\}$. It is easy to see that 1) the composite operator $(C\mathcal{O}) : \mathcal{C} \to \mathcal{C}$ is monotone; 2) $C\mathcal{O}X \subseteq X$, for every $X \in \mathcal{C}$. Suppose $\{C\mathcal{O}X_i\}_{i\in I}$ is an upward-directed subfamily of $\mathcal{D}$. In order to prove that $\bigcup_{i\in I} C\mathcal{O}X_i \in \mathcal{D}$, it suffices to prove that

$$\bigcup_{i\in I} C\mathcal{O}X_i = C\mathcal{O}(\bigcup_{i\in I} C\mathcal{O}X_i).$$

$(\subseteq)$ $(\forall j \in I)\ C\mathcal{O}X_j \subseteq \bigcup_{i\in I} C\mathcal{O}X_i$

$$\overset{1)}{\Longrightarrow} (\forall j \in I)\ C\mathcal{O}(C\mathcal{O}X_j) = C(\mathcal{O}C\mathcal{O})X_j \overset{a)}{=} C\mathcal{O}X_j \subseteq C\mathcal{O}(\bigcup_{i\in I} C\mathcal{O}X_i)$$

$$\Longrightarrow \bigcup_{i\in I} C\mathcal{O}X_i \subseteq C\mathcal{O}(\bigcup_{i\in I} C\mathcal{O}X_i).$$

($\supseteq$) Since $\{\mathcal{COX}\}_{i\in I} \subseteq \mathcal{C}$ and $\mathcal{C}$ is algebraic, $\bigcup_{i\in I} C\mathcal{O}X_i \in \mathcal{C}$. Therefore

$$C\mathcal{O}(\textstyle\bigcup_{i\in I} C\mathcal{O}X_i) \overset{2)}{\subseteq} \textstyle\bigcup_{i\in I} C\mathcal{O}X_i. \qquad \Box$$

To deal with the issue of intersection in the families of finite closure relations, both full and axiomatic (the latter will be introduced in the next section), we need the following technical lemma.

Lemma 4.1.3. *Let $\mathcal{C}$ be a closure system in A, $C : \mathcal{P}(A) \to \mathcal{C}$ be the closure operator associated with $\mathcal{C}$ and $\mathcal{O} : \mathcal{C} \to \mathcal{P}(A)$ be an operator such that*

a) $\mathcal{O}C\mathcal{O} = \mathcal{O}$,

b) $\mathcal{O}$ is meet-continuous,

Then TFAE

1. *$\{C\mathcal{O}X \mid X \in \mathcal{C}\}$ is a closure system in A,*
2. *the composite operator $(C\mathcal{O}) \colon \mathcal{C} \to \mathcal{C}$ is meet-continuous,*
3. *for every family $\{X_i\}_{i\in I} \subseteq \mathcal{C}$: $\bigcap_{i\in I}(\mathcal{O}X_i)^{\mathcal{C}} = (\bigcap_{i\in I}\mathcal{O}X_i)^{\mathcal{C}}$.*

Proof. Denote $\mathcal{D} = \{\mathrm{C}\mathcal{O}X \mid X \in \mathcal{C}\}$.

($1 \Rightarrow 2$) Suppose $\{X_i\}_{i\in I}$ is a non-empty family of members of $\mathcal{C}$. Then, since $\mathcal{D}$ is a closure system in A, there is $X \in \mathcal{C}$ such that $C\mathcal{O}X = \bigcap_{i\in I} C\mathcal{O}X_i$ $(*)$. Then

$$\mathcal{O}C\mathcal{O}X = \mathcal{O}(C\mathcal{O}X) \overset{(*)}{=} \mathcal{O}(\textstyle\bigcap_{i\in I} C\mathcal{O}X_i) \overset{b)}{=} \textstyle\bigcap_{i\in I} \mathcal{O}C\mathcal{O}X_i \overset{a)}{=} \textstyle\bigcap_{i\in I} \mathcal{O}X_i.$$

$$\text{Therefore } C(\mathcal{O}C\mathcal{O})X = C(\textstyle\bigcap_{i\in I} \mathcal{O}X_i). \quad (**)$$

Thus $\bigcap_{i\in I} C\mathcal{O}X_i \overset{(*)}{=} C\mathcal{O}X \overset{a)}{=} C\mathcal{O}C\mathcal{O}X \overset{(**)}{=} C(\bigcap_{i\in I} \mathcal{O}X_i) \overset{b)}{=} C\mathcal{O}(\bigcap_{i\in I} X_i)$.

($2 \Rightarrow 3$) $\bigcap_{i\in I} C\mathcal{O}X_i \overset{2.}{=} C\mathcal{O}(\bigcap_{i\in I} X_i) \overset{a)}{=} C(\bigcap_{i\in I} \mathcal{O}X_i)$.

($3 \Rightarrow 1$) $\bigcap_{i\in I} C\mathcal{O}X_i \overset{3.}{=} C(\bigcap_{i\in I} \mathcal{O}X_i)$. $\Box$

4.2 Fully adequate Gentzen systems

A number of important non-protoalgebraic 1st-level deductive system that have a natural algebraic semantics also have a so-called fully adequate Gentzen system associated with them, the conjunction-disjunction fragment of the classical propositional logic being a paradigmatic example. In order to give a criterion for the existence of a fully adequate Gentzen system and provide a particular Gentzen axiomatization for it, we will proceed as follows: we define for an arbitrary 1st-level deductive system $\mathcal{S}$ the set $\mathbf{Fcr}\,\mathcal{S}$ of *full closure relations for* $\mathcal{S}$, that are closure relations counterparts for so-called *generalized full models for* $\mathcal{S}$ on $\mathbf{Fm}_{\mathcal{L}}$. If $\mathcal{S}$ has a fully adequate Gentzen system $\mathbf{Fcr}\,\mathcal{S}$ turns out to be a weakly algebraizable 2nd-level deductive system. In fact, the set of theories of the fully adequate Gentzen system is just $\mathbf{Fcr}\,\mathcal{S}$ itself, whenever $\mathcal{S}$ has theorems, and $\mathbf{Fcr}\,\mathcal{S} \cup \{\mathrm{Fm}^{+}_{\mathcal{L}}\}$ if it does not. Then, using properties of weakly algebraizable 2nd-level deductive systems, we will extract a particular axiomatization for the fully adequate Gentzen system.

Recall that in Chapter 2, we have defined for a 1st-level deductive system $\mathcal{S}$ the set $\mathbf{Gcr}\,\mathcal{S}$ of general closure relations for $\mathcal{S}$, which is essentially the set of theories of the trivial Gentzen axiomatization of $\mathcal{S}$ and standard Gentzen rules. This somewhat trivial Gentzen system plays nevertheless an important role of a framework for defining more interesting constructions.

Definition 4.2.1. The set of *full closure relations* for a 1st-level deductive system $\mathcal{S}$ is

$$\mathbf{Fcr}\,\mathcal{S} := \{(\mathbf{\Omega}\,\mathcal{A})^{\mathbf{Gcr}\,\mathcal{S}} \mid \mathcal{A} \in \mathbf{Gcr}\,\mathcal{S}\}.$$

An element of $\mathbf{Fcr}\,\mathcal{S}$ is called *a full closure relation for* $\mathcal{S}$. □

Note that the full closure relations for $\mathcal{S}$ arise from the strong Galois connection between $\mathrm{Con}\,\mathbf{Fm}_{\mathcal{L}} \subseteq \mathcal{P}(\mathrm{Fm}^{+}_{\mathcal{L}})$ and the closure system $\mathbf{Gcr}\,\mathcal{S} \subseteq \mathcal{P}(\mathrm{Fm}^{+}_{\mathcal{L}})$, inheriting properties formulated in Lemma 4.1.1

Proposition 4.2.2. *For any 1st-level deductive system* $\mathcal{S}$

1. $\mathbf{Fcr}\,\mathcal{S} \subseteq \mathbf{Gcr}\,\mathcal{S}$,
2. $\mathbf{Fcr}\,\mathcal{S} = \{(\theta)^{\mathbf{Gcr}\,\mathcal{S}} \mid \theta \in \mathrm{Con}\,\mathbf{Fm}_{\mathcal{L}}\}$,
3. $\mathcal{A} \in \mathbf{Fcr}\,\mathcal{S} \implies \mathcal{A} = \mathbf{R}_{\mathcal{L}}\{T \in \mathrm{Th}\,\mathcal{S} \mid \mathbf{\Omega}\,\mathcal{A} \subseteq \mathbf{\Omega}\,T\}$,
4. $\mathbf{Fcr}\,\mathcal{S}$ *is a closure system in* $\mathrm{Fm}^{+}_{\mathcal{L}}$ *iff for every non-empty family* $\{\mathcal{F}_i\}_{i\in I} \subseteq \mathbf{Fcr}\,\mathcal{S}$

$$\textstyle\bigcap_{i\in I}\mathcal{F}_i = (\bigcap_{i\in I}\mathbf{\Omega}\mathcal{F}_i)^{\mathbf{Gcr}\,\mathcal{S}}.$$

Proof. (1) By definition of $\mathbf{Fcr}\,\mathcal{S}$.

(2) ($\subseteq$) It holds by definition of $\mathbf{Fcr}\,\mathcal{S}$, since $\mathbf{\Omega}\,\mathcal{A} \in \mathrm{Con}\,\mathbf{Fm}_{\mathcal{L}}$.

(2) ($\supseteq$) Suppose $\mathcal{A} = (\theta)^{\mathbf{Gcr}\,\mathcal{S}}$, for some $\theta \in \mathrm{Con}\,\mathbf{Fm}_{\mathcal{L}}$. Then

$$\theta \overset{3.1.5}{\subseteq} \mathbf{\Omega}\,\mathcal{A} \overset{3.1.5}{\subseteq} \mathcal{A} \implies \mathcal{A} = (\theta)^{\mathbf{Gcr}\,\mathcal{S}} \subseteq (\mathbf{\Omega}\,\mathcal{A})^{\mathbf{Gcr}\,\mathcal{S}} \subseteq \mathcal{A}^{\mathbf{Gcr}\,\mathcal{S}} = \mathcal{A} \implies \mathcal{A} = (\mathbf{\Omega}\,\mathcal{A})^{\mathbf{Gcr}\,\mathcal{S}}.$$

(3) Let $\mathcal{A} \in \mathbf{Fcr}\,\mathcal{S}$. Since $\mathcal{A}$ is a general closure relation for $\mathcal{S}$, then $\mathcal{A} = \mathbf{R}_{\mathcal{L}}\,\mathcal{C}$, for some algebraic closure system $\mathcal{C} \subseteq \mathrm{Th}\,\mathcal{S}$. Let $\mathcal{D} = \{T \in \mathrm{Th}\,\mathcal{S} \mid \mathbf{\Omega}\,\mathcal{A} \subseteq \mathbf{\Omega}\,T\}$. Then it is straightforward to show, that $\mathcal{D}$ is an algebraic closure system on $\mathrm{Fm}_{\mathcal{L}}$. Also $\mathcal{C} \subseteq \mathcal{D}$, because

$$(\forall T \in \mathcal{C})\ \mathbf{\Omega}\,\mathcal{A} = \mathbf{\Omega}\,\mathbf{R}_{\mathcal{L}}\,\mathcal{C} \overset{3.1.9}{=} \widetilde{\mathbf{\Omega}}\,\mathcal{C} \overset{def}{=} \bigcap_{S \in \mathcal{C}} \mathbf{\Omega}\,S \subseteq \mathbf{\Omega}\,T.$$

Thus $\mathcal{A} = \mathbf{R}_{\mathcal{L}}\,\mathcal{D}$, because

$$(\supseteq)\quad \mathcal{C} \subseteq \mathcal{D} \implies \mathbf{R}_{\mathcal{L}}\,\mathcal{D} \subseteq \mathbf{R}_{\mathcal{L}}\,\mathcal{C} = \mathcal{A},$$

$$(\subseteq)\quad \mathbf{\Omega}\,\mathcal{A} \subseteq \bigcap\{\mathbf{\Omega}\,T \mid T \in \mathcal{D}\} \overset{def}{=} \widetilde{\mathbf{\Omega}}\,\mathcal{D} \overset{3.1.9}{=} \mathbf{\Omega}\,\mathbf{R}_{\mathcal{L}}\,\mathcal{D} \overset{3.1.5}{\subseteq} \mathbf{R}_{\mathcal{L}}\,\mathcal{D}$$

$$\implies \mathcal{A} \overset{4.2.1}{=} (\mathbf{\Omega}\,\mathcal{A})^{\mathbf{Gcr}\,\mathcal{S}} \subseteq (\mathbf{R}_{\mathcal{L}}\,\mathcal{D})^{\mathbf{Gcr}\,\mathcal{S}} = \mathbf{R}_{\mathcal{L}}\,\mathcal{D}.$$

(5) It suffices to check conditions of Lemma 4.1.3 for operators $\mathbf{\Omega} : \mathbf{Gcr}\,\mathcal{S} \to \mathrm{Con}\,\mathbf{Fm}_{\mathcal{L}}$ and $\mathrm{C} : \mathrm{Con}\,\mathbf{Fm}_{\mathcal{L}} \to \mathbf{Gcr}\,\mathcal{S}$, where C is the closure operator associated with the closure system $\mathbf{Gcr}\,\mathcal{S}$.

a) $\mathbf{\Omega} : \mathbf{Gcr}\,\mathcal{S} \to \mathrm{Con}\,\mathbf{Fm}_{\mathcal{L}}$ is monotone, hence meet-continuous.

b) By definition, $(\mathrm{C}\,\mathbf{\Omega}) : \mathbf{Fcr}\,\mathcal{S} \to \mathbf{Fcr}\,\mathcal{S}$ acts as identity on $\mathbf{Fcr}\,\mathcal{S}$, hence $\mathbf{\Omega}\,\mathrm{C}\,\mathbf{\Omega} = \mathbf{\Omega}$. □

It follows from Proposition 4.2.2(2) that the largest full closure relation always exists and it is obviously $(1_{\mathbf{Fm}_{\mathcal{L}}})^{\mathbf{Gcr}\,\mathcal{S}}$. It is easy to see that it is equal to $\mathrm{Fm}^{+}_{\mathcal{L}}$ if $\mathcal{S}$ has theorems, and to $\mathrm{Fm}^{+}_{\mathcal{L}} \setminus \triangleright\,\mathrm{Fm}_{\mathcal{L}}$ if $\mathcal{S}$ does not. In other words:

1) $\mathbf{Type}(\mathbf{Fcr}\,\mathcal{S}) = \omega$ iff $\mathcal{S}$ has theorems;

2) $\mathbf{Type}(\mathbf{Fcr}\,\mathcal{S}) = \omega \setminus \{0\}$ iff $\mathcal{S}$ has no theorems.

4.3 A criterion for the existence of the fully adequate Gentzen system

Let $\mathbf{A}$ be an algebra. The subset $F \subseteq A$ is called an *$\mathcal{S}$-filter on* $\mathbf{A}$ if $h^{-1}F \in \mathrm{Th}\,\mathcal{S}$ for every homomorphism $h : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$. The set of all filters on $\mathbf{A}$ is denoted by $\mathcal{F}i_{\mathcal{S}}\mathbf{A}$.

A *basic full 2nd-level model for* $\mathcal{S}$ is a 2nd-level matrix $\langle \mathbf{A}, \mathbf{R}_A(\mathcal{F}i_{\mathcal{S}}\mathbf{A})\rangle$; a *full 2nd-level model for* $\mathcal{S}$ is a matrix $\mathfrak{B} = \langle \mathbf{B}, \mathcal{B}\rangle$ such that there exists a basic full model $\mathcal{A}$ for which $\mathfrak{A} \preccurlyeq \mathfrak{B}$. It is easy to see that if $\mathfrak{A}$ is a full model for $\mathcal{S}$ and $\mathfrak{A} \preccurlyeq \mathfrak{C}$, then $\mathfrak{C}$ is also a full model for $\mathcal{S}$. Let $\mathbf{FGMod}\,\mathcal{S}$ be the class of all full 2nd-level models of $\mathcal{S}$.

Proposition 4.3.1. *Let* $\mathcal{S}$ *be a 1st-level deductive system. Then*

1. $\mathbf{Fcr}\,\mathcal{S} = \{\mathcal{A} \mid \langle \mathbf{Fm}_{\mathcal{L}}, \mathcal{A}\rangle \in \mathbf{FGMod}\,\mathcal{S}\}$,
2. $\mathbf{Fcr}\,\mathcal{S}$ *is closed under inverse surjective substitutions.*

Proof. (1) ($\subseteq$) Suppose $\mathcal{A} \in \mathbf{Fcr}\,\mathcal{S}$. Then, by Proposition 4.2.2(3), $\mathcal{A} = \mathcal{R}_{\mathcal{L}}\mathcal{C}$, where $\mathcal{C} = \{T \in \mathrm{Th}\,\mathcal{S} \mid \boldsymbol{\Omega}\,\mathcal{A} \subseteq \boldsymbol{\Omega}\,T\}$. Suppose $\mathbf{A} = \mathbf{Fm}_{\mathcal{L}} / \boldsymbol{\Omega}\,\mathcal{A}$ and let $h : \mathbf{Fm}_{\mathcal{L}} \twoheadrightarrow \mathbf{A}$ be the canonical homomorphism associated with this factor. Since h is a surjective homomorphism and $\boldsymbol{\Omega}\,\mathcal{A} = h^{-1}1_{\mathbf{A}}$, then $\mathcal{C} = h^{-1}\,\mathcal{F}i_{\mathcal{S}}\mathbf{A}$. Therefore

$$\mathcal{A} = \mathbf{R}_{\mathcal{L}}\,\mathcal{C} = \mathbf{R}_{\mathcal{L}}\,h^{-1}\,\mathcal{F}i_{\mathcal{S}}\mathbf{A} \overset{2.1.4}{=} h^{-1}\,\mathbf{R}_A(\mathcal{F}i_{\mathcal{S}}\mathbf{A}).$$

Thus $\langle \mathbf{Fm}_{\mathcal{L}}, \mathcal{A}\rangle \preccurlyeq \langle \mathbf{A}, \mathbf{R}_A(\mathcal{F}i_{\mathcal{S}}\mathbf{A})\rangle$ and therefore $\langle \mathbf{Fm}_{\mathcal{L}}, \mathcal{A}\rangle$ is a full 2nd-level model for $\mathcal{S}$, since $\langle \mathbf{A}, \mathbf{R}_A(\mathcal{F}i_{\mathcal{S}}\mathbf{A})\rangle$ is a basic full model for $\mathcal{S}$.

($\supseteq$) Suppose $\langle \mathbf{Fm}_{\mathcal{L}}, \mathcal{A}\rangle \in \mathbf{FGMod}\,\mathcal{S}$. Then there is an algebra $\mathbf{A}$ and a surjective homomorphism $h : \mathbf{Fm}_{\mathcal{L}} \twoheadrightarrow \mathbf{A}$ such that $\langle \mathbf{Fm}_{\mathcal{L}}, \mathcal{A}\rangle = h^{-1}\langle \mathbf{A}, \mathbf{R}_A(\mathcal{F}i_{\mathcal{S}}\mathbf{A})\rangle$. Let $\mathcal{C} = \{T \in \mathrm{Th}\,\mathcal{S} \mid \boldsymbol{\Omega}\,\mathcal{A} \subseteq \boldsymbol{\Omega}\,T\}$. Then $\mathcal{A} = h^{-1}\,\mathbf{R}_A(\mathcal{F}i_{\mathcal{S}}\mathbf{A}) \overset{2.1.4}{=} \mathbf{R}_{\mathcal{L}}\,h^{-1}\,\mathcal{F}i_{\mathcal{S}}\mathbf{A} = \mathbf{R}_{\mathcal{L}}\,\mathcal{C}$, so $\mathcal{A} \in \mathbf{Fcr}\,\mathcal{S}$, by Proposition 4.2.2(3).

(2) Suppose $\mathcal{A} \in \mathbf{Fcr}\,\mathcal{S}$ and σ is a surjective substitution. Then, by (1), $\langle \mathbf{Fm}_{\mathcal{L}}, \mathcal{A}\rangle$ is a full 2nd-level model for $\mathcal{S}$. Since $\langle \mathbf{Fm}_{\mathcal{L}}, \sigma^{-1}\mathcal{A}\rangle \preccurlyeq \langle \mathbf{Fm}_{\mathcal{L}}, \mathcal{A}\rangle$, therefore $\langle \mathbf{Fm}_{\mathcal{L}}, \sigma^{-1}\mathcal{A}\rangle$ is a full 2nd-level model for $\mathcal{S}$, hence, by (1), $\sigma^{-1}\mathcal{A} \in \mathbf{Fcr}\,\mathcal{S}$. □

We will also need

Lemma 4.3.2. *[14, Lemma 2.12] Let* $\mathcal{S}$ *be a deductive system of language type* $\mathcal{L}$*,* $\mathbf{A}$ *be an algebra of type* $\mathcal{L}$ *and* $\mathbf{K} = \{\langle \mathbf{B}, \mathbf{R}_B(\mathcal{F}i_{\mathcal{S}}\mathbf{B})\rangle \mid \mathbf{B} \subseteq_\omega \mathbf{A}\}$*. Then* $\mathbf{K}$ *is an upward-directed by* $\subseteq$ *set of 2nd-level matrices and* $\bigcup \mathbf{K} = \langle \mathbf{A}, \mathbf{R}_A(\mathcal{F}i_{\mathcal{S}}\mathbf{A})\rangle$.

Let $\mathbf{Inc}_{\mathcal{L}}$ be the class of all models of the form $\langle \mathbf{A}, A^{+}\rangle$, where $\mathbf{A}$ is an algebra of type $\mathcal{L}$.

Definition 4.3.3. A Gentzen system $\mathcal{G}$ is *fully adequate for a deductive system* $\mathcal{S}$ if

$$\mathbf{Mod}\,\mathcal{G} = \mathbf{FGMod}\,\mathcal{S} \cup \mathbf{Inc}_{\mathcal{L}}. \qquad \square$$

The reason for including the models $\mathbf{Inc}_{\mathcal{L}}$ is that when $\mathcal{S}$ has no theorems, $\mathbf{Type}(\mathbf{Fcr}\,\mathcal{S}) \neq \omega$, therefore $\mathbf{Fcr}\,\mathcal{S}$ cannot be a Gentzen system simply because of syntactical considerations. However the following holds

Theorem 4.3.4. *If $\mathcal{S}$ has a fully adequate Gentzen system, then $\mathbf{Fcr}\,\mathcal{S}$ is a weakly algebraizable 2nd-level deductive system, therefore $\mathbf{Fcr}\,\mathcal{S}$ has a graded congruence basis (see Definition 3.2.2).*

Proof. Suppose $\mathcal{S}$ has a fully adequate Gentzen system $\mathcal{G}$, so, by definition,

$$\mathbf{Mod}\,\mathcal{G} = \mathbf{FGMod}\,\mathcal{S} \cup \mathbf{Inc}_{\mathcal{L}}.$$

Restricting the last equality to 2nd-level matrices on $\mathbf{Fm}_{\mathcal{L}}$, we get that $\mathrm{Th}\,\mathcal{G} = \mathbf{Fcr}\,\mathcal{S} \cup \{\mathrm{Fm}_{\mathcal{L}}^{+}\}$. $\mathbf{Fcr}\,\mathcal{S}$ has the largest element, it is obviously $(1_{\mathbf{Fm}_{\mathcal{L}}})^{\mathbf{Gcr}\,\mathcal{S}}$. Since $\mathcal{G}$ is a Gentzen system, $\mathrm{Th}\,\mathcal{G}$ and, hence $\mathbf{Fcr}\,\mathcal{S}$, is closed under non-empty intersections. Also we have, by Lemma 4.1.2, that $\mathbf{Fcr}\,\mathcal{S}$ is algebraic, and, by Proposition 4.3.1(2), it is closed under inverse surjective substitutions, therefore, by Lemma 2.2.4, it is closed under inverse arbitrary substitutions, and hence it is a 2nd-level deductive system. Also $\boldsymbol{\Omega} : \mathbf{Fcr}\,\mathcal{S} \to \mathrm{Con}\,\mathbf{Fm}_{\mathcal{L}}$ is injective; hence $\mathbf{Fcr}\,\mathcal{S}$ is a weakly algebraizable 2nd-level deductive system, and, by Theorem 3.2.3, it has a graded congruence basis. $\square$

The following is a technical definition.

Definition 4.3.5. Let $\mathcal{E} = \bigcup_{n \in N} \varepsilon_n$ be a graded congruence basis for $\mathbf{Fcr}\,\mathcal{S}$. A 2nd-level matrix $\mathfrak{A} = \langle \mathbf{A}, \mathcal{A} \rangle$ is *congruence definable by $\mathcal{E}$* if $N = \mathbf{Type}\,\mathcal{A}$ and

$$\bar{a} \rhd a \in \mathcal{A} \iff \varepsilon_{|\bar{a}|}(\bar{a}, a) \subseteq \boldsymbol{\Omega}_{\mathbf{A}}\mathcal{A}.$$

Lemma 4.3.6. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be 2nd-level matrices such that $\mathfrak{B} \preccurlyeq \mathfrak{A}$. Then*

$$\mathfrak{A} \textit{ is congruence definable by } \mathcal{E} \textit{ iff } \mathfrak{B} \textit{ is congruence definable by } \mathcal{E}.$$

Proof. Suppose $\mathfrak{A} = \langle \mathbf{A}, \mathcal{A} \rangle \preccurlyeq \mathfrak{B} = \langle \mathbf{B}, \mathcal{B} \rangle$. Then, by definition, there exists a homomorphism $h : \mathbf{A} \twoheadrightarrow \mathbf{B}$ such that $\mathbf{B} = h\mathbf{A}$ and $\mathcal{A} = h^{-1}\mathcal{B}$. Thus, for each pair of sequents $\bar{a} \rhd a \in A^+$, $\bar{b} \rhd b \in B^+$, such that $h(\bar{a} \rhd a) = \bar{b} \rhd b$,

$$\bar{b} \rhd b = h(\bar{a} \rhd a) \in \mathcal{A} \iff \bar{a} \rhd a \in h^{-1}\mathcal{B} = \mathcal{A}. \quad (*)$$

($\Rightarrow$) Suppose $\mathcal{B}$ is congruence definable by $\mathcal{E}$. Then

$$\bar{a} \rhd a \in \mathcal{A} \overset{(*)}{\iff} h(\bar{a} \rhd a) \in \mathcal{B} \overset{4.3.5}{\iff} \varepsilon^{\mathbf{B}}_{|\bar{a}|}(h\bar{a}, ha) \subseteq \boldsymbol{\Omega}_{\mathbf{B}}\, \mathcal{B}$$

$$\iff h\varepsilon^{\mathbf{A}}_{|\bar{a}|}(\bar{a}, a) \subseteq \boldsymbol{\Omega}_{\mathbf{B}}\, \mathcal{B} \iff \varepsilon^{\mathbf{A}}_{|\bar{a}|}(\bar{a}, a) \subseteq h^{-1}\boldsymbol{\Omega}_{\mathbf{B}}\mathcal{B} = \boldsymbol{\Omega}_{\mathbf{A}} h^{-1}\mathcal{B} = \boldsymbol{\Omega}_{\mathbf{A}}\mathcal{A}.$$

($\Leftarrow$) Suppose $\mathcal{A}$ is congruence definable by $\mathcal{E}$. For any given $\bar{b} \rhd b \in \mathcal{B}$, let $\bar{a} \rhd a \in h^{-1}(\bar{b} \rhd b)$. Then

$$\bar{b} \rhd b \in \mathcal{B} \overset{(*)}{\iff} \bar{a} \rhd a \in \mathcal{A} \iff \varepsilon^{\mathbf{A}}_{|\bar{a}|}(\bar{a}, a) \subseteq \boldsymbol{\Omega}_{\mathbf{A}}\mathcal{A} = \boldsymbol{\Omega}_{\mathbf{A}} h^{-1}\mathcal{B} = h^{-1}\boldsymbol{\Omega}_{\mathbf{B}}\mathcal{B}$$

$$\iff h\varepsilon^{\mathbf{A}}_{|\bar{a}|}(\bar{a}, a) = \varepsilon^{\mathbf{B}}_{|\bar{a}|}(h\bar{a}, ha) = \varepsilon^{\mathbf{B}}_{|\bar{b}|}(\bar{b}, b) \subseteq \boldsymbol{\Omega}_{\mathbf{B}}\mathcal{B}. \qquad \square$$

Definition 4.3.7. If $\mathcal{E}$ is a graded congruence basis for $\mathbf{Fcr}\,\mathcal{S}$, define

$$G_{\mathcal{E}} := \left\{ \bar{x} \rhd x \vdash t\alpha \rhd t\beta \mid t \in \mathcal{T}_{\mathcal{L}}, \alpha \rhd \beta \in \varepsilon_{|\bar{x}|}(\bar{x}, x) \right\},$$

$$\mathcal{G}_{\mathcal{E}}(\mathcal{S}) := \begin{cases} \mathcal{G}_{\mathcal{E}} \cup (\mathrm{CR}) \cup (\vdash \mathbf{R}_{\mathcal{L}}\, \mathrm{Th}\,\mathcal{S}), & \text{if } \mathrm{Thm}\, S \neq \emptyset; \\ \mathcal{G}_{\mathcal{E}} \cup (\mathrm{CR}) \cup (\vdash \mathbf{R}_{\mathcal{L}}\, \mathrm{Th}\,\mathcal{S}) \cup \displaystyle\bigcup_{|\bar{x}| \in \omega} \frac{\rhd y}{\bar{x} \rhd x}, & \text{if } \mathrm{Thm}\, S = \emptyset. \end{cases} \qquad \square$$

Directly from the definition of $\mathcal{G}_{\mathcal{E}}(\mathcal{S})$, it follows that

1. if $\mathfrak{A} = \langle \mathbf{A}, \mathcal{A} \rangle \in \mathrm{Th}\, \mathcal{G}_{\mathcal{E}}(\mathcal{S})$, then $\mathcal{A}$ is a finite closure relation on A,
2. for every $\bar{a} \rhd a \in A^+$, if $\bar{a} \rhd a \in \mathcal{A}$, then $\varepsilon^{\mathbf{A}}_{|\bar{a}|}(\bar{a}, a) \subseteq \boldsymbol{\Omega}\, \mathcal{A}$.

Note that for defining $\mathcal{G}_{\mathcal{E}}(\mathcal{S})$ in the case when $\mathcal{S}$ has no theorems (i.e., when $\mathbf{Fcr}\,\mathcal{S}$ is a 2nd-level non Gentzen deductive system), we used the idea of Lemma 2.2.2.

Lemma 4.3.8. *If $\mathcal{E}$ is a graded congruence basis for* $\mathbf{Fcr}\,\mathcal{S}$*, then every full 2nd-level model of $\mathcal{S}$ is congruence definable by $\mathcal{E}$.*

Proof. The proof consists of three succinct steps:

(1) Let $\mathfrak{A} = \langle \mathbf{A}, \mathcal{A} \rangle$ be a full 2nd-level model for $\mathcal{S}$ over a finite algebra $\mathbf{A}$. Then $\mathfrak{A} \preccurlyeq \mathfrak{B}$, for some $\mathfrak{B} = \langle \mathrm{Fm}_{\mathcal{L}}, \mathcal{B} \rangle$, where $\mathcal{B} \in \mathbf{Fcr}\,\mathcal{S}$. By definition of the congruence basis, $\mathfrak{B}$ is congruence definable by $\mathcal{E}$, therefore, according to Lemma 4.3.6, $\mathfrak{A}$ is also congruence definable by $\mathcal{E}$.

(2) Suppose $\mathfrak{A} = \langle \mathbf{A}, \mathcal{A} \rangle$ is a basic full model for $\mathcal{S}$. Let

$$\mathbf{K} = \{ \langle \mathbf{B}, \mathbf{R}_B(\mathcal{F}i_{\mathcal{S}}\mathbf{B}) \rangle \mid \mathbf{B} \subseteq_\omega \mathbf{A} \}.$$

By (1), each $\mathfrak{B} \in \mathbf{K}$ is congruence definable by $\mathcal{E}$, hence $\mathfrak{B} \in \mathbf{Mod}\,\mathcal{G}_{\mathcal{E}}(\mathcal{S})$. By Lemma 2.3.4, $\bigcup \mathbf{K} \in \mathbf{Mod}\,\mathcal{G}_{\mathcal{E}}(\mathcal{S})$, therefore, by Lemma 4.3.2, the basic full model $\mathfrak{A} = \bigcup \mathbf{K} \in \mathbf{Mod}\,\mathcal{G}_{\mathcal{E}}(\mathcal{S})$, This secures one direction in Definition 4.3.5.

For the other direction, consider $\bar{a} \rhd a \in A^{n+1}$ such that $n \in N$ and suppose that $\varepsilon^{\mathbf{A}}_{|\bar{a}|}(\bar{a}, a) \subseteq \boldsymbol{\Omega}\,\mathcal{A}$. Let $\mathbf{B} = \mathbf{A}\langle \bar{a}, a \rangle$ be the subalgebra of $\mathbf{A}$ generated by the set of elements $\{\bar{a}, a\}$ and let $\mathcal{B} = \mathcal{A} \cap B^{+}$. Then

$$\mathcal{T}_{\mathbf{B}}\varepsilon^{\mathbf{B}}_{|\bar{a}|}(\bar{a}, a) = \mathcal{T}_{\mathbf{A}}\varepsilon^{\mathbf{A}}_{|\bar{a}|}(\bar{a}, a) \cap B^2 \subseteq (\boldsymbol{\Omega}\,\mathcal{A}) \cap B^{+} \subseteq \mathcal{A} \cap B^{+}.$$

By Lemma 3.1.8, $\mathcal{T}_{\mathbf{B}}\varepsilon^{\mathbf{B}}_{|\bar{a}|}(\bar{a}, a) \subseteq \boldsymbol{\Omega}(\mathcal{A} \cap B^{+}) = \boldsymbol{\Omega}\,\mathcal{B}$, hence, by (1), $\bar{a} \rhd a \in \mathcal{B}$. But $\mathcal{B} \subseteq \mathcal{A} \cap B^{+} \subseteq \mathcal{A}$, so $\bar{a} \rhd a \in \mathcal{A}$, as needed.

(3) Suppose $\mathfrak{A}$ is a full 2nd-level model of $\mathcal{S}$. Then $\mathfrak{B} \preccurlyeq \mathfrak{A}$ for some basic full 2nd-level model of $\mathfrak{B}$. By (2), $\mathfrak{B}$ is congruence definable by $\mathcal{E}$. Therefore, by Lem. 4.3.6, $\mathfrak{A}$ is congruence definable by $\mathcal{E}$. □

Now we are ready to prove the main theorem of this section.

Theorem 4.3.9.

If $\mathcal{E}$ is a graded congruence basis for $\mathbf{Fcr}\,\mathcal{S}$, then $\mathcal{G}_{\mathcal{E}}(\mathcal{S})$ is fully adequate for $\mathcal{S}$.

Proof. We need to prove that $\mathbf{Mod}\,\mathcal{G}_{\mathcal{E}}(\mathcal{S}) = \mathbf{FGMod}\,\mathcal{S} \cup \mathbf{Inc}_{\mathcal{L}}$.

($\supseteq$) Let $\mathfrak{A} = \langle \mathbf{A}, \mathcal{A} \rangle$ be a 2nd-level matrix. If $\mathfrak{A} \in \mathbf{Inc}_{\mathcal{L}}$ then it is a model of any Gentzen system, hence a model of $\mathcal{G}_{\mathcal{E}}(\mathcal{S})$. If $\mathfrak{A} = \langle \mathbf{A}, \mathcal{A} \rangle \in \mathbf{FGMod}\,\mathcal{S}$, then $\mathfrak{A}$ is a 2nd-level model for $\mathcal{S}$, hence a general closure relation for $\mathcal{S}$, therefore a model of for the Gentzen system $(\mathrm{CR}) \cup (\vdash \mathbf{R}_{\mathcal{L}}\,\mathrm{Th}\,\mathcal{S})$. Also, by Lemma 4.3.8, $\mathfrak{A}$ is congruence definable by $\mathcal{E}$, therefore a model of $\mathcal{G}_{\mathcal{E}}$. Finally, if $\mathcal{S}$ has no theorems, then $\mathcal{A}$ contains no sequent of the form $\rhd a$, therefore the rules $\rhd y \vdash \bar{x} \rhd x$ hold vacuously in $\mathfrak{A}$.

($\subseteq$) Suppose $\mathfrak{A} = \langle \mathbf{A}, \mathcal{A} \rangle \in \mathbf{Mod}\, \mathcal{G}_{\mathcal{E}}(\mathcal{S})$, where $\mathcal{E} = \bigcup_{n \in N} \varepsilon_n$. Then $\mathfrak{A}$ is a 2nd-level model for $\mathcal{S}$, since all the rules of $(\vdash \mathbf{R}_{\mathcal{L}}\, \mathrm{Th}\, \mathcal{S}) \cup (\mathrm{CR})$ hold in $\mathfrak{A}$. Let $\mathcal{B} = \langle \mathbf{A}, \mathcal{B} \rangle$ be a full 2nd-level model of $\mathcal{S}$, such that $\mathbf{\Omega}\, \mathcal{B} = \mathbf{\Omega}\, \mathcal{A}$. Then $\mathcal{B} \subseteq \mathcal{A}$. Also, for every $\bar{a} \rhd a$ such that $|\bar{a}| \in N$,

$$\bar{a} \rhd a \in \mathcal{A} \implies \mathcal{T}^{\mathbf{A}} \varepsilon^{\mathbf{A}}_{|\bar{a}|}(\bar{a}, a) \subseteq \mathcal{A} \overset{3.1.8}{\iff} \varepsilon^{\mathbf{A}}_{|\bar{a}|}(\bar{a}, a) \subseteq \mathbf{\Omega}\, \mathcal{A} = \mathbf{\Omega}\, \mathcal{B} \overset{4.3.8}{\iff} \bar{a} \rhd a \in \mathcal{B}.$$

If $|\bar{a}| \notin N$, for some $s = \bar{a} \rhd a \in \mathcal{A}$, then $N = \omega \setminus \{0\}$ and $s = \rhd a$. Since $N = \mathbf{Type}(\mathbf{Fcr}\, \mathcal{S}) = \omega \setminus \{0\}$, then $\mathcal{S}$ has no theorems, therefore all the rules $\rhd y \vdash \bar{x} \rhd x$ hold in $\mathfrak{A}$, hence $\mathcal{A} = A^{+}$. Thus, either $\mathcal{B} = \mathcal{A}$, and $\mathfrak{A} \in \mathbf{FGMod}\, S$, or $\mathcal{A} = A^{+}$ and $\mathfrak{A} \in \mathbf{Inc}_{\mathcal{L}}$. □

4.4 Protoalgebraic deductive systems with fully adequate Gentzen systems

Protoalgebraic deductive systems, that have fully adequate Gentzen systems, were studied in [14]. It was found there that the existence of fully adequate Gentzen system is equivalent to the existence of *Leibnitz-generating parameterized graded deduction-detachment system* (LPGDD system) $\mathbf{\Delta}$ *for* $\mathcal{S}$, where $\mathbf{\Delta} = \bigcup_{n \in \omega} \Delta_n \subseteq \mathbf{Fm}_{\mathcal{L}}$.

We can formulate a variety (namely, 18 = 3 × 6) of necessary conditions for a deductive system with a fully adequate Gentzen system, depending on properties of $\mathcal{G}_{\mathcal{E}}(\mathcal{S})$ and $\mathcal{S}$. We will list here only two extreme cases:

$\mathcal{G}_{\mathcal{E}}(\mathcal{S})$	$\mathcal{S}$	Relation
WA2	PA1	$X \vdash_{\mathcal{G}_{\mathcal{E}}(\mathcal{S})} Y \overset{WA}{\iff} \mathcal{E}X \vDash_{\mathcal{S}} \mathcal{E}Y \overset{PA}{\iff} (\Delta\mathcal{T}\mathcal{E})X \vdash_{\mathcal{S}} (\Delta\mathcal{T}\mathcal{E})Y$
FA1	FA1	$X \vdash_{\mathcal{G}_{\mathcal{E}}(\mathcal{S})} Y \overset{FA}{\iff} \mathcal{E}X \vDash_{\mathcal{S}} \mathcal{E}Y \overset{FA}{\iff} (\Delta\tilde{\mathcal{J}}\mathcal{E})X \vdash_{\mathcal{S}} (\Delta\tilde{\mathcal{J}}\mathcal{E})Y$

Thus, according to Definitions [14]

$$\mathbf{\Delta} = \textstyle\bigcup_{n \in \omega} \Delta(\mathcal{T}\varepsilon_n(\bar{x}, x))$$

is a Leibnitz-generating parameterized graded deduction-detachment system for $\mathcal{S}$.

Proposition 4.4.1. *If* $\mathbf{Fcr}\,\mathcal{S}$ *has a graded congruence basis, then* $\mathbf{Fcr}\,\mathcal{S}$ *is a closure system in* $\mathrm{Fm}_{\mathcal{L}}^{+}$.

Proof. Suppose $\{\mathcal{F}_i\}_{i\in I} \subseteq \mathbf{Fcr}\,\mathcal{S}$ and $\bigcup_{n\in N} \varepsilon_n(\bar{x}, x)$ is a graded congruence basis for $\mathbf{Fcr}\,\mathcal{S}$. By Proposition 4.2.2(4) it suffices to show that $\bigcap_{i\in I} \mathcal{F}_i = \Big(\mathbf{\Omega}(\bigcap_{i\in I} \mathcal{F}_i)\Big)^{\mathbf{Gcr}\,S}$. Thus

$$\begin{aligned} \bar{\alpha} \rhd \alpha \in \textstyle\bigcap_{i\in I} \mathcal{F}_i &\iff (\forall i \in I)\, \bar{\alpha} \rhd \alpha \in \mathcal{F}_i \\ &\iff (\forall i \in I)\, \varepsilon_{|\bar{\alpha}|}(\bar{\alpha}, \alpha) \in \mathbf{\Omega}\,\mathcal{F}_i \iff \varepsilon_{|\bar{\alpha}|}(\bar{\alpha}, \alpha) \in \textstyle\bigcap_{i\in I} \mathbf{\Omega}\,\mathcal{F}_i \\ &\iff \bar{\alpha} \rhd \alpha \in (\textstyle\bigcap_{i\in I} \mathbf{\Omega}\,\mathcal{F}_i)^{\mathbf{Gcr}\,\mathcal{S}} = \Big(\mathbf{\Omega}(\textstyle\bigcap_{i\in I} \mathcal{F}_i)\Big)^{\mathbf{Gcr}\,S} \in \mathbf{Fcr}\,\mathcal{S}. \end{aligned}$$

□

5. AXIOMATIC CLOSURE RELATIONS

Historically the examples of 2nd-level deductive systems were provided by the studies of 1st-level deductive systems. In particular, in this section we will show that *a multiterm deduction-detachment theorem* holds in a 1st-level deductive system with theorems if and only if the set of *axiomatic closure relations of* $\mathcal{S}$ forms a 2nd-level deductive system. Axiomatic finite closure relation for $\mathcal{S}$ can be defined as a general closure relation generated in a closure system of all general closure relations by sets of trivial sequents (i.e., the sequents of the form $\rhd\alpha$, $\alpha \in \mathrm{Fm}_{\mathcal{L}}$).

5.1 Axiomatic closure relations

Define for every $X \subseteq \mathbf{Fm}_{\mathcal{L}}$ and every $\mathcal{A} \subseteq \mathrm{Fm}^{+}_{\mathcal{L}}$

$$\rhd X := \{\, \rhd \alpha \mid \alpha \in X\}, \quad \mathrm{Thm}\, \mathcal{A} := \{\alpha \in \mathrm{Fm}_{\mathcal{L}} \mid \rhd \alpha \in \mathcal{A}\}, \quad \mathbf{\Theta}\, \mathcal{A} := \{\, \rhd \alpha \in \mathrm{Fm}_{\mathcal{L}} \mid \rhd \alpha \in \mathcal{A}\}.$$

Thus we get operators

$$(\rhd) : \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}) \to \mathcal{P}(\mathrm{Fm}^{1}_{\mathcal{L}}), \quad \mathrm{Thm} : \mathcal{P}(\mathrm{Fm}^{+}_{\mathcal{L}}) \to \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}), \quad \mathbf{\Theta} : \mathcal{P}(\mathrm{Fm}^{+}_{\mathcal{L}}) \to \mathcal{P}(\mathrm{Fm}^{1}_{\mathcal{L}}),$$

where $\mathrm{Fm}^{1}_{\mathcal{L}}$ by definition is $\{\langle\alpha\rangle \mid \alpha \in \mathrm{Fm}_{\mathcal{L}}\}$. Mnemonically, $\mathbf{\Theta}$ stands for "Theorems".

Definition 5.1.1. For a deductive system $\mathcal{S}$, define a set of *axiomatic closure relations of* $\mathcal{S}$:

$$\mathbf{Acr}\, \mathcal{S} := \{(\rhd T)^{\mathbf{Gcr}\, \mathcal{S}} \mid T \in \mathrm{Th}\, \mathcal{S}\}.$$

An element of $\mathbf{Acr}\, \mathcal{S}$ is called *an axiomatic closure relations for* $\mathcal{S}$. □

We will see that the operator $\mathbf{\Theta} : \mathbf{Acr}\, \mathcal{S} \to \mathcal{P}(\mathrm{Fm}^{+}_{\mathcal{L}})$ plays for axiomatic closure relations a role, to some degree, similar to that of the Leibnitz operator $\mathbf{\Omega} : \mathbf{Fcr}\, \mathcal{S} \to \mathrm{Con}\, \mathbf{Fm}_{\mathcal{L}}$ for full closure relations of $\mathcal{S}$.

Proposition 5.1.2. *For any 1st-level deductive system* $\mathcal{S}$

1. $\mathbf{Acr}\,\mathcal{S} \subseteq \mathbf{Gcr}\,\mathcal{S}$,
2. $\mathcal{A} \in \mathbf{Gcr}\,\mathcal{S} \implies \mathrm{Thm}\,\mathcal{A} \in \mathrm{Th}\,\mathcal{S}$,
3. $\mathbf{Acr}\,\mathcal{S} = \{(\triangleright X)^{\mathbf{Gcr}\,\mathcal{S}} \mid X \subseteq \mathrm{Fm}_{\mathcal{L}}\}$,
4. $\mathbf{Acr}\,\mathcal{S} = \{\mathbf{R}\,[T)_{\mathrm{Th}\,\mathcal{S}} \mid T \in \mathrm{Th}\,\mathcal{S}\}$.
5. *For every* $X \subseteq \mathrm{Fm}_{\mathcal{L}}$,

$$\bar{\alpha} \triangleright \alpha \in (\triangleright X)^{\mathbf{Gcr}\,\mathcal{S}} \iff \alpha \in \{\bar{\alpha}\}^{\mathcal{S}} \vee X^{\mathcal{S}} \iff X, \bar{\alpha} \vdash_{\mathcal{S}} \alpha.$$

Proof. 1. By definition.

2. $\mathcal{A} \overset{def}{=} (\mathbf{\Theta}\,\mathcal{A})^{\mathbf{Gcr}\,\mathcal{S}} \implies \mathbf{\Theta}\,\mathcal{A} = \mathbf{\Theta}(\mathbf{\Theta}\,\mathcal{A})^{\mathbf{Gcr}\,\mathcal{S}}$.

3. Suppose $\mathcal{A} \in \mathbf{Gcr}\,\mathcal{S}$, then, by definition, $\mathcal{A} = \mathbf{R}\,\mathcal{C}$, for some algebraic closure system $\mathcal{C} \subseteq \mathrm{Th}\,\mathcal{S}$. We have $\mathrm{Thm}\,\mathcal{A} = \bigcap \mathcal{C}$, because

$$\alpha \in \bigcap \mathcal{C} \iff \triangleright \alpha \in \mathbf{R}\,\mathcal{C} = \mathcal{A} \iff \alpha \in \mathrm{Thm}\,\mathcal{A}.$$

Therefore, since $\mathcal{C} \subseteq \mathrm{Th}\,\mathcal{S}$, $\mathrm{Thm}\,\mathcal{A} \in \mathrm{Th}\,\mathcal{S}$.

3. If $\mathcal{A} \in \mathbf{Acr}\,\mathcal{S}$, then $\mathcal{A} = (\mathbf{\Theta}\,\mathcal{A})^{\mathbf{Gcr}\,\mathcal{S}} = (\triangleright\,\mathrm{Thm}\,\mathcal{A})^{\mathbf{Gcr}\,\mathcal{S}}$. For the other direction, suppose $\mathcal{A} = (\triangleright X)^{\mathbf{Gcr}\,\mathcal{S}}$, for some $X \subseteq \mathrm{Fm}_{\mathcal{L}}$. Then $\mathcal{A} = (\mathbf{\Theta}\,\mathcal{A})^{\mathbf{Gcr}\,\mathcal{S}}$, because

($\supseteq$) $\mathbf{\Theta}\,\mathcal{A} \subseteq \mathcal{A} \implies (\mathbf{\Theta}\,\mathcal{A})^{\mathbf{Gcr}\,\mathcal{S}} \subseteq \mathcal{A}^{\mathbf{Gcr}\,\mathcal{S}} = \mathcal{A}$,

($\subseteq$) $\mathcal{A} = (\triangleright X)^{\mathbf{Gcr}\,\mathcal{S}} \implies \triangleright X \subseteq \mathcal{A} \implies \triangleright X \subseteq \mathbf{\Theta}\,\mathcal{A} \implies \mathcal{A} = (\triangleright X)^{\mathbf{Gcr}\,\mathcal{S}} \subseteq (\mathbf{\Theta}\,\mathcal{A})^{\mathbf{Gcr}\,\mathcal{S}}$.

4. Suppose $\mathcal{A} \in \mathbf{Acr}\,\mathcal{S}$. Then, by 3), $\mathcal{A} = (\triangleright T)^{\mathbf{Gcr}\,\mathcal{S}}$, where $T = \mathrm{Thm}\,\mathcal{A} \in \mathrm{Th}\,\mathcal{S}$. Let $\mathcal{C} = [T)_{\mathrm{Th}\,\mathcal{S}}$. Being a finite closure relation for $\mathcal{S}$, $\mathcal{A} = \mathbf{R}\,\mathcal{D}$, for some algebraic closure system $\mathcal{D} \subseteq \mathrm{Th}\,\mathcal{S}$. Then $\mathcal{A} = \mathbf{R}\,\mathcal{C}$, because

($\supseteq$) $T \overset{(3)}{=} \bigcap \mathcal{D} \Longrightarrow \mathcal{D} \subseteq [T)_{\mathrm{Th}\,\mathcal{S}} = \mathcal{C} \implies \mathbf{R}\,\mathcal{C} \subseteq \mathbf{R}\,\mathcal{D} = \mathcal{A}$,

($\subseteq$) $\mathbf{\Theta}\,\mathbf{R}\,\mathcal{D} = \triangleright(\bigcap \mathcal{D}) = \triangleright T = \triangleright(\bigcap \mathcal{C}) = \mathbf{\Theta}\,\mathbf{R}\,\mathcal{C} \Longrightarrow \triangleright T \subseteq \mathbf{R}\,\mathcal{C} \Longrightarrow \mathcal{A} = (\triangleright T)^{\mathbf{Gcr}\,\mathcal{S}} \subseteq \mathbf{R}\,\mathcal{C}$.

5. $\bar{\alpha} \triangleright \alpha \in (\triangleright T)^{\mathbf{Gcr}\,\mathcal{S}} \overset{(3)}{=} \mathbf{R}\,[T)_{\mathrm{Th}\,\mathcal{S}}$

$$\iff \alpha \in \{\bar{\alpha}\}^{[T)_{\mathrm{Th}\,\mathcal{S}}} = (T \cup \{\bar{\alpha}\})^{\mathcal{S}} = T \vee \{\bar{\alpha}\}^{\mathcal{S}} \iff T, \bar{\alpha} \vdash_{\mathcal{S}} \alpha. \quad \square$$

Lemma 5.1.3. $\mathbf{Acr}\,\mathcal{S}$ *is a closure system iff for every family* $\{\mathcal{A}_i\}_{i\in I} \subseteq \mathbf{Acr}\,\mathcal{S}$

$$\textstyle\bigcap_{i\in I}\mathcal{A}_i = (\bigcap_{i\in I}\boldsymbol{\Theta}\mathcal{A}_i)^{\mathbf{Gcr}\,\mathcal{S}}.$$

Proof. It follows directly from the implications

$$\begin{aligned}
&(\Rightarrow) \quad \boldsymbol{\Theta}(\textstyle\bigcap_{i\in I}\mathcal{A}_i) = \bigcap_{i\in I}\boldsymbol{\Theta}\mathcal{A}_i \\
&\qquad \Longrightarrow \textstyle\bigcap_{i\in I}\mathcal{A}_i \overset{5.1.2(2)}{=} \Big(\boldsymbol{\Theta}(\bigcap_{i\in I}\mathcal{A}_i)\Big)^{\mathbf{Gcr}\,\mathcal{S}} = (\bigcap_{i\in I}\boldsymbol{\Theta}\mathcal{A}_i)^{\mathbf{Gcr}\,\mathcal{S}}. \\
&(\Leftarrow) \quad \textstyle\bigcap_{i\in I}\boldsymbol{\Theta}\mathcal{A}_i \subseteq \triangleright\mathrm{Fm}_{\mathcal{L}} \Longrightarrow \bigcap_{i\in I}\mathcal{A}_i = (\bigcap_{i\in I}\boldsymbol{\Theta}\mathcal{A}_i)^{\mathbf{Gcr}\,\mathcal{S}} \in \mathbf{Acr}\,\mathcal{S}. \quad \square
\end{aligned}$$

5.2 Deduction-Detachment Theorem

The following is a standard definition.

Definition 5.2.1. A deductive system $\mathcal{S}$ has a *deduction-detachment theorem* (DDT_Δ) with respect to a finite (may be empty) set $\Delta(x,y)$ of formulas of two variables if

$$\begin{array}{lll}
(1) & x, \Delta(x,y) \vdash_{\mathcal{S}} y & \Delta\text{-}\textit{detachment}, \\[1ex]
(2) & \dfrac{\Gamma, x \vdash_{\mathcal{S}} y}{\Gamma \vdash_{\mathcal{S}} \Delta(x,y)} & \Delta\text{-}\textit{deduction}. \qquad \square
\end{array}$$

Lemma 5.2.2. *Suppose* $\mathbf{Acr}\,\mathcal{S}$ *for some deductive system* $\mathcal{S}$ *is a closure system, and let* $\Delta(x,y)$ *be a nonempty set of formulas of two variables. Then* $\mathcal{S}$ *has* DDT_Δ *iff*

$$\{x \triangleright y\}^{\mathbf{Acr}\,\mathcal{S}} = (\triangleright\Delta)^{\mathbf{Acr}\,\mathcal{S}}.$$

Proof. For readability sake, we write $\Delta(x,y)$ as Δ inside closure operators.

(1) $x \triangleright y \in (\triangleright\Delta)^{\mathbf{Acr}\,\mathcal{S}} \overset{5.1.1}{=} (\triangleright\Delta)^{\mathbf{Gcr}\,\mathcal{S}}$

$\overset{5.1.2(5)}{\Longrightarrow} y \in \{x\}^{\mathcal{S}} \vee (\Delta)^{\mathcal{S}} \Longrightarrow x, \Delta(x,y) \vdash_{\mathcal{S}} y.$ / Δ-detachment

(2) $\bar{z}, x \triangleright y \in \mathcal{A} \in \mathbf{Acr}\,\mathcal{S} \overset{5.1.2(5)}{\Longrightarrow} x \triangleright y \in (\triangleright(\{\bar{z}\}^{\mathcal{S}} \vee \mathrm{Thm}\,\mathcal{A}))^{\mathbf{Acr}\,\mathcal{S}}$

$\Longrightarrow (\triangleright\Delta)^{\mathbf{Gcr}\,\mathcal{S}} \overset{(*)}{=} \{x \triangleright y\}^{\mathbf{Acr}\,\mathcal{S}} \subseteq (\triangleright(\{\bar{z}\}^{\mathcal{S}} \vee \mathrm{Thm}\,\mathcal{A}))^{\mathbf{Acr}\,\mathcal{S}}$

$\Longrightarrow \Delta(x,y) \subseteq \{\bar{z}\}^{\mathcal{S}} \vee \mathrm{Thm}\,\mathcal{A} \overset{5.1.2(5)}{\Longrightarrow} \bar{z} \vdash_{\mathcal{S}} \Delta(x,y).$ / Δ-deduction $\square$

Note that the inconsistent 1st-level deductive system $\mathcal{S} = \langle \mathrm{Fm}_{\mathcal{L}}, \{\mathrm{Fm}_{\mathcal{L}}\}\rangle$ over the language $\mathcal{L}$ has DDT_{Δ} with respect to any finite set $\Delta \subseteq \mathrm{Fm}_{\mathcal{L}}$ of formulas, because

$$(\triangleright \Delta)^{\mathbf{Gcr}\,\mathcal{S}} = \mathrm{Fm}^{+}_{\mathcal{L}} = \{x \triangleright y\}^{\mathbf{Acr}\,\mathcal{S}}.$$

We also define that the almost inconsistent deductive system $\mathcal{S} = \langle \mathrm{Fm}_{\mathcal{L}}, \{\emptyset, \mathrm{Fm}_{\mathcal{L}}\}\rangle$ over $\mathcal{L}$ has $\mathrm{DDT}_{\emptyset}$, because

$$\{x \triangleright y\}^{\mathbf{Acr}\,\mathcal{S}} = \mathbf{R}_{\mathcal{L}}\{\emptyset, \mathrm{Fm}_{\mathcal{L}}\} = \mathrm{Fm}^{+}_{\mathcal{L}} \setminus \mathrm{Fm}^{1}_{\mathcal{L}} = (\emptyset)^{\mathbf{Gcr}\,\mathcal{S}}.$$

Theorem 5.2.3. *Let $\mathcal{S}$ be a 1st-level deductive system with theorems. Then $\mathbf{Acr}\,\mathcal{S}$ is a Gentzen system iff $\mathcal{S}$ has a multiterm deduction-detachment theorem.*

Proof. In view of the remarks above, it suffices to prove the theorem for $\mathcal{S}$ that is neither inconsistent nor almost inconsistent.

($\Rightarrow$) Suppose $\mathbf{Acr}\,\mathcal{S}$ is a closure system, then there is a closure of the set $\{x \triangleright y\}$ in $\mathbf{Acr}\,\mathcal{S}$. If $\{x \triangleright y\}^{\mathbf{Acr}\,\mathcal{S}} = (\emptyset)^{\mathbf{Gcr}\,\mathcal{S}}$, then

$$\{x \triangleright y\}^{\mathbf{Acr}\,\mathcal{S}} = (\emptyset)^{\mathbf{Gcr}\,\mathcal{S}} \implies x \triangleright y \in (\emptyset)^{\mathbf{Gcr}\,\mathcal{S}} \overset{5.1.2(5)}{\implies} x \vdash_{\mathcal{S}} y,$$

so $\mathcal{S}$ is either inconsistent or almost inconsistent, a contradiction with the assumption. Thus $\{x \triangleright y\}^{\mathbf{Acr}\,\mathcal{S}} = (\triangleright T)^{\mathbf{Gcr}\,\mathcal{S}}$, for some $T \in \mathrm{Th}\,\mathcal{S}$, such that $T \neq \mathrm{Thm}\,\mathcal{S}$. Since $\{x \triangleright y\}^{\mathbf{Acr}\,\mathcal{S}}$ is compact in $\mathbf{Acr}\,\mathcal{S}$, there is a finite subset $\mathcal{O} \subseteq T$, such that $\{x \triangleright y\}^{\mathbf{Acr}\,\mathcal{S}} = \mathcal{O}^{\mathbf{Gcr}\,\mathcal{S}}$. Suppose σ is any substitutions such that $\sigma\{x, y\} = \{x, y\}$ and $\sigma(\mathrm{Var} \setminus \{x, y\}) \subseteq \{x, y\}$ and let $\Delta(x, y) = \sigma\mathcal{O}$. Since $\mathbf{Acr}\,\mathcal{S}$ is a 2nd-level deductive system, it is invariant under inverse substitutions, therefore

$$\{x \triangleright y\}^{\mathbf{Acr}\,\mathcal{S}} = \{\sigma x \triangleright \sigma y\}^{\mathbf{Acr}\,\mathcal{S}} = (\sigma\mathcal{O})^{\mathbf{Acr}\,\mathcal{S}} = (\triangleright\Delta(x, y))^{\mathbf{Acr}\,\mathcal{S}}.$$

Therefore, by Lemma 5.2.2, $\mathcal{S}$ has DDT_{Δ}.

($\Leftarrow$) Suppose $\mathcal{S}$ has DDT_{Δ}, where $\Delta \neq \emptyset$. Δ can be viewed as a function $\Delta : \mathrm{Fm}^{2}_{\mathcal{L}} \subseteq \mathrm{Fm}^{+}_{\mathcal{L}} \to \mathcal{P}(\mathrm{Fm}_{\mathcal{L}})$. Furthermore it can be extended to a function from $\mathrm{Fm}^{+}_{\mathcal{L}}$ to $\mathcal{P}(\mathrm{Fm}_{\mathcal{L}})$ inductively by

$$\Delta(\triangleright\alpha) := \alpha, \quad \Delta(\bar{\alpha}, \alpha_{|\bar{\alpha}|} \triangleright \alpha) := \Delta(\bar{\alpha}, \Delta(\alpha_{|\bar{\alpha}|}, \alpha)) = \textstyle\bigcup_{\delta\in\Delta}\{\Delta(\bar{\alpha} \triangleright \delta(\alpha_{|\bar{\alpha}|}, \alpha))\},$$

and further, in the usual way, to a complex function $\Delta : \mathcal{P}(\mathrm{Fm}^{+}_{\mathcal{L}}) \to \mathcal{P}(\mathrm{Fm}_{\mathcal{L}})$. Then for every

$\mathcal{A} \in \mathbf{Acr}\,\mathcal{S}$ holds

$$(1)\quad \rhd\alpha \in \mathcal{A} \iff \Delta(\rhd\alpha) \overset{def}{=} \alpha \in \operatorname{Thm}\mathcal{A}$$

$$(2)\quad \bar{\alpha}, \alpha_{|\bar{\alpha}|} \rhd \alpha \in \mathcal{A} \overset{5.1.2(5)}{\iff} \operatorname{Thm}\mathcal{A}, \bar{\alpha}, \alpha_{|\bar{\alpha}|} \vdash_{\mathcal{S}} \alpha \overset{5.2.1}{\iff} \operatorname{Thm}\mathcal{A}, \bar{\alpha} \vdash_{\mathcal{S}} \Delta(\alpha_{|\bar{\alpha}|}, \alpha)$$
$$\overset{5.1.2(5)}{\iff} \bar{\alpha} \rhd \Delta(\alpha_{|\bar{\alpha}|}, \alpha) \subseteq \mathcal{A} \iff \ldots \iff \rhd\Delta(\bar{\alpha}, \alpha_{|\bar{\alpha}|} \rhd \alpha) \subseteq \mathcal{A}$$
$$\iff \Delta(\bar{\alpha}, \alpha_{|\bar{\alpha}|} \rhd \alpha) \subseteq \operatorname{Thm}\mathcal{A}.$$

In other words: $\qquad \bar{\alpha} \rhd \alpha \in \mathcal{A} \iff \Delta(\bar{\alpha} \rhd \alpha) \in \operatorname{Thm}\mathcal{A}.$

Then, for every family $\{\mathcal{A}_i\}_{i\in I} \subseteq \mathbf{Acr}\,\mathcal{S}$,

$$\bar{\alpha} \rhd \alpha \in \textstyle\bigcap_{i\in I}\mathcal{A}_i \iff (\forall\, i \in I)\ \bar{\alpha} \rhd \alpha \in \mathcal{A}_i$$
$$\iff (\forall\, i \in I)\ \Delta(\bar{\alpha} \rhd \alpha) \subseteq \operatorname{Thm}\mathcal{A}_i \iff \Delta(\bar{\alpha} \rhd \alpha) \subseteq \textstyle\bigcap_{i\in I} \operatorname{Thm}\mathcal{A}_i$$
$$\iff \bar{\alpha} \rhd \alpha \in (\rhd(\textstyle\bigcap_{i\in I} \operatorname{Thm}\mathcal{A}_i))^{\mathbf{Gcr}\,\mathcal{S}} = (\textstyle\bigcap_{i\in I}\mathbf{\Theta}\mathcal{A}_i)^{\mathbf{Gcr}\,\mathcal{S}} \in \mathbf{Acr}\,\mathcal{S}.$$

Thus, by Lemma 5.1.3, $\mathbf{Acr}\,\mathcal{S}$ is closed under arbitrary intersections, hence a closure system.

Now suppose $\mathcal{A} \in \mathbf{Acr}\,\mathcal{S}$ and σ is any substitution. Then

$$\bar{\alpha} \rhd \alpha \in \sigma^{-1}\mathcal{A} \iff \sigma(\bar{\alpha} \rhd \alpha) \in \mathcal{A} \iff \Delta(\sigma\bar{\alpha} \rhd \sigma\alpha) = \sigma\Delta(\bar{\alpha} \rhd \alpha) \subseteq \operatorname{Thm}\mathcal{A}$$
$$\iff \Delta(\bar{\alpha} \rhd \alpha) \subseteq \sigma^{-1}(\operatorname{Thm}\mathcal{A}) \iff \bar{\alpha} \rhd \alpha \in (\sigma^{-1}\mathbf{\Theta}\mathcal{A})^{\mathbf{Gcr}\,\mathcal{S}} \in \mathbf{Acr}\,\mathcal{S}.$$

Thus, in addition to being a closure system, $\mathbf{Acr}\,\mathcal{S}$ is closed under inverse substitutions, therefore it is a 2nd-level deductive system. □

A multiterm deduction-detachment theorem is rather a strong property of 1st-level deductive systems. It is interesting to investigate weaker and constituent conditions of it. One was suggested in [9, p. 164]:

Definition 5.2.4. A complete lattice L is *infinitely meet-distributive over compact elements* if for every compact element $c \in L$ and every family $\{a_i\}_{i\in I} \subseteq L$

$$c \vee \left(\textstyle\bigwedge_{i\in I} a_i\right) = \textstyle\bigwedge_{i\in I}(c \vee a_i). \qquad \square$$

J. Czelakowski in [9, Theorem 2.6.8.] had shown that for a protoalgebraic deductive system $\mathcal{S}$ to have a multiterm Deduction-Detachment theorem is equivalent to that the lattice $\operatorname{Th}\mathcal{S}$ is infinitely meet-distributive over compact elements, therefore the condition that $\mathbf{Acr}\,\mathcal{S}$ is invariant can be dropped under conditions of protoalgebraicity.

Theorem 5.2.5. $\mathbf{Acr}\,\mathcal{S}$ *is a closure system iff the lattice* $\mathrm{Th}\,\mathcal{S}$ *is infinitely meet-distributive over compact elements.*

Proof. In this theorem $\vee$ denotes the join in the complete lattice $\mathrm{Th}\,\mathcal{S}$.

($\Rightarrow$) Suppose $\mathbf{Acr}\,\mathcal{S}$ is closed under arbitrary intersections. Let $\{T_i\}_{i\in I} \subseteq \mathrm{Th}\,\mathcal{S}$ and $\{\bar{\alpha}\} \subseteq_\omega \mathrm{Fm}_{\mathcal{L}}$. Then for any $\alpha \in \mathrm{Fm}_{\mathcal{L}}$

$$\alpha \in \bigcap_{i\in I}(\{\bar{\alpha}\}^{\mathcal{S}} \vee T_i) \iff (\forall i \in I)\ \alpha \in \{\bar{\alpha}\}^{\mathcal{S}} \vee T_i$$

$$\iff (\forall i \in I)\ \bar{\alpha} \rhd \alpha \in (\rhd T_i)^{\mathbf{Gcr}\,\mathcal{S}} \iff \bar{\alpha} \rhd \alpha \in \bigcap_{i\in I}(\rhd T_i)^{\mathbf{Gcr}\,\mathcal{S}}$$

$$\overset{5.1.3}{\iff} \bar{\alpha} \rhd \alpha \in \big(\bigcap_{i\in I}(\rhd T_i)\big)^{\mathbf{Gcr}\,\mathcal{S}} = \big(\rhd(\bigcap_{i\in I} T_i)\big)^{\mathbf{Gcr}\,\mathcal{S}} \iff \alpha \in \{\bar{\alpha}\}^{\mathcal{S}} \vee (\bigcap_{i\in I} T_i).$$

Thus $\bigcap_{i\in I}(\{\bar{\alpha}\}^{\mathcal{S}} \vee T_i) = \{\bar{\alpha}\}^{\mathcal{S}} \vee (\bigcap_{i\in I} T_i)$.

($\Leftarrow$) By contradiction. Suppose $\mathrm{Th}\,\mathcal{S}$ is infinitely meet-distributive over compact elements, but there is a family $\{\mathcal{A}_i\}_{i\in I} \subseteq \mathbf{Acr}\,\mathcal{S}$ and a sequent $\bar{\alpha} \rhd \alpha$ such that $\bar{\alpha} \rhd \alpha \in \bigcap_{i\in I}\mathcal{A}_i$ and $\bar{\alpha} \rhd \alpha \notin (\bigcap_{i\in I} \mathrm{Thm}\,\mathcal{A}_i)^{\mathbf{Gcr}\,\mathcal{S}}$. Then

1) $\bar{\alpha} \rhd \alpha \notin (\bigcap_{i\in I} \mathbf{\Theta}\,\mathcal{A}_i)^{\mathbf{Gcr}\,\mathcal{S}} \overset{5.1.2(5)}{\iff} \alpha \notin \{\bar{\alpha}\}^{\mathcal{S}} \vee (\bigcap_{i\in I} \mathrm{Thm}\,\mathcal{A}_i)$,

2) $\bar{\alpha} \rhd \alpha \in \bigcap_{i\in I}\mathcal{A}_i \iff (\forall i \in I)\ \bar{\alpha} \rhd \alpha \in \mathcal{A}_i \iff (\forall i \in I)\ \alpha \in \{\bar{\alpha}\}^{\mathcal{S}} \vee \mathrm{Thm}\,\mathcal{A}_i$
$\iff \alpha \in \bigcap_{i\in I}(\{\bar{\alpha}\}^{\mathcal{S}} \vee \mathrm{Thm}\,\mathcal{A}_i)$.

But, by assumption, $\{\bar{\alpha}\}^{\mathcal{S}} \vee (\bigcap_{i\in I} \mathrm{Thm}\,\mathcal{A}_i) = \bigcap_{i\in I}(\{\bar{\alpha}\}^{\mathcal{S}} \vee \mathrm{Thm}\,\mathcal{A}_i)$, a contradiction. ☐

Note that $\mathrm{Th}\,\mathcal{S}$ is always infinitely join-distributive over compact elements

$$\{\bar{\alpha}\}^{\mathcal{S}} \cap (\bigvee_{i\in I} \mathrm{Thm}\,\mathcal{A}_i) = \bigvee_{i\in I}(\{\bar{\alpha}\}^{\mathcal{S}} \cap \mathrm{Thm}\,\mathcal{A}_i),$$

by [9, Proposition 2.5.1], since $\mathrm{Th}\,\mathcal{S}$ is algebraic.

Corollary 5.2.6. *[14, Corollary 5.7] Let* $\mathcal{S}$ *be a weakly algebraizable 1st-level deductive system. Then* $\mathcal{S}$ *has a fully adequate Gentzen system iff it has a multiterm deduction-detachment theorem.*

Proof. It follows directly from Proposition 5.2.3, and the fact that $\mathbf{Fcr}\,\mathcal{S} = \mathbf{Acr}\,\mathcal{S}$ for any weakly algebraizable 1st-level deductive system $\mathcal{S}$. ☐

Suppose $\mathcal{S}$ is a 1st-level deductive system. If $\mathcal{F} \in \mathbf{Fcr}\,\mathcal{S}$, then $\mathrm{Thm}\,\mathcal{F} \in \mathrm{Th}\,\mathcal{S}$ is called a *Leibnitz theory of* $\mathcal{S}$. The set of all Leibnitz theories for $\mathcal{S}$ is denoted by $\mathrm{Th}^{L}\mathcal{S}$.

When $\mathcal{S}$ is protoalgebraic, every Leibnitz theory for $\mathcal{S}$ is always the largest among theories with the same Leibnitz (1st-level) congruence.

Corollary 5.2.7. *Suppose* $\mathcal{S}$ *has a deduction-detachment theorem. Then* $\mathcal{S}$ *has a fully adequate Gentzen system, whenever* $\mathrm{Th}^{L}\mathcal{S}$ *is closed under non-empty intersections.*

Proof. Suppose $\{\mathcal{F}_i\}_{i\in I} \subseteq \mathbf{Fcr}\,\mathcal{S}$, then $\{\mathrm{Thm}\,\mathcal{F}_i\}_{i\in I} \subseteq \mathrm{Th}^{L}\,\mathcal{S}$. Since $\mathcal{S}$ is protoalgebraic: $\mathbf{Fcr}\,\mathcal{S} \subseteq \mathbf{Acr}\,\mathcal{S}$, so $\{\mathcal{F}_i\}_{i\in I} \subseteq \mathbf{Acr}\,\mathcal{S}$. By Theorem 5.2.5

$$\textstyle\bigcap_{i\in I}\mathcal{F}_i = (\bigcap_{i\in I} \mathrm{Thm}\,\mathcal{F}_i)^{\mathbf{Gcr}\,\mathcal{S}} \in \mathbf{Fcr}\,\mathcal{S},$$

since, by assumption, $\bigcap_{i\in I} \mathrm{Thm}\,\mathcal{F}_i \in \mathrm{Th}^{L}\,\mathcal{S}$. □

CONCLUSION

The string of three conceptually and technically interdependent results forms the backbone of this thesis.

1. A syntactical characterization for weakly algebraizable Gentzen systems (Thm. 3.2.3).

We use this characterization to obtain the canonical form of a fully adequate Gentzen system (Thm. 4.3.9). It is also important for providing a syntactic characterizations for algebraizable Gentzen systems and completing Herrmann-style [16, 17] algebraic hierarchy of Gentzen systems.

2. A general criterion for the existence of fully adequate Gentzen systems for Hilbert deductive systems (Thm. 4.3.4 and Thm. 4.3.9).

This criterion generalizes a number of diverse results. That includes: the criterion for the existence of a fully adequate Gentzen system for a protoalgebraic Hilbert system [14]; as well as the known sufficient conditions [12, Thm. 4.27 and Thm. 4.45].

3. A new characterization of the deduction-detachment theorem (Thm. 5.2.3).

The deduction-detachment theorem is often formulated by using Gentzen rules, but it was not known what abstract Gentzen system is axiomatized by those rules. The characterization provided in Thm. 5.2.3 for the first time links the deduction-detachment theorem with abstract Gentzen system of axiomatic closure systems. The former were systematically studied in this thesis.

Future plans and related problems. There is a number of related facts that have been left unexplained in this thesis, and which may be interesting to investigate.

1. It was proven in [12, Thm's. 4.27, 4.45] that

a) a self-extensional deductive system with conjunction has a fully adequate Gentzen system;

b) a self-extensional deductive system with implication has a fully adequate Gentzen system.

Although they are similar in formulation and have apparent closeness in proofs, the conditions a) and b) have so far resisted the unification. The uniform method, if it exists, would reveal common properties of conjunction and implication and also might help better distinguish them.

Problem: Find a condition that would entail both a) and b).

2. As a by-product of the method employed in [12], it was proven that every self-extensional deductive system with conjunction or implication is fully self-extensional. However it is known that it is not always the case [1]. This problem relates to the question when the validity of Gentzen rules is preserved by all full models.

Problem: It would be interesting to derive the result directly using the technique of this thesis.

3. The graded congruence basis $\mathcal{E} = \bigcup_{n \in N} \varepsilon_n$ (Def. 3.2.2) in its most general form consists of "grades" $\varepsilon_n = \{\alpha_i(\bar{x}, x) \lhd\!\rhd \beta_i(\bar{x}, x)\}_{i \in I}$, where $\alpha_i(\bar{x}, x)$, $\beta_i(\bar{x}, x)$ are some, seemingly unrelated, formulas. In all observed cases though, the shapes of α_i and β_i are regular and related, and also are built by using a binary operation. Compare, for instance, the "grades" of the graded congruence bases for self-extensional logic with conjunction and implication

$$\varepsilon_n^{\wedge} = \{\, x_0 \wedge (x_1 \wedge \ldots (x_{n-1} \wedge x_{n-1}) \ldots)) \lhd\!\rhd x_0 \wedge (x_1 \wedge \ldots (x_{n-1} \wedge x) \ldots)) \,\} \quad [12]$$

$$\varepsilon_n^{\rightarrow} = \{\, x_0 \rightarrow (x_1 \rightarrow \ldots (x_{n-1} \rightarrow x_{n-1}) \ldots)) \lhd\!\rhd x_0 \rightarrow (x_1 \rightarrow \ldots (x_{n-1} \rightarrow x) \ldots)) \,\} \quad [12]$$

Note also the visual coincidence between $\varepsilon_n^{\wedge}$ and $\varepsilon_n^{\rightarrow}$.

Problem: Explain this phenomenon.

4. In modal propositional logics (consider K for instance) the set $\mathcal{M}$ (Thm. 3.3.4) can be replaced by unary polynomials of the form $\{\Box^n x\}_{n \in \omega}$, where $\Box$ is the "necessity" connective. The reason and the exact conditions for possibility of such replacement are not known.

Problem: Find those conditions.

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