

Terwilliger algebras of wreath products of association schemes

by

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ABSTRACT

In this thesis, we study the \mathcal{T} -algebras of symmetric association schemes that are obtained as the wreath product of $H(1, m)$ for $m \geq 2$. We find that the D -class association scheme $K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_D}$ formed by taking the wreath product of one-class association schemes $K_{n_i} = H(1, n_i)$ has the triple-regularity property. We determine the dimension of the \mathcal{T} -algebra for the association scheme $K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_D}$. We also show that the wreath power $(K_m)^{\wr D} = K_m \wr K_m \wr \cdots \wr K_m$, D copies of K_m , is formally self-dual. We give a complete description of the irreducible \mathcal{T} -modules and the structure of \mathcal{T} -algebra for $(K_m)^{\wr D}$ for $m \geq 2$ by essentially studying the irreducible modules of 2 copies of K_m and then extending it to the general case for D copies of K_m . Through these calculations we obtain that the \mathcal{T} -algebra for $(K_m)^{\wr D}$ is $M_{D+1}(\mathbb{C}) \oplus \mathbb{C}^{\oplus \frac{1}{2}D(D+1)}$ for $m \geq 3$, and $M_{D+1}(\mathbb{C}) \oplus \mathbb{C}^{\oplus \frac{1}{2}D(D-1)}$ for $m = 2$.

CHAPTER 1. A GENERAL OVERVIEW AND INTRODUCTION

We first present a general historical overview of the study of association schemes and their Terwilliger algebras. We will then give a brief overview of the organization of this dissertation.

The origins of association schemes lie in the work of R. C. Bose and his co-researchers. Association schemes have fascinated mathematicians for several decades because they are important mathematical objects that bear close connections with other combinatorial objects such as codes, designs and distance regular graphs. In 1973 Delsarte recognized and used association schemes as the basic underlying structures of coding theory and design theory [11]. Through association schemes several combinatorial objects have been viewed from an algebraic point of view [4], [11].

During 1992 and 1993 Paul Terwilliger introduced a new tool called the subconstituent algebra for studying the underlying structure of association schemes through a series of three papers [21], [22], and [23]. Terwilliger introduced a method for studying commutative association schemes by defining a new algebra that is noncommutative, finite dimensional, semisimple \mathbb{C} -algebra. The subconstituent algebra is now popularly referred to as the ‘‘Terwilliger’’ algebra. His idea is summarized in the following way.

Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a commutative association scheme. Let A_0, A_1, \dots, A_D denote the associate matrices of \mathcal{X} . Then these matrices form a basis for a semisimple commutative subalgebra \mathcal{M} of the full $|X| \times |X|$ matrix algebra $M_{|X|}(\mathbb{C})$ over \mathbb{C} . Let x be an arbitrary fixed element of X , and consider the partition $X = R_0(x) \cup R_1(x) \cup \dots \cup R_D(x)$ where $R_i(x) = \{y \in X : (x, y) \in R_i\}$, the i -th subconstituent of \mathcal{X} . Let V be the $|X|$ -dimensional vector space $\mathbb{C}^{|X|} = \bigoplus_{x \in X} \mathbb{C}\hat{x}$ of column vectors \hat{x} corresponding to $x \in X$. Let E_i^* be the orthogonal projection map from V to the i -th subconstituent $V_i^* = \bigoplus_{y \in R_i(x)} \mathbb{C}\hat{y}$.

Then E_0^*, \dots, E_D^* form a basis for the semisimple commutative subalgebra \mathcal{M}^* of $M_{|X|}(\mathbb{C})$ called the dual Bose-Mesner algebra. The Terwilliger algebra $\mathcal{T}(= \mathcal{T}(x))$ with respect to x is the algebra generated by \mathcal{M} and \mathcal{M}^* . Terwilliger showed that V decomposes into an orthogonal direct sum of irreducible \mathcal{T} -modules. While studying the Terwilliger algebra of an association scheme the aim is to identify all the irreducible \mathcal{T} -modules in this sum. In general that seems too difficult.

Terwilliger in his papers showed that when many intersection numbers or Krein parameters vanished in an association scheme, the dimension of \mathcal{T} was relatively lower and the problem took a form which was relatively easier to handle. In particular he described many tools needed to study the Terwilliger algebras of P -polynomial and Q -polynomial association schemes.

Since 1993 a lot of work has been done to study the Terwilliger algebras of distance regular graphs. For example, the \mathcal{T} -algebras for P - and Q -polynomial association schemes have been studied in [6], [25]. The structure of the \mathcal{T} -algebra of group association schemes has been studied in [3] and [1]. In [13] detailed study of the irreducible \mathcal{T} -modules for $H(d, 2)$ is shown. In [20], the irreducible \mathcal{T} -modules of Doob schemes, the schemes coming from the direct products of copies of $H(2, 4)$ and/or Shrikhande graphs, are studied. In [15] the structure of the \mathcal{T} -algebra of a Hamming scheme $H(d, q)$ is given as symmetric d -tensors of the \mathcal{T} -algebra of $H(1, q)$ which are all isomorphic for $q > 2$. It is also shown that the \mathcal{T} -algebra of $H(d, q)$ is decomposed as a direct sum of \mathcal{T} -algebra of hypercubes in [15].

In this thesis, we study the \mathcal{T} -algebra of symmetric association schemes that are obtained as the wreath product of $H(1, m)$ for $m \geq 2$. We find that the D -class association scheme $K_{n_1} \wr K_{n_2} \wr \dots \wr K_{n_D}$ formed by taking the wreath product of one-class association schemes $K_{n_i} = H(1, n_i)$ has the triple-regularity property in the sense of [14] and [17]. Based on this fact, we determine the dimension of the \mathcal{T} -algebra for the association scheme $K_{n_1} \wr K_{n_2} \wr \dots \wr K_{n_D}$. We then find that the wreath power $(K_m)^{\wr D} = K_m \wr K_m \wr \dots \wr K_m$, D copies of K_m , is formally self-dual, and its \mathcal{T} -algebra is isomorphic to $M_{D+1}(\mathbb{C}) \oplus \mathbb{C}^{\oplus \frac{1}{2}D(D+1)}$, the direct sum of the $(D+1)^2$ -dimensional full matrix algebra $M_{D+1}(\mathbb{C})$ and \mathbb{C} with multiplicities $\frac{1}{2}D(D+1)$ over the complex field. We also give a complete description of the irreducible \mathcal{T} -modules of $(K_2)^{\wr D}$

which behaves a little differently from the general case $(K_m)^{\wr D}$ for $m \geq 3$.

The organization of this dissertation is as follows. In chapter 2 we provide some basic definitions and facts about the theory of association schemes and their related Terwilliger algebras. Chapter 3 reviews the wreath product of association schemes briefly and then we calculate the dimensions of some subalgebras of the wreath product of one-class association schemes. In chapter 4 we describe the irreducible modules of the scheme $(K_m)^{\wr D}$ and find that the \mathcal{T} -algebra is $M_{D+1}(\mathbb{C}) \oplus \mathbb{C}^{\oplus \frac{1}{2}D(D+1)}$ for $m \geq 3$. Chapter 5 describes the irreducible modules of $(K_2)^{\wr D}$; it is found out that the \mathcal{T} -algebra is $M_{D+1}(\mathbb{C}) \oplus \mathbb{C}^{\oplus \frac{1}{2}D(D-1)}$. Finally we wrap up the dissertation in chapter 6 and discuss some future research problems. Appendix 1 provides the basics of association schemes and Appendix 2 the basics related to their Terwilliger algebras.

CHAPTER 2. PRELIMINARIES: COMMUTATIVE ASSOCIATION SCHEMES AND THEIR TERWILLIGER ALGEBRAS

In this chapter we will recall some basic definitions and facts about the theory of association schemes and related algebras.

2.1 The Bose-Mesner Algebra

We begin by recalling some basic facts about association schemes [2]. Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ be a commutative association scheme of class D with $|X| = n$. The $D + 1$ relations R_i are conveniently described by their $\{0, 1\}$ -**adjacency matrices** A_0, A_1, \dots, A_D defined by $(A_i)_{xy} = 1$ if $(x, y) \in R_i$; 0 otherwise. The **intersection numbers** p_{ij}^h are defined in terms of the relations for the scheme by the following formula:

$$p_{ij}^h = |\{z \in X : (x, z) \in R_i, (z, y) \in R_j\}|$$

where (x, y) is a fixed member of the relation R_h . The definition of an association scheme is equivalent to the following four axioms:

(1) $A_0 = I$, (2) $A_0 + A_1 + \dots + A_D = J$, (3) $A_i^t = A_{i'}$ for some $i' \in \{0, 1, \dots, D\}$, (4) $A_i A_j = \sum_{h=0}^D p_{ij}^h A_h$, where $I = I_n$ and $J = J_n$ are the $n \times n$ identity matrix and all-one matrix, respectively, and A^t denotes the transpose of A . If the scheme is **symmetric**, then $A_{i'} = A_i$ for all i . **Commutativity** of the scheme asserts that $p_{ij}^h = p_{ji}^h$, and thus $A_i A_j = A_j A_i$.

If the scheme is commutative, the adjacency matrices generate a $(D + 1)$ -dimensional commutative subalgebra $\mathcal{M} = \langle A_0, A_1, \dots, A_D \rangle$ of the full matrix algebra $M_n(\mathbb{C})$ over the field of complex numbers \mathbb{C} . The algebra \mathcal{M} is called the **Bose-Mesner algebra** of the scheme. The Bose-Mesner algebra for a commutative association scheme, being semisimple,

admits a second basis E_0, E_1, \dots, E_D of primitive idempotents. In other words,

$$(1) E_0 = \frac{1}{n}J, (2) E_0 + E_1 + \dots + E_D = I, (3) E_i E_j = \delta_{ij} E_i$$

$$(0 \leq i, j \leq D).$$

Note that \mathcal{M} is also closed under the Hadamard (entrywise) multiplication “ \circ ” of matrices.

So there are nonnegative real numbers q_{ij}^h called the **Krein parameters**, such that

$$(4) E_i \circ E_j = \frac{1}{n} \sum_{h=0}^D q_{ij}^h E_h.$$

There exist two sets of the $(D+1)^2$ complex numbers $p_j(i)$ and $q_j(i)$ according to the $D+1$ expressions

$$A_j = \sum_{i=0}^D p_j(i) E_i$$

$$E_j = \frac{1}{n} \sum_{i=0}^D q_j(i) A_i.$$

The number $p_j(i)$ is characterized by the relation $A_j E_i = p_j(i) E_i$. That is, $p_j(i)$ is the eigenvalue of A_j , associated with the eigenspace spanned by the columns of E_i , occurring with the multiplicity $m_i = \text{rank}(E_i)$. We define P to be the $(D+1) \times (D+1)$ matrix whose (i, j) -entry is $p_j(i)$. P is referred to as the **character table** (or **first eigenmatrix**) of the association scheme \mathcal{X} .

Let V be the n -dimensional vector space $\mathbb{C}^{|X|} = \bigoplus_{x \in X} \mathbb{C} \hat{x}$ of column vectors. Here for each $x \in X$, we denote by \hat{x} the column vector with 1 in the x -th position, and 0 elsewhere. Then V can be written as an orthogonal direct sum of; $V_i = E_i V$, which are the maximal common eigenspaces of A_0, A_1, \dots, A_D . We define the **valencies** k_0, k_1, \dots, k_D of \mathcal{X} by the following formula:

$$k_i = |\{z \in X : (x, z) \in R_i\}| = p_{ii}^0.$$

We have the following identities:

$$\begin{aligned}
k_i &= p_i(0); \quad \sum_{j=0}^D k_j = n; \\
m_i &= \frac{n}{\sum_{j=0}^D |p_j(i)|^2 \cdot k_j^{-1}}; \quad \sum_{i=0}^D m_i = n; \\
\sum_{j=0}^D p_j(i) &= 0, \text{ for } i = 1, 2, \dots, D.
\end{aligned}$$

2.2 The Dual Bose-Mesner Algebra

Let X be an n -element set for a positive integer n . Let $M_n(\mathbb{C})$ denote the \mathbb{C} -algebra of matrices with complex entries whose rows and columns are indexed by X . By the **standard module** of X , we mean the vector space $V = \mathbb{C}^n$ of column vectors whose coordinates are indexed by X . Observe that $M_n(\mathbb{C})$ acts on V by left multiplication. We endow V with the Hermitian inner product defined by $\langle u, v \rangle = u^t \bar{v}$ for $u, v \in V$.

For a given association scheme $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ of order $|X| = n$, fix a ‘base element’ $x \in X$. Let $R_i(x) = \{y \in X : (x, y) \in R_i\}$, and $V_i^* = V_i^*(x) = \bigoplus_{y \in R_i(x)} \mathbb{C}\hat{y}$. Both $R_i(x)$ and $V_i^*(x)$ are referred to as the i -th **subconstituent** of \mathcal{X} with respect to x . Let $E_i^* = E_i^*(x)$ be the orthogonal projection map from $V = \bigoplus_{i=0}^D V_i^*$ to the i -th subconstituent V_i^* . So, E_i^* can be represented by the diagonal matrix given by

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } (x, y) \in R_i \\ 0 & \text{if } (x, y) \notin R_i \end{cases}$$

Then we have

$$E_0^* + E_1^* + \dots + E_D^* = I,$$

$$E_i^* E_j^* = E_i^* \circ E_j^* = \delta_{ij} E_i^* \quad (0 \leq i, j \leq D).$$

The matrices $E_0^*, E_1^*, \dots, E_D^*$ are linearly independent, and they form a basis for a subalgebra $\mathcal{M}^* = \mathcal{M}^*(x) = \langle E_0^*, E_1^*, \dots, E_D^* \rangle$ of $M_n(\mathbb{C})$.

Let $A_i^* = A_i^*(x)$ be the diagonal matrix in $M_n(\mathbb{C})$ with (y, y) -entry $(A_i^*)_{yy} = n \cdot (E_i^*)_{yx}$.

Then we have (1) $A_0^* = I$, (2) $A_0^* + A_1^* + \cdots + A_D^* = nE_0^*$, (3) $A_i^*A_j^* = \sum_{h=0}^D q_{ij}^h A_h^* = A_j^*A_i^*$.

Furthermore, we also have

$$A_j^* = \sum_{i=0}^D q_j(i) E_i^*$$

$$E_j^* = \frac{1}{n} \sum_{i=0}^D p_j(i) A_i^*.$$

Thus, $A_0^*, A_1^*, \dots, A_D^*$ forms a second basis for \mathcal{M}^* . The algebra \mathcal{M}^* is a commutative, semisimple subalgebra of $M_n(\mathbb{C})$. This algebra is called the **dual Bose-Mesner algebra of \mathcal{X} with respect to x** .

2.3 The Terwilliger Algebra

Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme and fix a vertex $x \in X$, and let $\mathcal{T} = \mathcal{T}(x)$ denote the subalgebra of $M_n(\mathbb{C})$ generated by the Bose-Mesner algebra \mathcal{M} and the dual Bose-Mesner algebra $\mathcal{M}^* = \mathcal{M}^*(x)$. We call \mathcal{T} the **Terwilliger algebra of \mathcal{X} with respect to x** . By a **module** for \mathcal{T} , we mean a subspace $W \subseteq \mathbb{C}^{|X|}$ such that $BW \subseteq W$ for all $B \in \mathcal{T}$. A \mathcal{T} -module is said to be **irreducible** whenever W is nonzero and W contains no other \mathcal{T} -modules other than 0 and W .

By a **\mathcal{T} -isomorphism** from W to W' , we mean a vector space isomorphism $\sigma : W \rightarrow W'$ such that $(\sigma B - B\sigma)W = 0$ for all $B \in \mathcal{T}$. The modules W, W' are said to be **\mathcal{T} -isomorphic** whenever there exists a \mathcal{T} -isomorphism from W to W' .

The standard module decomposes into irreducible \mathcal{T} -modules in a manner that reflects the structure of \mathcal{T} . The next subsection will be useful when we describe the irreducible \mathcal{T} -modules for our scheme.

2.3.1 The Central Primitive Idempotents of \mathcal{T}

Definition 2.3.1. Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme and fix a vertex $x \in X$, let $\mathcal{T} = \mathcal{T}(x)$. By the **center** of \mathcal{T} , we mean the subalgebra of \mathcal{T} given by

$$\text{Center}(\mathcal{T}) := \{B \in \mathcal{T} : BC = CB \quad \forall C \in \mathcal{T}\}$$

Lemma 2.3.2. [24] Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme, fix a vertex $x \in X$, and let $\mathcal{T} = \mathcal{T}(x)$ and abbreviate $\mathcal{C} = \text{Center}(\mathcal{T})$. Then there exists a set $\Phi = \Phi(x)$ and a basis $\{e_\lambda : \lambda \in \Phi\}$ for \mathcal{C} such that

1. $I = \sum_{\lambda \in \Phi} e_\lambda$,
2. $e_\lambda e_\mu = \delta_{\lambda\mu} e_\lambda$ ($\forall \lambda \in \Phi, \forall \mu \in \Phi$)

We refer to e_λ as the **central primitive idempotents** of \mathcal{T} .

Let $V = \mathbb{C}^{|X|}$ denote the standard module. The standard module decomposes into irreducible \mathcal{T} -modules in a manner which reflects the structure of \mathcal{T} .

Lemma 2.3.3. [21] Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme, fix a vertex $x \in X$, and let $\mathcal{T} = \mathcal{T}(x)$. Let $\{e_\lambda : \lambda \in \Phi\}$ be the central primitive idempotents of \mathcal{T} .

1. $V = \sum_{\lambda \in \Phi} e_\lambda V$ (Orthogonal direct sum). Moreover, $e_\lambda : V \rightarrow e_\lambda V$ is an orthogonal projection for all $\lambda \in \Phi$.
2. For each irreducible \mathcal{T} -module W , there is a unique $\lambda \in \Phi$ such that $W \subseteq e_\lambda V$. We refer to λ as the **type** of W .
3. Let W and W' denote irreducible \mathcal{T} -modules. Then W and W' are \mathcal{T} -isomorphic if and only if W and W' have the same type.
4. For all $\lambda \in \Phi$, $e_\lambda V$ can be decomposed as an orthogonal direct sum of irreducible \mathcal{T} -modules of type λ .

5. Referring to 4, the number of irreducible \mathcal{T} -modules in the decomposition is independent of the decomposition. We shall denote this number by $\text{mult}(e_\lambda)$ (or simply $\text{mult}(\lambda)$) and refer to it as the multiplicity (in V) of the irreducible \mathcal{T} -module of type λ .

2.3.2 Some Triple Products in the Terwilliger Algebra

In this subsection we look at the triple products $E_i^* A_j E_h^*$ a little closely. We can view $E_i^* A_j E_h^*$ as a linear map from $V_h^* \rightarrow V_i^*$ such that $E_i^* A_j E_h^* \hat{y} = \sum_{z \in R_i(x) \cap R_j(y)} \hat{z}$ for each $\hat{y} \in V_h^*$. Terwilliger proved in [21] that

Proposition 2.3.4. *Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme and fix a vertex $x \in X$, let $\mathcal{T} = \mathcal{T}(x)$. $E_i^* A_j E_h^* = 0$ if and only if $p_{ij}^h = 0$ for $(0 \leq i, j, h \leq D)$.*

Note that A_i and E_i^* can be written in terms of the triple products $E_i^* A_j E_h^*$ for $(0 \leq i \leq D)$. Thus, $E_i^* A_j E_h^*$ generates the \mathcal{T} -algebra. It is often easier to find the irreducible modules if we work with the triple products $E_i^* A_j E_h^*$ instead of A_i and E_i^* .

CHAPTER 3. WREATH PRODUCTS OF ONE-CLASS ASSOCIATION SCHEMES

In this chapter, we recall some basic facts of the wreath product of association schemes and then investigate the Terwilliger algebra of the product. In particular, we calculate the dimension of some subalgebra of the Terwilliger algebra of the wreath product of one-class association schemes.

3.1 The Wreath Product of Association Schemes

Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ and $\mathcal{Y} = (Y, \{S_j\}_{0 \leq j \leq e})$ be association schemes of order $|X| = m$ and $|Y| = n$. The **wreath product** $\mathcal{X} \wr \mathcal{Y}$ of \mathcal{X} and \mathcal{Y} is defined on the set $X \times Y$; but we take $Y = \{y_1, y_2, \dots, y_n\}$, and regard $X \times Y$ as the disjoint union of n copies X_1, X_2, \dots, X_n of X , where $X_j = X \times \{y_j\}$. The relations on $X_1 \cup X_2 \cup \dots \cup X_n$ is defined by the following rule:

- For any j , the relations between the elements of X_j are determined by the association relations between the first coordinates in \mathcal{X} .
- The relations between the elements that belong to two different sets, say X_i and X_j , are determined by the association relation of the second coordinates y_i and y_j in \mathcal{Y} and the relation is independent from the first coordinates.

That is, the relations W_0, W_1, \dots, W_{d+e} of $\mathcal{X} \wr \mathcal{Y}$ are defined by

- $W_0 = \{((x, y), (x, y)) : (x, y) \in X \times Y\}$
- $W_k = \{((x_1, y), (x_2, y)) : (x_1, x_2) \in R_k, y \in Y\}$, for $1 \leq k \leq d$; and

- $W_k = \{((x_1, y_1), (x_2, y_2)) : x_1, x_2 \in X, (y_1, y_2) \in S_{k-d}\}$ for $d+1 \leq k \leq d+e$.

It is easy to see that $\mathcal{X} \wr \mathcal{Y} = (X \times Y, \{W_k\}_{0 \leq k \leq d+e})$ is an association scheme. It is also clear that $\mathcal{X} \wr \mathcal{Y}$ is commutative (resp. symmetric) if and only if \mathcal{X} and \mathcal{Y} are. Let A_0, A_1, \dots, A_d and C_0, C_1, \dots, C_e be the adjacency matrices of \mathcal{X} and those of \mathcal{Y} , respectively. Then the adjacency matrices of $\mathcal{X} \wr \mathcal{Y}$ are given by

$$C_0 \otimes A_0, C_0 \otimes A_1, C_0 \otimes A_2, \dots, C_0 \otimes A_d, C_1 \otimes J_m, C_2 \otimes J_m, \dots, C_e \otimes J_m.$$

Here “ \otimes ” denotes the Kronecker product $A \otimes B = (a_{ij}B)$ of two matrices $A = (a_{ij})$ and B . So, the Bose-Mesner algebra of $\mathcal{X} \wr \mathcal{Y}$ is defined by the usual matrix operations:

$$(A \otimes B)(C \otimes D) = AC \otimes BD, \quad (A \otimes B) \circ (C \otimes D) = (A \circ C) \otimes (B \circ D),$$

$$(A + B) \otimes (C + D) = A \otimes C + B \otimes C + A \otimes D + B \otimes D.$$

A wreath product scheme is conveniently described by its **relation table**. Let \mathcal{X} be the scheme of order m with its adjacency matrices A_0, A_1, \dots, A_d , as above. Then by the **relation table of \mathcal{X}** we mean the matrix $R(\mathcal{X})$ described by

$$R(\mathcal{X}) = \sum_{i=0}^d i \cdot A_i$$

i.e. the (x, y) -entry of $R(\mathcal{X})$ is i if and only if $(x, y) \in R_i$. It is easy to see that, with the above ordering of the association relations of $\mathcal{X} \wr \mathcal{Y}$, the **relation table** of the wreath product is described by

$$R(\mathcal{X} \wr \mathcal{Y}) = I_n \otimes R(\mathcal{X}) + \{R(\mathcal{Y}) + d(J_n - I_n)\} \otimes J_m.$$

3.2 The D -class Association Scheme $K_{n_1} \wr K_{n_2} \wr \dots \wr K_{n_D}$

Let K_n denote the one-class association scheme $([n], \{R_0, R_1\})$ where $[n] = \{1, 2, \dots, n\}$; i.e., let $A_1 = J - I$ be the associate matrix of the non-diagonal relation R_1 . Let $\mathcal{X} =$

$(X, \{R_i\}_{0 \leq i \leq D})$ denote the D -class scheme $K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_D}$ with $X = [n_1] \times [n_2] \times \cdots \times [n_D] = \{(a_1, a_2, \dots, a_D) : a_i \in [n_i], \text{ for } i = 1, 2, \dots, D\}$. Let $(1, 1, \dots, 1) \in X$ be a fixed base vertex x of \mathcal{X} . Without loss of generality, we can arrange the vertices as follows:

- $R_0(x) = \{x\}$
- $R_1(x) = \{(a, 1, 1, \dots, 1) : a \in \{2, 3, \dots, n_1\}\}$
- for $i = 2, 3, \dots, D$,
 $R_i(x) = \{(a_1, a_2, \dots, a_{i-1}, b, 1, 1, \dots, 1) : a_k \in [n_k] \text{ for } 1 \leq k \leq i-1, b \in [n_i] - \{1\}\}$.

We observe that $k_i = |R_i(x)| = (n_i - 1) \prod_{k=1}^{i-1} n_k$. We can arrange the rows and columns of the relation table of \mathcal{X} by the order of parts in the partition $X = R_0(x) \cup R_1(x) \cup \cdots \cup R_D(x)$.

Example 3.2.1. The following is the relation table for the wreath product of three association schemes K_2 , K_2 and K_3 .

$$R(K_2 \wr K_2 \wr K_3) = \left(\begin{array}{cc|cc|cccccccc} 0 & 1 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 1 & 0 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ \hline 2 & 2 & 0 & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 1 & 0 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ \hline 3 & 3 & 3 & 3 & 0 & 1 & 2 & 2 & 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 1 & 0 & 2 & 2 & 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 2 & 2 & 0 & 1 & 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 2 & 2 & 1 & 0 & 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 0 & 1 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 1 & 0 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 2 & 2 & 0 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 2 & 2 & 1 & 0 & 0 \end{array} \right)$$

Lemma 3.2.2. Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D}) = K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_D}$. Then the complete list of nonzero p_{ij}^h where $h, i, j \in \{0, 1, 2, \dots, D\}$ is as follows:

1. For $h = 0, k_0 = p_{00}^0 = 1, k_1 = p_{11}^0 = n_1 - 1$, and

$$k_j = p_{jj}^0 = (n_j - 1) \prod_{l=1}^{j-1} n_l \text{ for } j = 2, 3, \dots, D.$$

2. For $h = 1, 2, \dots, D$,

$$(a) p_{hh}^h = (n_h - 2) \prod_{l=1}^{h-1} n_l,$$

$$(b) p_{jj}^h = (n_j - 1) \prod_{l=1}^{j-1} n_l \text{ for } h + 1 \leq j \leq D,$$

$$(c) p_{hj}^h = p_{jh}^h = (n_j - 1) \prod_{l=1}^{j-1} n_l \text{ for } 1 \leq j \leq h - 1,$$

$$(d) p_{0h}^h = p_{h0}^h = 1.$$

Proof. It is straightforward to calculate the intersection numbers. □

Due to this lemma we have the following list of non-zero triple products in \mathcal{T} .

Theorem 3.2.3. *Let $\mathcal{X} = K_{n_1} \wr K_{n_2} \wr \dots \wr K_{n_D}$. Then the following is the complete list of nonzero $E_i^* A_j E_h^*$ among all $h, i, j \in \{0, 1, 2, \dots, D\}$*

1. $E_i^* A_i E_0^*$ for $(0 \leq i \leq D)$,
2. $E_h^* A_h E_h^*$ if and only if $n_h \geq 3$ for $1 \leq h \leq D$,
3. $E_j^* A_j E_h^*$ for $2 \leq h + 1 \leq j \leq D$,
4. $E_j^* A_h E_h^*$ for $1 \leq j + 1 \leq h \leq D$,
5. $E_h^* A_j E_h^*$ for $1 \leq j + 1 \leq h \leq D$.

Proof. Immediate from the above lemma and by Proposition 2.3.4. □

3.3 The Dimension of the \mathcal{T} -algebra of $K_{n_1} \wr K_{n_2} \wr \dots \wr K_{n_D}$

Let $\mathcal{X} = K_{n_1} \wr K_{n_2} \wr \dots \wr K_{n_D}$ be a D -class association scheme we discussed in the previous section. Let $\mathcal{T}_0(x)$ be the linear subspace of $\mathcal{T}(x)$ spanned by $E_i^* A_j E_h^*$, $(0 \leq i, j, h \leq D)$.

Clearly, $\mathcal{T}(x)$ is generated by $\mathcal{T}_0(x)$ as an algebra since $\mathcal{T}_0(x)$ contains A_i and E_i^* for all i , but in general, $\mathcal{T}_0(x)$ may be a proper subspace of $\mathcal{T}(x)$.

Theorem 3.3.1. *For $K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_D}$ and for any $x \in X$, the dimension of $\mathcal{T}_0(x)$ is given by*

$$(D+1)^2 + \frac{1}{2}D(D+1) - b$$

where b is the number of the factors K_{n_i} with $n_i = 2$. In particular,

$$(D+1)^2 + \frac{1}{2}D(D-1) \leq \dim(\mathcal{T}_0(x)) \leq (D+1)^2 + \frac{1}{2}D(D+1).$$

Proof. From Theorem 3.2.3 the number of non-zero triple products can be counted as $D+1$ from 1, $D-b$ from 2, $\frac{1}{2}D(D-1)$ from 3, and $D(D+1)$ from 4 and 5. As they are independent of each other we have the $\dim(\mathcal{T}_0)$ as desired. The case where all of $n_i = 2$ gives the lower bound of $\dim(\mathcal{T}_0)$ as in this case $b = D$. The upper bound is given by the case where none of $n_i = 2$. In such a situation $b = 0$. This completes the proof. □

We are now ready to show that the Terwilliger algebra \mathcal{T} for $K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_D}$ is indeed the same as \mathcal{T}_0 . Not all association schemes have this property. An association scheme that has this property is said to be **triply regular**. This concept has been studied earlier in [21, 17, 14]. Due to the following results of Munemasa, in order to show that $\mathcal{T} = \mathcal{T}_0$, it suffices to show that all triple products $A_i E_h^* A_j$ belong to \mathcal{T}_0 .

Proposition 3.3.2. [17] Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ be a D -class commutative association scheme. Then the following properties are equivalent.

1. \mathcal{X} is **triply regular**; i.e., by the definition, for any $x, y, z \in X$ with $(x, y) \in R_i$, $(x, z) \in R_m$ and $(y, z) \in R_n$, the cardinality of the set $R_i(x) \cap R_j(y) \cap R_h(z)$ depends only on the choice of $h, i, j, l, m, n \in \{0, 1, \dots, D\}$.
2. For any $h, i, j \in \{0, 1, \dots, D\}$, $A_i E_j^* A_h \in \mathcal{T}_0$.

3. $\mathcal{T} = \mathcal{T}_0$.

Lemma 3.3.3. *Let $\mathcal{X} = K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_D}$ be a D -class association scheme of order $n_1 n_2 \cdots n_D$. Then $A_i E_h^* A_j = (A_i E_h^*)(E_h^* A_j)$ for $i, j, h \in \{0, 1, \dots, D\}$.*

Proof. The proof follows from the fact that $E_i^* E_j^* = \delta_{ij} E_i^*$.

□

Lemma 3.3.4. *Let $\mathcal{X} = K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_D}$. For $i, h \in \{0, 1, \dots, D\}$,*

1. *if $i = h$, then $A_i E_h^* = \sum_{j=0}^h E_j^* A_h E_h^*$;*
2. *if $i < h$, then $A_i E_h^* = E_h^* A_i E_h^*$;*
3. *if $i > h$, then $A_i E_h^* = E_i^* A_i E_h^*$.*

Proof. 1. If $i = h$, then the nonzero entries of $A_i E_h^*$ are the nonzero entries of the columns of A_i indexed by vertices in $R_h(x)$. The rest of the entries are zero. The columns of A_i indexed by vertices in $R_h(x)$ have 1 in rows indexed by the vertices in $R_0(x) \cup R_1(x) \cup \cdots \cup R_{h-1}(x)$. In addition the rows indexed by the vertices $R_h(x)$ have zero in the diagonal blocks of size $n_1 n_2 \cdots n_{h-1} \times n_1 n_2 \cdots n_{h-1}$, and one elsewhere. The entries indexed by rows with vertices in $R_{h+1}(x) \cup R_{h+2}(x) \cup \cdots \cup R_D(x)$ all have entries zero. These essentially add up to $\sum_{j=0}^h E_j^* A_h E_h^*$.

2. If $i < h$, then the nonzero entries of $A_i E_h^*$ are the nonzero entries of the columns of A_i indexed by vertices in $R_h(x)$. The rest of the entries are zero. The columns of A_i indexed by vertices in $R_h(x)$ have one in rows indexed by the vertices in $R_h(x)$. The rest of the entries are zero. This completes the proof of part (2).

3. If $i > h$, then the nonzero entries of $A_i E_h^*$ are the nonzero entries of the columns of A_i indexed by vertices in $R_h(x)$. The rest of the entries are zero. The columns of A_i indexed by vertices in $R_h(x)$ have one in rows indexed by the vertices in $R_i(x)$. The rest of the entries are zero.

□

Lemma 3.3.5. Let $\mathcal{X} = K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_D}$. For $i, h \in \{0, 1, \dots, D\}$,

1. if $i = h$, then $E_h^* A_i = \sum_{j=0}^h E_h^* A_h E_j^*$;
2. if $i < h$, then $E_h^* A_i = E_h^* A_i E_h^*$;
3. if $i > h$, then $E_h^* A_i = E_h^* A_i E_i^*$.

Proof. The proof is immediate from the fact that the transpose of $A_i E_h^*$ is $E_h^* A_i$. □

Lemma 3.3.6. Let $\mathcal{X} = K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_D}$. For $i, j, h \in \{0, 1, \dots, D\}$, if $i = h = j$, then

$$\begin{aligned} A_i E_h^* A_j &= (n_h - 1) \left(\prod_{k=1}^{h-1} n_k \right) \sum_{m=0}^D \sum_{l=0}^{h-1} \sum_{n=0}^{h-1} E_l^* A_m E_n^* \\ &+ (n_h - 2) \left(\prod_{k=1}^{h-1} n_k \right) \left\{ \sum_{m=0}^D \sum_{n=0}^{h-1} E_h^* A_m E_n^* + \sum_{m=0}^t \sum_{l=0}^h E_l^* A_m E_h^* \right\}. \end{aligned}$$

Proof. Lemmas 3.3.3, 3.3.4(1) and 3.3.5(1) gives us

$$\begin{aligned} A_i E_h^* A_j &= k_h \sum_{m=0}^D \sum_{l=0}^{h-1} \sum_{n=0}^{h-1} E_l^* A_m E_n^* \\ &+ (k_h - (k_0 + \cdots + k_{h-1})) \left\{ \sum_{m=0}^D \sum_{n=0}^{h-1} E_h^* A_m E_n^* + \sum_{m=0}^D \sum_{l=0}^h E_l^* A_m E_h^* \right\} \end{aligned}$$

$k_h = n_1 n_2 \cdots n_h - n_1 n_2 \cdots n_{h-1}$ and $k_h - (k_0 + \cdots + k_{h-1}) = n_1 n_2 \cdots n_h - n_1 n_2 \cdots n_{h-1} - n_1 n_2 \cdots n_{h-1}$. This gives us the required result. □

Lemma 3.3.7. Let $\mathcal{X} = K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_D}$. For $i, j, h \in \{0, 1, \dots, D\}$

1. (a) if $i \neq j$ and $i > j$
 - i. if $i = h$, then

$$A_i E_h^* A_j = (n_j - 1) \left(\prod_{k=1}^{j-1} n_k \right) \sum_{m=0}^D \sum_{l=0}^{h-1} E_l^* A_m E_h^*;$$

ii. if $j = h$, then

$$\begin{aligned} A_i E_h^* A_j &= (n_h - 1) \left(\prod_{k=1}^{h-1} n_k \right) \sum_{m=0}^D \sum_{n=0}^{h-1} E_i^* A_m E_n^* \\ &\quad + (n_h - 2) \left(\prod_{k=1}^{h-1} n_k \right) \sum_{m=0}^D E_i^* A_m E_h^*; \end{aligned}$$

(b) if $i \neq j$ and $i < j$

i. if $i = h$, then

$$\begin{aligned} A_i E_h^* A_j &= (n_h - 1) \left(\prod_{k=1}^{h-1} n_k \right) \sum_{m=0}^D \sum_{l=0}^{h-1} E_l^* A_m E_j^* \\ &\quad + (n_h - 2) \left(\prod_{k=1}^{h-1} n_k \right) \sum_{m=0}^D E_h^* A_m E_j^*; \end{aligned}$$

ii. if $j = h$, then

$$A_i E_h^* A_j = (n_i - 1) \left(\prod_{k=1}^{i-1} n_k \right) \sum_{m=0}^D \sum_{n=0}^{h-1} E_h^* A_m E_n^*;$$

2. if $i = j$ but none of i or j is equal to h

(a) if $i > h$, then

$$A_i E_h^* A_j = (n_h - 1) \left(\prod_{l=1}^{h-1} n_l \right) \sum_{m=0}^D E_i^* A_m E_i^*;$$

(b) if $i < h$, then

$$A_i E_h^* A_j = (n_i - 1) \left(\prod_{l=1}^{i-1} n_l \right) \sum_{m=0}^D E_h^* A_m E_h^*.$$

Proof. 1(a)(i) Lemmas 3.3.3, 3.3.4(1) and 3.3.5(2) gives us

$$A_i E_h^* A_j = k_j \sum_{m=0}^D \sum_{l=0}^{h-1} E_l^* A_m E_h^*,$$

and the result follows.

1(b)(i) Lemmas 3.3.3, 3.3.4(1) and 3.3.5(3) gives us

$$A_i E_h^* A_j = k_h \sum_{m=0}^t \sum_{l=0}^{h-1} E_l^* A_m E_j^* \\ + (k_h - (k_0 + \cdots + k_{h-1})) \sum_{m=0}^D E_h^* A_m E_j^*,$$

and the result follows.

1(a)(ii) The proof of this result is same as 1(b)(i).

1(b)(ii) The proof of this result is same as 1(a)(i).

2(a) Lemmas 3.3.3, 3.3.4(3) and 3.3.5(3) gives us

$$A_i E_h^* A_j = k_h \sum_{m=0}^D E_i^* A_m E_i^*,$$

and the result follows.

(b) Lemmas 3.3.3, 3.3.4(2) and 3.3.5(2) gives us

$$A_i E_h^* A_j = k_h \sum_{m=0}^D E_h^* A_m E_h^*,$$

and the result follows. □

Lemma 3.3.8. *Let $\mathcal{X} = K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_D}$. For $i, j, h \in \{0, 1, \dots, D\}$ assume that $i \neq j \neq h$.*

1. *If $i > h$ and $j > h$, then*

$$A_i E_h^* A_j = (n_h - 1) \left(\prod_{l=1}^{h-1} n_l \right) \sum_{m=0}^D E_i^* A_m E_j^*.$$

2. *If $i < h$ and $j > h$, then*

$$A_i E_h^* A_j = (n_i - 1) \left(\prod_{l=1}^{i-1} n_l \right) \sum_{m=0}^D E_h^* A_m E_j^*.$$

3. If $i > h$ and $j < h$, then

$$A_i E_h^* A_j = (n_j - 1) \left(\prod_{l=1}^{j-1} n_l \right) \sum_{m=0}^D E_i^* A_m E_h^*.$$

4. If $i < h$, $j < h$,

(a) and if $i < j$, then

$$A_i E_h^* A_j = (n_i - 1) \left(\prod_{l=1}^{i-1} n_l \right) E_h^* A_j E_h^*,$$

(b) and if $i > j$, then

$$A_i E_h^* A_j = (n_j - 1) \left(\prod_{l=1}^{j-1} n_l \right) E_h^* A_i E_h^*.$$

Proof. The proofs of 1, 2 and 3 are similar to Lemma 3.3.7.

4 (a) Lemmas 3.3.3, 3.3.4(2) and 3.3.5(2) give us nonzero entries of $A_i E_h^* A_j$ occur in the rows and columns indexed by the vertices $R_h(x)$. Consider the diagonal blocks of size $k_j \times k_j$ inside $A_i E_h^*$ indexed by the rows and columns of vertices in $R_h(x)$. These diagonal blocks have k_i 1's in each row and column. The off diagonal entries are all zero. In a similar manner consider diagonal blocks of size $k_j \times k_j$ inside $A_i E_h^* A_j$ indexed by the rows and columns of vertices in $R_h(x)$. These diagonal blocks are all zero and the off diagonal entries are one. This observation gives us

$$A_i E_h^* A_j = (n_i - 1) \left(\prod_{l=1}^{i-1} n_l \right) E_h^* A_j E_h^*$$

(b) The result is same as part (a).

□

Theorem 3.3.9. *The D -class association scheme $K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_D}$ is triply regular and its Terwilliger algebra $\mathcal{T}(x)$ equals to $\mathcal{T}_0(x)$ for any $x \in X$.*

Proof. Lemmas 3.3.6, 3.3.7 and 3.3.8 along with Proposition 3.3.2 provide the proof.

□

Theorem 3.3.10. *Let $\mathcal{X} = K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_D}$ be a D -class (symmetric) association scheme of order $n_1 n_2 \cdots n_D$. Then the Terwilliger algebra $\mathcal{T}(x)$ is generated by the following nonzero linearly independent matrices $E_i^* A_j E_h^*$ among all $h, i, j \in \{0, 1, 2, \dots, D\}$*

1. $E_i^* A_i E_0^*$ for $(0 \leq i \leq D)$,
2. $E_h^* A_h E_h^*$ if and only if $n_h \geq 3$ for $1 \leq h \leq D$,
3. $E_j^* A_j E_h^*$ for $2 \leq h+1 \leq j \leq D$,
4. $E_j^* A_h E_h^*$ for $1 \leq j+1 \leq h \leq D$,
5. $E_h^* A_j E_h^*$ for $1 \leq j+1 \leq h \leq D$.

Proof. From Lemma 3.2.3 and Theorem 3.3.9. □

Theorem 3.3.11. *Let $\mathcal{X} = K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_D}$ be a D -class (symmetric) association scheme of order $n_1 n_2 \cdots n_D$. Then the dimension of the Terwilliger algebra $\mathcal{T}(x)$ is $(D+1)^2 + \frac{1}{2}D(D+1) - b$ where b denotes the number of K_2 's.*

Proof. From Lemma 3.3.1 and Theorem 3.3.9. □

3.4 Building the basis of D copies from $D - 1$ copies of K_{n_i}

In the previous section we showed that the nonzero triple products $E_i^* A_j E_k^*$ form a basis of the space \mathcal{T} in $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D}) = K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_D}$. In this section we build the nonzero $E_i^* A_j E_k^*$'s of $(K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_{D-1}}) \wr K_{n_D}$ from those of the $(D - 1)$ -class association scheme $K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_{D-1}}$.

As a notational convention, for given

$$X = [n_1] \times [n_2] \times \cdots \times [n_l] = \{(a_1, a_2, \dots, a_l) : a_i \in [n_i], 1 \leq i \leq l\},$$

we always choose the first vertex $x = (1, 1, \dots, 1)$ as the base vertex. Let us denote the triple products $E_i^* A_j E_k^*$'s of the l -class association scheme as $[E_i^* A_j E_k^*]^l$ in order to differentiate them when we discuss wreath products with different number of factors. The next theorem shows how we can obtain $D^2 + \frac{1}{2}D(D-1) - b$ triple products $[E_i^* A_j E_k^*]^D$ from the triple products $[E_i^* A_j E_k^*]^{D-1}$.

Lemma 3.4.1. *Let $(K_{n_1} \wr K_{n_2} \wr \dots \wr K_{n_{D-1}}) \wr K_{n_D}$ be the D -class association scheme formed by taking the wreath product of the $D-1$ class association scheme $K_{n_1} \wr K_{n_2} \wr \dots \wr K_{n_{D-1}}$ and the 1-class association scheme K_{n_D} . For $i, j, h \in \{0, 1, \dots, D-1\}$*

$$[E_i^* A_j E_h^*]^D = \left[\begin{array}{c|c} [E_i^* A_j E_h^*]^{D-1} & 0 \\ \hline 0 & 0 \end{array} \right]$$

Proof. From the definition of the wreath product of two association schemes the triple products $[E_i^* A_j E_k^*]^D$ have $(n_D - 1) \prod_{l=1}^{D-1} n_l$ additional rows and columns corresponding to the elements that are in the D -th relation from the base vertex in comparison to the triple products $[E_i^* A_j E_k^*]^{D-1}$. It is easy to see that $[E_i^* A_j E_k^*]^{D-1}$ is viewed as a principal submatrix of $[E_i^* A_j E_k^*]^D$ with additional blocks of all 0 entries. \square

In addition to the above triple products, the scheme $(K_{n_1} \wr K_{n_2} \wr \dots \wr K_{n_{D-1}}) \wr K_{n_D}$ has additional triple products; among those, by Lemma 3.2.3, the following $3D+1$ triple products are non-zero if $n_D \geq 3$ and $3D$ are non-zero excluding $E_D^* A_D E_D^*$ if $n_D = 2$. Furthermore, we have the following descriptions for the triple products.

Lemma 3.4.2. *In $(K_{n_1} \wr K_{n_2} \wr \dots \wr K_{n_{D-1}}) \wr K_{n_D}$, consider the above notation.*

(1) *For $i, h \in \{0, 1, \dots, D-1\}$, $[E_i^* A_D E_D^*]^D$ (resp. $[E_D^* A_D E_h^*]^D$) has all-ones matrix J placed at rows indexed by the vertices in $R_i(x)$ (resp. $R_D(x)$) and columns indexed by $R_D(x)$ (resp. $R_h(x)$), and has zero for all other entries.*

(2) *For $j \in \{0, 1, \dots, D-1\}$, $[E_D^* A_j E_D^*]^D = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & T \end{array} \right]$ where $T = I_{n_{D-1}} \otimes [A_j]^{D-1}$, with $[A_j]^{D-1}$ being the j -th adjacency matrix of $K_{n_1} \wr K_{n_2} \wr \dots \wr K_{n_{D-1}}$.*

(3)

$$[E_D^* A_t E_D^*]^D = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & T \end{array} \right]$$

where $T = (J_{n_D} - I_{n_D}) \otimes I_{n_1 n_2 \dots n_{D-1}}$ if $n_D \neq 2$ and $T = 0$ if $n_D = 2$.

Proof. (1) From the construction of the wreath product, the vertices in $R_0(x) \cup \dots \cup R_{D-1}(x)$ and those in $R_D(x)$ are in relation D with each other. This proves the result.

(2) The j -th adjacency matrix A_j for the scheme $(K_{n_1} \wr K_{n_2} \wr \dots \wr K_{n_{D-1}}) \wr K_{n_D}$ is $I_{n_D} \otimes [A_i]^{D-1}$. $[E_D^* A_j E_D^*]^D$ has nonzero entries placed at rows indexed by the vertices in $R_D(x)$ and columns indexed by $R_D(x)$. Combining these facts we have

$$[E_D^* A_j E_D^*]^D = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & T \end{array} \right]$$

(3) The matrix $[E_D^* A_D E_D^*]^D$ has nonzero entries placed at rows indexed by the vertices in $R_D(x)$ and columns indexed by $R_D(x)$. Further the diagonal blocks of size $n_1 n_2 \dots n_{D-1} \times n_1 n_2 \dots n_{D-1}$ have zero as their entries. The off diagonal entries are 1. This gives us the result. \square

CHAPTER 4. STRUCTURE OF TERWILLIGER ALGEBRAS OF WREATH POWERS OF ONE-CLASS ASSOCIATION SCHEMES

In this chapter we will describe the Terwilliger algebras of the 1-class association scheme K_m , the 2-class association scheme $K_m \wr K_m$, and the 3-class association scheme $(K_m)^{\wr 3}$. We end the chapter by describing the Terwilliger algebras of the 4-class association scheme $(K_m)^{\wr 4}$ and the D -class association scheme $(K_m)^{\wr D}$ for $m \geq 3$. The D -class association scheme $(K_2)^{\wr D}$ will be dealt separately in chapter 5 because there are fewer nonzero generators $E_i^* A_j E_h^*$ in this case. It is worth noting that $K_m \wr K_m$ is a strongly regular graph and is a P -polynomial scheme whereas $(K_m)^{\wr 3}$ is neither strongly regular nor a P -polynomial scheme and that makes the description of the Terwilliger algebra much more complicated.

4.1 The Terwilliger Algebra of K_m

Let $\mathcal{X} = (X, \{R_0, R_1\}) = K_m$ be a 1-class association scheme of order m . Let $X = [m] = \{1, 2, \dots, m\}$ be the vertex set. Let $x = 1$ be the fixed base vertex. The relations of the association schemes are as follows

1. $R_0(x) = \{x\}$
2. $R_1(x) = \{i : 2 \leq i \leq m\}$

The relation table is given by

$$J_m - I_m = \begin{pmatrix} 0 & \mathbf{u}^t \\ \mathbf{u} & B \end{pmatrix}$$

where \mathbf{u} is the $m - 1$ dimensional column vector all of whose entries are one and $B = J_{m-1} - I_{m-1}$.

In the rest of the section we will describe the Terwilliger algebra for the scheme K_m .

Lemma 4.1.1. *Let $\mathcal{X} = (X, \{R_0, R_1\}) = K_m$ be a 1-class association scheme of order m . Then the Terwilliger algebra $\mathcal{T}(x)$ is generated by the matrices $E_0^*A_0E_0^*$, $E_0^*A_1E_1^*$, $E_1^*A_1E_0^*$, $E_1^*A_0E_1^*$ and $E_1^*A_1E_1^*$.*

Proof. Theorem 3.3.10 gives us the result. □

Note that K_m is a particular case of the Hamming scheme $H(d, m)$ with $d = 1$. The Terwilliger algebra of the hypercube and the Hamming scheme have been extensively studied in [13] and [15].

The next theorem describes the Terwilliger algebra of the scheme K_m .

Theorem 4.1.2. [15] *Let $\mathcal{X} = (X, \{R_0, R_1\}) = K_m$ be a 1-class association scheme of order m . Then*

$$\mathcal{T}(x) = \begin{cases} M_2(\mathbb{C}) & \text{if } m = 2 \\ \mathbb{C} \oplus M_2(\mathbb{C}) & \text{if } m > 2 \end{cases}$$

Proof. Every matrix in the subalgebra $\mathcal{T}(x)$ of the matrix algebra $M_m(\mathbb{C})$ is generated by $E_0^*A_0E_0^*$, $E_0^*A_1E_1^*$, $E_1^*A_1E_0^*$, $E_1^*A_0E_1^*$ and $E_1^*A_1E_1^*$. Let $E = E_0^*A_0E_0^*$, $L = E_0^*A_1E_1^*$, $M = \frac{1}{m-1}(E_1^*A_0E_1^* + E_1^*A_1E_1^*)$ and $N = \frac{1}{m-1}E_1^*A_1E_0^*$. The multiplication table is given by

$$\left(\begin{array}{c|cccc} 0 & E & L & M & N \\ \hline E & E & L & 0 & 0 \\ L & 0 & 0 & L & E \\ M & 0 & 0 & M & N \\ N & N & M & 0 & 0 \end{array} \right)$$

Define a subalgebra \mathcal{U} of $\mathcal{T}(x)$ generated by E, L, M, N . Define the isomorphism that takes

$$E \mapsto e_{11}; \quad L \mapsto e_{12}; \quad M \mapsto e_{22}; \quad N \mapsto e_{21}$$

where e_{ij} is the standard basis. Now if $R = E_1^* A_0 E_1^* - M$, it turns out that $RX = 0$ for all $X \in \mathcal{U}$. This gives us $\mathcal{T}(x) = \mathbb{C}R \oplus \mathcal{U}$. If $m = 2$, $R = 0$. if $m > 2$, $R \neq 0$. Thus, we have

$$\mathcal{T}(x) = \begin{cases} M_2(\mathbb{C}) & \text{if } m = 2 \\ \mathbb{C} \oplus M_2(\mathbb{C}) & \text{if } m > 2 \end{cases}$$

□

4.2 The Scheme $K_m \wr K_m$

Let $\mathcal{X} = (X, \{R_0, R_1, R_2\}) = K_m \wr K_m$ be a 2-class association scheme of order m^2 . Let $X = \{(x, y) : x, y \in [m]\}$ be the vertex set. Let $x = (1, 1)$ be the fixed base vertex. The relations of the association schemes are as follows

1. $R_0(x) = \{x\}$
2. $R_1(x) = \{(i, j) : 1 \leq i \leq m, 2 \leq j \leq m\}$
3. $R_2(x) = \{(i, 1) : 2 \leq i \leq m\}$

In order to study the Terwilliger algebra we will have to study the adjacency matrices of the scheme in details. Let the adjacency matrices A_i of \mathcal{X} be decomposed according to the partition $X = R_0(x) \cup R_1(x) \cup R_2(x)$. Then,

$$A_1 = \begin{pmatrix} 0 & \mathbf{u}_1^t & \mathbf{0}^t \\ \mathbf{u}_1 & B_1 & J \\ \mathbf{0} & J^t & B_2 \end{pmatrix}$$

where \mathbf{u}_1 is the all-one column vector of size $m(m-1)$, $\mathbf{0}$ is the all-zero column vector of size $(m-1)$, J is the $m(m-1) \times (m-1)$ all-one matrix, $B_1 = J_{m(m-1)} - (I_{m-1} \otimes J_m)$. B_2 is a

$(m - 1) \times (m - 1)$ zero matrix. A_2 is given by

$$A_2 = \begin{pmatrix} 0 & \mathbf{0}^t & \mathbf{u}_2^t \\ \mathbf{0} & C_1 & O \\ \mathbf{u}_2 & O^t & C_2 \end{pmatrix}$$

where $\mathbf{0}$ is the all-zero column vector of size $m(m - 1)$; \mathbf{u}_2 is the all-one column vector of size $(m - 1)$; O is a zero matrix of size $m(m - 1) \times (m - 1)$; $C_1 = I_{m-1} \otimes (J_m - I_m)$ and $C_2 = J_{m-1} - I_{m-1}$.

Lemma 4.2.1. \mathcal{X} is a P -polynomial scheme.

Proof. Direct computation shows that $A_2 = \frac{1}{m(m-1)}A_1^2 - \frac{m-2}{m-1}A_1 - I$. □

Lemma 4.2.2. For $\mathcal{X} = K_m \wr K_m$, A_1 is the adjacency matrix of a complete multipartite strongly regular graph with parameters $(m^2, m(m - 1), m(m - 2), m(m - 1))$. The first and the second eigenmatrices P and Q are the following :

$$P = Q = \begin{pmatrix} 1 & m(m - 1) & m - 1 \\ 1 & 0 & -1 \\ 1 & -m & m - 1 \end{pmatrix}$$

Proof. The first eigenmatrix is constructed from the parameters easily [19].

$$P = \begin{pmatrix} 1 & m(m - 1) & m - 1 \\ 1 & 0 & -1 \\ 1 & -m & m - 1 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} m^2 & 0 & 0 \\ 0 & m^2 & 0 \\ 0 & 0 & m^2 \end{pmatrix}$$

By using the definition of P and Q , we have the identity $PQ = m^2I$ (cf.[2, p.60]). Thus ,

$$Q = \begin{pmatrix} 1 & m(m-1) & m-1 \\ 1 & 0 & -1 \\ 1 & -m & m-1 \end{pmatrix}$$

□

Theorem 4.2.3. [4, Lemma 1.3.1] *Let Γ be a strongly regular graph with parameters (v, k, λ, μ) that is neither a clique nor a coclique. Then the eigenvalues of Γ are k, r, s with $r \geq 0, s \leq -1$, where r, s are the roots of the quadratic equation $\theta^2 + (\mu - \lambda)\theta + (\mu - k) = 0$.*

Lemma 4.2.4. *Let $\mathcal{X} = K_m \wr K_m$ be a 2-class association scheme of order m^2 . The values of r and s are 0 and $-m$, respectively.*

Proof. $\lambda = p_{11}^1 = m(m-2)$; $\mu = p_{11}^2 = m(m-1)$, and $k = p_{ii}^0 = m(m-1)$. With these values, the equation $\theta^2 + (\mu - \lambda)\theta + (\mu - k) = 0$ becomes $\theta^2 + m\theta + 0 = 0$ which gives us the values of r and s respectively.

□

Cameron, Goethals and Seidel introduced the concept of restricted vectors for strongly regular graphs. Tomiyama and Yamazaki extended the concept to describe the subconstituent algebra of 2-class association scheme constructed from a strongly regular graph. In the following section we will describe the Terwilliger algebra of our specific scheme $K_m \wr K_m$.

Definition 4.2.5. An eigenvalue of B_i for $i = 1, 2$ is called **restricted** whenever it has an eigenvector orthogonal to the all-one vector of size k_i .

Lemma 4.2.6. [5] *Let Γ be a strongly regular graph with eigenvalues k, r and s . Then*

1. *Suppose \mathbf{y} is a restricted eigenvector of B_1 with an eigenvalue θ . Then \mathbf{y} is the eigenvector of JJ^t with the eigenvalue $(r - \theta)(\theta - s)$, and $J^t\mathbf{y}$ is the zero vector or the restricted eigenvector of B_2 with the eigenvalue $r + s - \theta$. In particular, $J^t\mathbf{y}$ is zero if and only if $\theta \in \{r, s\}$.*

2. Suppose \mathbf{z} is a restricted eigenvector of B_2 with an eigenvalue θ' . Then \mathbf{z} is the eigenvector of $J^t J$ with the eigenvalue $(r - \theta')(\theta' - s)$, and $J\mathbf{z}$ is the zero vector or the restricted eigenvector of B_1 with the eigenvalue $r + s - \theta'$. In particular, $J\mathbf{z}$ is zero if and only if $\theta' \in \{r, s\}$.

Lemma 4.2.7. *Let $\mathcal{X} = K_m \wr K_m$ be a 2-class association scheme of order m^2 . The eigenvalues of B_1 are $m(m - 2)$, $-m$ and 0 with multiplicities 1 , $m - 2$ and $m^2 - 2m + 1$ respectively. For B_2 , 0 is the only eigenvalue with multiplicity $m - 1$.*

Proof. B_1 is the adjacency matrix of the strongly regular graph with parameters $(m(m - 1), m(m - 2), m(m - 3), m(m - 2))$. Hence the result follows from Lemma 4.2.3. □

Lemma 4.2.8. [25] *Let $\mathcal{X} = K_m \wr K_m$ be a 2-class association scheme of order m^2 . Let $\mathcal{T}(x)$ denote the Terwilliger algebra. Then the following hold*

1. *Let \mathbf{y} be a restricted eigenvector of B_1 with an eigenvalue θ . Then the vector space W over \mathbb{C} which is spanned by $(0, \mathbf{y}^t, \mathbf{0}^t)^t$ is a thin irreducible \mathcal{T} -module over \mathbb{C} and dimension of W is 1 if $\theta \in \{0, -m\}$.*
2. *Let \mathbf{z} be a restricted eigenvector of B_2 with an eigenvalue θ' . Then the vector space W' over \mathbb{C} which is spanned by $(0, \mathbf{0}^t, \mathbf{z}^t)^t$ is a thin irreducible \mathcal{T} -module over \mathbb{C} and dimension of W' is 1 if $\theta' \in \{0, -m\}$.*

Proof. 1. \mathbf{y} is the restricted eigenvector of B_1 with eigenvalue θ . Thus, $B_1 \mathbf{y} = \theta \mathbf{y}$. This gives us $(B_1 \mathbf{y})^t = (\theta \mathbf{y})^t$. Hence, $(0, (B_1 \mathbf{y})^t, \mathbf{0}^t)^t \in \text{span} \{(0, \mathbf{y}^t, \mathbf{0}^t)^t\}$. The vector space W over \mathbb{C} is spanned by $(0, \mathbf{y}^t, \mathbf{0}^t)^t$. Now, $A_1(0, \mathbf{y}^t, \mathbf{0}^t)^t = (0, (B_1 \mathbf{y})^t, \mathbf{0}^t)$ since \mathbf{y} is orthogonal to the all-one vector of size $m(m - 1)$ and the eigenvalues of restricted eigenvectors of B_1 are 0 and $-m$. Also, by [5] $J^t \mathbf{y} = \mathbf{0}$. Thus, W is A_1 -invariant and hence also \mathcal{M} -invariant. W is also \mathcal{M}^* -invariant.

2. Similar to (1)

□

The next lemma is a result from [25] suited to our scheme.

Lemma 4.2.9. [25, Lemma 3.3] *Let $\mathcal{X} = K_m \wr K_m$ be a 2-class association scheme of order m^2 and V denote the standard module. There exist irreducible modules $\{W_i\}_{1 \leq i \leq m(m-1)}$ and $\{W'_j\}_{2 \leq j \leq m-2}$ such that $V = (\oplus W_i) \oplus (\oplus W'_j)$.*

Proof. B_1 is a real symmetric matrix of order $m(m-1) \times m(m-1)$. Using Gram Schmidt process we can find eigenvectors $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{m(m-1)}$ such that $\{\mathbf{y}_i\}_{1 \leq i \leq m(m-1)}$ span $E_1^*V \cong \mathbb{C}^{m(m-1)}$, \mathbf{y}_1 is an all-one vector, $\langle \mathbf{y}_i, \mathbf{y}_j \rangle = 0$ for $i \neq j$. Let θ_i be the eigenvalue of B_1 with respect to the eigenvector \mathbf{y}_i for $2 \leq i \leq m(m-1)$. Now, $\theta_i \in \{0, -m\}$ for $2 \leq i \leq m(m-1)$. Let W_i denote the linear span of $(\langle \mathbf{y}_i, \mathbf{y}_1 \rangle, \mathbf{0}^t, \mathbf{0}^t)^t$; $(0, \mathbf{y}_i^t, \mathbf{0}^t)^t$; $(0, \mathbf{0}^t, (J^t \mathbf{y}_i)^t)^t$ over \mathbb{C} . Then W_i is a thin irreducible \mathcal{T} -module and $W_i \cap W_j = \{\mathbf{0}\}$ for $i \neq j$. Also,

$$\dim W_i = \begin{cases} 3 & \text{if } i = 1 \\ 1 & \text{if } 2 \leq i \leq m(m-1) \end{cases}$$

Observe that W_1 is the primary module generated by $(\langle \mathbf{y}_1, \mathbf{y}_1 \rangle, \mathbf{0}^t, \mathbf{0}^t)^t$; $(0, \mathbf{y}_1^t, \mathbf{0}^t)^t$ and $(0, \mathbf{0}^t, (J^t \mathbf{y}_1)^t)^t$ over \mathbb{C} and for $2 \leq i \leq m(m-1)$, W_i are generated by $(0, \mathbf{y}_i^t, \mathbf{0}^t)^t$. Note that $\mathbf{z}_1 = J^t \mathbf{y}_1$ is an eigenvector of B_2 . Let $\mathbf{z}_2, \dots, \mathbf{z}_{m-1}$ be the eigenvectors of B_2 such that $\mathbf{z}_1, \dots, \mathbf{z}_{m-1}$ span $E_2^*V \cong \mathbb{C}^{m-1}$ and $\langle \mathbf{z}_i, \mathbf{z}_j \rangle = 0$ for $1 \leq i \neq j \leq m-1$. Let W'_i be the linear span of $(0, \mathbf{0}^t, \mathbf{z}_i^t)^t$ over \mathbb{C} for $2 \leq i \leq m-1$. W'_i is a thin irreducible \mathcal{T} -module of dimension 1. Thus, we have the decomposition $V = (\oplus W_i) \oplus (\oplus W'_j)$ as desired.

□

Lemma 4.2.10. [25, Lemma 3.4] *Let $\mathcal{X} = K_m \wr K_m$ be a 2-class association scheme of order m^2 . Let $\{\theta_i\}_{1 \leq i \leq m(m-1)}$, $\{\theta'_i\}_{2 \leq i \leq m-1}$, $\{W_i\}_{1 \leq i \leq m(m-1)}$ and $\{W'_i\}_{2 \leq i \leq m-1}$. Then the following hold.*

1. *For all i with $2 \leq i \leq m(m-1)$, W_1 and W_i are not \mathcal{T} -isomorphic.*
2. *For all i and j with $1 \leq i \leq m(m-1)$ and $2 \leq j \leq m-1$, W_i and W'_j are not \mathcal{T} -isomorphic.*

3. For i and j with $2 \leq i, j \leq m(m-1)$, W_i and W_j are \mathcal{T} -isomorphic if and only if $\theta_i = \theta_j$.
4. For i and j with $2 \leq i, j \leq m-1$, W'_i and W'_j are \mathcal{T} -isomorphic.

In order to describe the Terwilliger algebra of $K_m \wr K_m$, let Λ denote the index set for the isomorphism classes of irreducible \mathcal{T} -modules. Lemma 4.2.9 and 4.2.10 gives us the orthogonal decomposition of the standard module V into irreducible \mathcal{T} -modules. Let us group the isomorphic irreducible modules together, say V_λ for $\lambda \in \Lambda$. Then, $V = \bigoplus_{\lambda \in \Lambda} V_\lambda$. By Lemma 2.3.3 we know that two irreducible modules in the decomposition of the standard module V are isomorphic if and only if they are the same type. We can conclude that for each subspace V_λ there is a unique central idempotent e_λ such that $V_\lambda = e_\lambda V$. Let W be an irreducible \mathcal{T} -module in the decomposition of V . The map taking $L \in e_\lambda \mathcal{T}$ to the endomorphism $w \mapsto Lw$ where $w \in W$ is an isomorphism. Hence, we have $e_\lambda V \cong \text{End}_{\mathbb{C}} W$. But, $\text{End}_{\mathbb{C}} W$ is isomorphic to the $k \times k$ complex matrix algebra, where k is the dimension of W . Thus, $\mathcal{T} = \bigoplus_{\lambda \in \Lambda} e_\lambda \mathcal{T}$ is isomorphic to a direct sum of complex matrix algebras.

In what follows, we will use the notation $\mathbb{C}^{\oplus n}$ for the direct sum of n copies of $M_1(\mathbb{C})$ algebra.

Theorem 4.2.11. *Let $\mathcal{X} = K_m \wr K_m$ be a 2-class association scheme of order m^2 . Then dimension of \mathcal{T} is 12 and $\mathcal{T} \cong M_3(\mathbb{C}) \oplus \mathbb{C}^{\oplus 3}$.*

Proof. Lemma 4.2.9 and 4.2.10 gives us a decomposition of the standard module V into irreducible \mathcal{T} -modules. The following is the list of non-isomorphic modules.

1. The primary module W_1
2. Each of W_i ($2 \leq i \leq m(m-1)$) belong to one of the two types corresponding to the eigenvalues $\theta_i \in \{0, -m\}$
3. For $2 \leq j \leq m-1$, W'_j are isomorphic.

Thus, $\dim \mathcal{T} = 3^2 + 1^2 + 1^2 + 1^2 = 12$ and $\mathcal{T} \cong M_3(\mathbb{C}) \oplus \mathbb{C}^{\oplus 3}$ following from the explanation above.

□

Remark 4.2.12. In chapter 3 we concluded that the Terwilliger algebra \mathcal{T} is the same as $\mathcal{T}_0(x)$ as the scheme is triply regular. For all $i, j, h \in \{0, 1, \dots, D\}$ the nonzero $E_i^* A_j E_h^*$ are the generators of \mathcal{T} . Thus, the generators of the irreducible modules should be $E_i^* A_j E_k^*$ invariant. For the scheme $K_m \wr K_m$ we can conclude that for $2 \leq i \leq m(m-1)$, \mathbf{y}_i is $E_1^* A_0 E_1^*$, $E_1^* A_1 E_1^*$, $E_1^* A_2 E_1^*$, $E_0^* A_1 E_1^*$ and $E_2^* A_1 E_1^*$ invariant. For the same reason for $2 \leq i \leq m-1$, \mathbf{z}_i is $E_0^* A_2 E_2^*$, $E_1^* A_1 E_2^*$, $E_2^* A_0 E_2^*$ and $E_2^* A_2 E_2^*$ invariant.

Throughout the remainder, we will use the notation $(K_m)^{\wr i}$ to denote an i -class association scheme $K_m \wr \dots \wr K_m$ (i copies).

4.3 The Terwilliger Algebra of $(K_m)^{\wr 3}$

In this section we describe the irreducible modules of $(K_m)^{\wr 3}$ and show that $\mathcal{T} \cong M_4(\mathbb{C}) \oplus \mathbb{C}^{\oplus 6}$. We will extend the irreducible modules in $(K_m)^{\wr 2}$ to find irreducible modules of $(K_m)^{\wr 3}$. We would like to mention here that $K_m \wr K_m$ is a distance regular graph, a P -polynomial scheme. Thus, A_2 could be represented as a polynomial of degree 2 in A_1 . Hence, A_1 , E_0^* and E_1^* were the only generators of \mathcal{T} . As we move on to three copies of K_m , the scheme is no longer a distance regular graph and hence not a P -polynomial scheme. Now the adjacency matrices can no longer be written as a polynomial in A_1 . We have to consider all the generators A_0 , A_1 , A_2 , A_3 , E_0^* , E_1^* , E_2^* and E_3^* making the calculations much more complicated.

4.3.1 The Scheme $(K_m)^{\wr 3}$

Let $\mathcal{X} = (X, \{R_0, R_1, R_2, R_3\}) = (K_m)^{\wr 3}$. Let $X = \{(x, y, z) : x, y, z \in [m]\}$. Let $x = (1, 1, 1)$ be the fixed base vertex. The relations of the association scheme are

1. $R_0(x) = \{(1, 1, 1)\}$
2. $R_1(x) = \{(i, j, k) : 1 \leq i, j \leq m, 2 \leq k \leq m\}$
3. $R_2(x) = \{(i, j, 1) : 1 \leq i \leq m, 2 \leq j \leq m\}$
4. $R_3(x) = \{(i, 1, 1) : 2 \leq i \leq m\}$

In order to study the Terwilliger algebra of $(K_m)^{\text{I3}}$ we will have to study the relation table of the scheme in details. Let the relation table of \mathcal{X} be decomposed according to the partition $X = R_0(x) \cup R_1(x) \cup R_2(x) \cup R_3(x)$.

4.3.2 The Embedded Structure of $K_m \wr K_m$ in $(K_m)^{\text{I3}}$

The relation table of the $K_m \wr K_m$ can be build by combining the adjacency matrices A_1 and A_2 described in the earlier section. Let the block of the relation table indexed by the vertices in $R_1(x)$ be denoted as S . Then S has entries 0, 1 or 2. We now form a new block from S by replacing 1 with 2, 2 with 3 and 0 with 0 and call this S_{new} . It is not hard to see that S_{new} is the block of the relation table $(K_m)^{\text{I3}}$ that is indexed by the vertices in $R_2(x)$.

The relation table of $(K_m)^{\text{I3}}$ is given by

$$\begin{pmatrix} 0 & \mathbf{p}^t & 2\mathbf{q}^t & 3\mathbf{r}^t \\ \mathbf{p} & D_1 & J_1 & J_2 \\ 2\mathbf{q} & J_1^t & S_{new} & 2J_3 \\ 3\mathbf{r} & J_2^t & 2J_3^t & 3D_2 \end{pmatrix}$$

where \mathbf{p} is the all-one column vector of size $m^2(m-1)$, \mathbf{q} is the all-one column vector of size $m(m-1)$, \mathbf{r} is the all-one column vector of size $m-1$, J_1 is the all-one matrix of size $m^2(m-1) \times m(m-1)$, J_2 is the all-one matrix of size $m^2(m-1) \times (m-1)$, J_3 is the all-one matrix of size $m(m-1) \times (m-1)$ $D_1 = I_m \otimes S_{new} + (J_m - I_m) \otimes J_{m(m-1)}$ and $D_2 = J_{m-1} - I_{m-1}$. Here J_{m-1} is the all-one matrix of size $(m-1) \times (m-1)$ and I_{m-1} is the identity matrix of size $(m-1) \times (m-1)$.

4.3.3 The Primary \mathcal{T} -module and the Subconstituents E_0^*V , E_2^*V and E_3^*V

Theorem 4.3.1. *Let $\mathcal{X} = (K_m)^{\text{I3}}$ be a 3-class association scheme of order m^3 and V denote the standard module. The linear span of $(1, \mathbf{0}_1^t, \mathbf{0}_2^t, \mathbf{0}_3^t)^t$, $(0, \mathbf{w}_1^t, \mathbf{0}_2^t, \mathbf{0}_3^t)^t$, $(0, \mathbf{0}_1^t, \mathbf{y}_1^t, \mathbf{0}_3^t)^t$, $(0, \mathbf{0}_1^t, \mathbf{0}_2^t, \mathbf{z}_1^t)^t$ generates the primary \mathcal{T} -module. Here, \mathbf{w}_1 is the column vector of dimension $m^2(m-1)$ with all entries one, $\mathbf{0}_1, \mathbf{0}_2, \mathbf{0}_3$ are zero column vectors of dimension $m^2(m-1)$, $m(m-1)$ and*

$(m - 1)$ respectively.

Proof. Straightforward. \square

Theorem 4.3.2. *Let $\mathcal{X} = (K_m)^{\wr 3}$ be a 3-class association scheme of order m^3 and V denote the standard module. There exist irreducible modules $\{W_i^3\}_{2 \leq i \leq (m-1)}$ such that $(0, \mathbf{0}_1^t, \mathbf{0}_2^t, \mathbf{z}_1^t)^t$ and $\{W_i^3\}_{2 \leq i \leq (m-1)}$ together constitute E_3^*V .*

Proof. We have seen in Lemma 4.2.9 that there exist vectors $\mathbf{z}_1, \dots, \mathbf{z}_{m-1}$ such that $\mathbf{z}_1, \dots, \mathbf{z}_{m-1}$ span \mathbb{C}^{m-1} and $\langle \mathbf{z}_i, \mathbf{z}_j \rangle = 0$ for $1 \leq i \neq j \leq m-1$. For $2 \leq i \leq m-1$, let W_i^3 be the linear span of $(0, \mathbf{0}_1^t, \mathbf{0}_2^t, \mathbf{z}_i^t)^t$ over \mathbb{C} where $\mathbf{0}_1, \mathbf{0}_2$ are zero column vectors of dimension $m^2(m-1)$ and $m(m-1)$ respectively. For $K_m \wr K_m$ the generators $E_i^* A_h E_i^*$ act on the modules W_i^3 and the significant nonzero actions are $E_2^* A_0 E_2^*$ and $E_2^* A_2 E_2^*$. The embedded structure of $K_m \wr K_m$ in $(K_m)^{\wr 3}$ ensures that for $(K_m)^{\wr 3}$ the generators $E_i^* A_h E_i^*$ act on the linear space $\{W_i^3\}_{2 \leq i \leq (m-1)}$ and the only nonzero actions are $E_3^* A_0 E_3^*$ and $E_3^* A_3 E_3^*$. It is obvious that $(0, \mathbf{0}_1^t, \mathbf{0}_2^t, \mathbf{z}_i^t)^t$ are $E_0^* A_3 E_3^*$, $E_1^* A_1 E_3^*$ and $E_2^* A_2 E_3^*$ invariant. Also, $\{W_i^3\}_{2 \leq i \leq (m-1)}$ is a irreducible \mathcal{T} -module of dimension 1. Hence, the result follows. \square

Theorem 4.3.3. *Let $\mathcal{X} = (K_m)^{\wr 3}$ be a 3-class association scheme of order m^3 and V denote the standard module. There exist irreducible modules $\{W_i^2\}_{2 \leq i \leq m(m-1)}$ such that $(0, \mathbf{0}_1^t, \mathbf{y}_1^t, \mathbf{0}_3^t)^t$ and $\{W_i^2\}_{2 \leq i \leq m(m-1)}$ constitute E_2^*V .*

Proof. We have seen in Lemma 4.2.9 that there exist vectors $\mathbf{y}_1, \dots, \mathbf{y}_{m(m-1)}$ such that $\mathbf{y}_1, \dots, \mathbf{y}_{m(m-1)}$ span $\mathbb{C}^{m(m-1)}$ and $\langle \mathbf{y}_i, \mathbf{y}_j \rangle = 0$ for $1 \leq i \neq j \leq m(m-1)$. For $2 \leq i \leq m(m-1)$, let W_i^2 be the linear span of $(0, \mathbf{0}_1^t, \mathbf{y}_i^t, \mathbf{0}_3^t)^t$ over \mathbb{C} where $\mathbf{0}_1, \mathbf{0}_3$ are zero column vectors of dimension $m^2(m-1)$ and $(m-1)$ respectively. For $K_m \wr K_m$ the generators $E_i^* A_h E_i^*$ act on the modules W_i^2 and the significant nonzero actions are $E_1^* A_0 E_1^*$, $E_1^* A_1 E_1^*$ and $E_1^* A_2 E_1^*$. The embedded structure of $K_m \wr K_m$ in $(K_m)^{\wr 3}$ ensures that for $(K_m)^{\wr 3}$ the generators $E_i^* A_h E_i^*$ act on the linear space $\{W_i^2\}_{2 \leq i \leq m(m-1)}$ and the nonzero actions are $E_1^* A_0 E_1^*$, $E_2^* A_2 E_2^*$ and $E_2^* A_3 E_2^*$. Also, $(0, \mathbf{0}_1^t, \mathbf{y}_i^t, \mathbf{0}_3^t)^t$ is $E_0^* A_2 E_2^*$, $E_1^* A_1 E_2^*$ and $E_3^* A_2 E_2^*$ invariant. Also, $\{W_i^2\}_{2 \leq i \leq m(m-1)}$ is a irreducible \mathcal{T} -module of dimension 1. Hence, the result follows.

□

Theorem 4.3.4. *Let $\mathcal{X} = (K_m)^{\wr 3}$ be a 3-class association scheme of order m^3 and V denote the standard module. $(1, \mathbf{0}_1^t, \mathbf{0}_2^t, \mathbf{0}_3^t)^t$ constitutes E_0^*V , where $\mathbf{0}_1, \mathbf{0}_2, \mathbf{0}_3$ are zero column vectors of dimension $m^2(m-1)$, $m(m-1)$ and $(m-1)$, respectively.*

4.3.4 The Subconstituent E_1^*V

In this section we will describe the irreducible modules that span E_1^*V . We know that $E_1^*V \cong \mathbb{C}^{m^2(m-1)}$. We will construct new vectors in the following way. Observe that any $m^2(m-1)$ dimensional column vector can be partitioned into m equal parts each of dimension $m(m-1)$. Let each part be denoted by the index j where $j \in \{1, 2, \dots, m\}$ so that the $m^2(m-1)$ dimensional column vector is of the form $(\mathbf{v}_1^t, \mathbf{v}_2^t, \dots, \mathbf{v}_m^t)^t$ where each $\mathbf{v}_j, j \in \{1, 2, \dots, m\}$ is a $m(m-1)$ dimensional column vector. For each $i \in \{2, 3, \dots, m(m-1)\}$ and $j \in \{1, 2, \dots, m\}$ let $\mathbf{u}_{j_i} = (\mathbf{v}_1^t, \mathbf{v}_2^t, \dots, \mathbf{v}_l^t, \dots, \mathbf{v}_m^t)^t$ denote the $m^2(m-1)$ dimensional column vector such that $\mathbf{v}_l = \delta_{lj}\mathbf{y}_i$ where δ_{lj} is the Kronecker delta.

Lemma 4.3.5. *Let $\mathcal{X} = (K_m)^{\wr 3}$ be a 3-class association scheme of order m^3 and V denote the standard module. There exist irreducible modules $\{W_i^1\}_{1 \leq i \leq m\{m(m-1)-1\}}$ where $\{W_i^1\}$ is the linear span of $(0, \mathbf{u}_{j_i}^t, \mathbf{0}_2^t, \mathbf{0}_3^t)^t$.*

Proof. For $K_m \wr K_m$ the generators $E_i^*A_hE_i^*$ act on the modules W_i and the only nonzero actions are $E_1^*A_0E_1^*$, $E_1^*A_1E_1^*$ and $E_1^*A_2E_1^*$. $D_1 = I \otimes S_{new} + (J - I) \otimes I$ implies that for $(K_m)^{\wr 3}$ the generators $E_i^*A_hE_i^*$ act on the linear space $\{W_i^1\}_{1 \leq i \leq m\{m(m-1)-1\}}$ where $\{W_i^1\}$ is the linear span of $(0, \mathbf{u}_{j_i}^t, \mathbf{0}_2^t, \mathbf{0}_3^t)^t$ and the only nonzero actions are $E_1^*A_0E_1^*$, $E_1^*A_2E_1^*$, $E_1^*A_3E_1^*$, $E_0^*A_1E_1^*$, $E_2^*A_1E_1^*$ and $E_3^*A_1E_1^*$. It follows that $\{W_i^1\}$ is an irreducible module of dimension 1. □

Lemma 4.2.11 tells us that among the irreducible modules $\{W_i\}_{2 \leq i \leq m(m-1)}$ of $K_m \wr K_m$ there are two non-isomorphic \mathcal{T} -modules of dimension 1. So far for $(K_m)^{\wr 3}$ we have the following non isomorphic \mathcal{T} -modules

1. The primary module of dimension 4.

2. Two non-isomorphic \mathcal{T} -modules of dimension 1 in E_1^*V .
3. Two non-isomorphic \mathcal{T} -modules of dimension 1 in E_2^*V .
4. One non-isomorphic \mathcal{T} -modules of dimension 1 in E_3^*V .

From Theorem 3.3.11 we have the dimension of $\mathcal{T}((K_m)^{\wr 3})$ is 22. From our above discussion the five non-isomorphic modules account for $4^2 + 1 + 1 + 1 + 1 + 1 = 21$ of the dimension. That leaves us with one non-isomorphic \mathcal{T} module of dimension 1 and multiplicity $m - 1$ in the sub constituent E_1^*V .

Lemma 4.3.6. *Pick any nonzero vector $\mathbf{v} \in E_1^*V$ which is orthogonal to all the irreducible modules $\{W_i^1\}_{1 \leq i \leq m\{m(m-1)-1\}}$. Then \mathbf{v} spans a one-dimensional irreducible \mathcal{T} -module. The actions of $E_1^*A_0E_1^*$, $E_1^*A_1E_1^*$, $E_1^*A_2E_1^*$, $E_1^*A_3E_1^*$, $E_1^*A_1E_1^*$, $E_2^*A_1E_1^*$ and $E_3^*A_1E_1^*$ are nonzero on this element and rest are all zero.*

Proof. The module is one-dimensional is accounted by the discussion above the lemma. The module is orthogonal and non-isomorphic to all the irreducible modules $\{W_i^1\}_{1 \leq i \leq m\{m(m-1)-1\}}$. \mathbf{v} is invariant under the actions of $E_1^*A_0E_1^*$, $E_1^*A_1E_1^*$, $E_1^*A_2E_1^*$, $E_1^*A_3E_1^*$, $E_1^*A_1E_1^*$, $E_2^*A_1E_1^*$ and $E_3^*A_1E_1^*$.

□

We will conclude the section by collecting all our non-isomorphic \mathcal{T} -modules and describing the Terwilliger algebra of $(K_m)^{\wr 3}$. The reasoning is similar to $K_m \wr K_m$.

Theorem 4.3.7. *Let $\mathcal{X} = (K_m)^{\wr 3}$ be a 3-class association scheme of order m^3 . Then dimension of \mathcal{T} is 22 and $\mathcal{T} \cong M_4(\mathbb{C}) \oplus \mathbb{C}^{\oplus 6}$.*

Proof. The dimension follows from Theorem 3.2.11. There are seven non-isomorphic \mathcal{T} -modules in $K_m \wr K_m \wr K_m$ one of which is the primary module of dimension 4 and six modules are each of dimension 1. Thus, $\mathcal{T} \cong M_4(\mathbb{C}) \oplus \mathbb{C}^{\oplus 6}$.

□

4.4 The Terwilliger Algebra of $(K_m)^4$

In this section we will study the Terwilliger algebra of the 4-class association scheme $(K_m)^4$ and show that the Terwilliger algebra $\mathcal{T} \cong M_5(\mathbb{C}) \oplus \mathbb{C}^{\oplus 10}$. We will use the embedded structure of $(K_m)^{l3}$ in $(K_m)^{l4}$ to describe the Terwilliger algebra.

4.4.1 The Scheme $(K_m)^4$

Let $\mathcal{X} = (X, \{R_0, R_1, R_2, R_3, R_4\}) = (K_m)^4$ be a 4-class association scheme of order m^4 . Let $X = \{(x_1, x_2, x_3, x_4) : x_1, x_2, x_3, x_4 \in [m]\}$. Let $x = (1, 1, 1, 1)$ be the fixed base vertex. The relations of the association scheme are

1. $R_0(x) = \{(1, 1, 1, 1)\}$
2. $R_1(x) = \{(i, j, k, l) : 1 \leq i, j, k \leq m, 2 \leq l \leq m\}$
3. $R_2(x) = \{(i, j, k, 1) : 1 \leq i, j \leq m, 2 \leq k \leq m\}$
4. $R_3(x) = \{(i, j, 1, 1) : 1 \leq i \leq m, 2 \leq j \leq m\}$
5. $R_4(x) = \{(i, 1, 1, 1) : 2 \leq i \leq m\}$

4.4.2 The Embedded Structure of $(K_m)^{l3}$ in $(K_m)^{l4}$

We will build the relation table of $(K_m)^{l4}$ from that of $(K_m)^{l3}$. We will form new blocks D_{C4} and S_{C4} from D_1 and S_{new} respectively in the following way. Replace 1 with 2, 2 with 3 and 3 with 4 in D_1 and S_{new} to form D_{C4} and S_{C4} respectively. D_{C4} is the block in the relation table of $(K_m)^{l4}$ that is indexed by the vertices in $R_2(x)$. S_{C4} is the block in the relation table of $(K_m)^{l4}$ that is indexed by the vertices in $R_3(x)$.

The relation table of $(K_m)^{l4}$ is given by

$$\begin{pmatrix} 0 & \mathbf{s}^t & 2\mathbf{p}^t & 3\mathbf{q}^t & 4\mathbf{r}^t \\ \mathbf{s} & D_3 & J_6 & J_5 & J_4 \\ 2\mathbf{p} & J_6^t & D_{C4} & 2J_1 & 2J_2 \\ 3\mathbf{q} & J_5^t & 2J_1^t & S_{C4} & 3J_3 \\ 4\mathbf{r} & J_4^t & J_2^t & 3J_3^t & 4D_2 \end{pmatrix}$$

where \mathbf{s} is the all-one column vector of size $m^3(m-1)$, J_4 is the all-one matrix of size $m^3(m-1) \times (m-1)$, J_5 is the all-one matrix of size $m^3(m-1) \times m(m-1)$, J_6 is the all-one matrix of size $m^3(m-1) \times m^2(m-1)$ $D_3 = I_m \otimes D_{C4} + (J_m - I_m) \otimes J_{m^2(m-1)}$ and $\mathbf{p}, \mathbf{q}, \mathbf{r}, J_1, J_2, J_3, D_2$ have the same meaning as described in section 4.3.2.

4.4.3 The Primary \mathcal{T} -module and the Subconstituents E_i^*V

Theorem 4.4.1. *Let $\mathcal{X} = (K_m)^{t4}$ be a 4-class association scheme of order m^4 and V denote the standard module. The linear span of $(1, \mathbf{0}_4^t, \mathbf{0}_1^t, \mathbf{0}_2^t, \mathbf{0}_3^t)^t$, $(1, \mathbf{t}_1^t, \mathbf{0}_1^t, \mathbf{0}_2^t, \mathbf{0}_3^t)^t$, $(0, \mathbf{0}_4^t, \mathbf{w}_1^t, \mathbf{0}_2^t, \mathbf{0}_3^t)^t$, $(0, \mathbf{0}_4^t, \mathbf{0}_1^t, \mathbf{y}_1^t, \mathbf{0}_3^t)^t$, $(0, \mathbf{0}_4^t, \mathbf{0}_1^t, \mathbf{0}_2^t, \mathbf{z}_1^t)^t$ generates the primary \mathcal{T} -module. Here, \mathbf{t}_1 is the column vector of dimension $m^3(m-1)$ with all entries one, $\mathbf{0}_4, \mathbf{0}_1, \mathbf{0}_2, \mathbf{0}_3$ are zero column vectors of dimension $m^3(m-1), m^2(m-1), m(m-1)$ and $(m-1)$ respectively. The vectors $\mathbf{w}_1, \mathbf{y}_1$ and \mathbf{z}_1 are as defined in previous sections.*

Proof. Straightforward. □

Theorem 4.4.2. *Let $\mathcal{X} = (K_m)^{t4}$ be a 4-class association scheme of order m^4 and V denote the standard module. There exist irreducible modules $\{W_i^4\}_{2 \leq i \leq (m-1)}$ which are linear span of $(0, \mathbf{0}_4^t, \mathbf{0}_1^t, \mathbf{0}_2^t, \mathbf{z}_1^t)^t$ over \mathbb{C} such that $(0, \mathbf{0}_4^t, \mathbf{0}_1^t, \mathbf{0}_2^t, \mathbf{z}_1^t)^t$ and $\{W_i^4\}_{2 \leq i \leq (m-1)}$ together constitute E_4^*V . Moreover, the generators $E_i^*A_nE_i^*$ act on the modules $\{W_i^4\}$ and the only nonzero actions are $E_4^*A_0E_4^*, E_4^*A_4E_4^*, E_0^*A_4E_4^*, E_1^*A_1E_4^*, E_2^*A_2E_4^*$ and $E_3^*A_3E_4^*$.*

Proof. Similar to Lemma 4.3.2 □

Theorem 4.4.3. *Let $\mathcal{X} = (K_m)^{l_4}$ be a 4-class association scheme of order m^4 and V denote the standard module. $(1, \mathbf{0}_4^t, \mathbf{0}_1^t, \mathbf{0}_2^t, \mathbf{0}_3^t)^t$ constitutes E_0^*V . Here, $\mathbf{0}_4, \mathbf{0}_1, \mathbf{0}_2, \mathbf{0}_3$ are zero column vectors of dimension $m^3(m-1)$, $m^2(m-1)$, $m(m-1)$ and $(m-1)$ respectively.*

Theorem 4.4.4. *Let $\mathcal{X} = (K_m)^{l_4}$ be a 4-class association scheme of order m^4 and V denote the standard module. There exist irreducible modules $\{W_i^5\}_{2 \leq i \leq m(m-1)}$ which are linear span of $(0, \mathbf{0}_4^t, \mathbf{0}_1^t, \mathbf{y}_i^t, \mathbf{0}_3^t)^t$ over \mathbb{C} such that $(0, \mathbf{0}_4^t, \mathbf{0}_1^t, \mathbf{y}_1^t, \mathbf{0}_3^t)^t$ and $\{W_i^5\}_{2 \leq i \leq m(m-1)}$ together constitute E_3^*V . Moreover, the generators $E_i^*A_hE_i^*$ act on the modules $\{W_i^4\}$ and the only nonzero actions are $E_3^*A_0E_3^*$, $E_3^*A_3E_3^*$, $E_3^*A_4E_3^*$, $E_0^*A_3E_3^*$, $E_1^*A_1E_3^*$, $E_2^*A_2E_3^*$ and $E_4^*A_3E_3^*$.*

Proof. Similar to Lemma 4.3.3 □

Theorem 4.4.5. *Let $\mathcal{X} = (K_m)^{l_4}$ be a 4-class association scheme of order m^4 and V denote the standard module. There exist irreducible modules $\{W_i^6\}_{1 \leq i \leq m\{m(m-1)-1\}}$ such that $(0, \mathbf{0}_4^t, \mathbf{w}_1^t, \mathbf{0}_2^t, \mathbf{0}_3^t)^t$ and $\{W_i^6\}_{1 \leq i \leq m\{m(m-1)-1\}}$ together constitute E_2^*V .*

Proof. We saw in the previous section that for $i \in \{2, 3, \dots, m(m-1)\}$ and $j \in \{1, 2, \dots, m\}$, $\mathbf{u}_{j_i} = (\mathbf{v}_1^t, \mathbf{v}_2^t, \dots, \mathbf{v}_i^t, \dots, \mathbf{v}_m^t)^t$ and $m-1$ vectors isomorphic to the vector described in Theorem 4.3.6 generated one dimensional modules for the first subconstituent for the scheme $(K_m)^{l_3}$. Let these $m^2(m-1) - 1$ vectors be denoted as g_i for $2 \leq i \leq m^2(m-1)$. For $2 \leq i \leq m^2(m-1)$, let W_i^6 be the linear span of $(0, \mathbf{0}_4^t, \mathbf{g}_i^t, \mathbf{0}_2^t, \mathbf{0}_3^t)^t$ over \mathbb{C} . For $(K_m)^{l_3}$ the generators $E_i^*A_hE_i^*$ act on the modules $\{W_i^1\}_{1 \leq i \leq m\{m(m-1)-1\}}$ and the $m-1$ isomorphic \mathcal{T} modules and the significant nonzero actions are $E_1^*A_0E_1^*$, $E_1^*A_1E_1^*$, $E_1^*A_2E_1^*$ and $E_1^*A_3E_1^*$. The embedded structure of $(K_m)^{l_3}$ in $(K_m)^{l_4}$ ensures that for $(K_m)^{l_4}$ the generators $E_i^*A_hE_i^*$ act on the modules $\{W_i^6\}_{2 \leq i \leq m^2(m-1)}$ and the nonzero actions are $E_2^*A_0E_2^*$, $E_2^*A_2E_2^*$, $E_2^*A_3E_2^*$, $E_2^*A_4E_2^*$, $E_0^*A_2E_2^*$, $E_1^*A_1E_2^*$, $E_3^*A_2E_2^*$ and $E_4^*A_2E_2^*$. Hence, the result follows. □

So far we have discussed the subconstituents E_0^*V , E_2^*V , E_3^*V , E_4^*V for the scheme $(K_m)^{l_4}$. We know that $E_1^*V \cong \mathbb{C}^{m^3(m-1)}$. Observe that any $m^3(m-1)$ dimensional column vector can be partitioned into m equal parts each of dimension $m^2(m-1)$. Let each part be denoted by

the index j where $j \in \{1, 2, \dots, m\}$ so that the $m^3(m-1)$ dimensional vector is of the form $(\mathbf{v}_1^t, \mathbf{v}_2^t, \dots, \mathbf{v}_m^t)^t$ where each \mathbf{v}_j , $j \in \{1, 2, \dots, m\}$ is a $m^2(m-1)$ dimensional column vector. For each $i \in \{2, 3, \dots, m^2(m-1)\}$ and $j \in \{1, 2, \dots, m\}$ let $\mathbf{h}_{j_i} = (\mathbf{v}_1^t, \mathbf{v}_2^t, \dots, \mathbf{v}_i^t, \dots, \mathbf{v}_m^t)^t$ denote the $m^3(m-1)$ dimensional column vector such that $\mathbf{v}_l = \delta_{lj}\mathbf{g}_i$ where δ_{lj} is the Kronecker delta.

Lemma 4.4.6. *Let $\mathcal{X} = (K_m)^{\wr 3}$ be a 3-class association scheme of order m^3 and V denote the standard module. There exist irreducible modules $\{W_i^{\wr 7}\}_{1 \leq i \leq m\{m^2(m-1)-1\}}$ where $\{W_i^{\wr 7}\}$ is the linear span of $(0, \mathbf{g}_{j_i}^t, \mathbf{0}_1^t, \mathbf{0}_2^t, \mathbf{0}_3^t)^t$.*

Proof. For $(K_m)^{\wr 3}$ the generators $E_i^* A_h E_i^*$ act on the modules W_i^1 and the only nonzero actions are $E_1^* A_0 E_1^*$, $E_1^* A_1 E_1^*$ and $E_1^* A_2 E_1^*$. $D_3 = I_m \otimes D_{C4} + (J_m - I_m) \otimes J_{m^2(m-1)}$ implies that for $(K_m)^{\wr 4}$ the generators $E_i^* A_h E_i^*$ act on the linear space $\{W_i^{\wr 7}\}_{1 \leq i \leq m\{m^2(m-1)-1\}}$ where $\{W_i^{\wr 7}\}$ is the linear span of $(0, \mathbf{g}_{j_i}^t, \mathbf{0}_1^t, \mathbf{0}_2^t, \mathbf{0}_3^t)^t$ and the only nonzero actions are $E_1^* A_0 E_1^*$, $E_1^* A_1 E_1^*$, $E_1^* A_2 E_1^*$, $E_1^* A_3 E_1^*$, $E_0^* A_1 E_1^*$, $E_2^* A_1 E_1^*$, $E_3^* A_1 E_1^*$ and $E_4^* A_1 E_1^*$. It follows that $\{W_i^{\wr 7}\}$ is an irreducible module of dimension 1. □

We have so far described the subconstituents $E_0^* V$, $E_2^* V$, $E_3^* V$, $E_4^* V$ completely and only part of $E_1^* V$. For $(K_m)^{\wr 4}$ we have the following non-isomorphic \mathcal{T} -modules

1. The primary module of dimension 5.
2. Three non-isomorphic \mathcal{T} -modules of dimension 1 in $E_1^* V$.
3. Three non-isomorphic \mathcal{T} -modules of dimension 1 in $E_2^* V$.
4. Two non-isomorphic \mathcal{T} -modules of dimension 1 in $E_3^* V$.
5. One non-isomorphic \mathcal{T} -modules of dimension 1 in $E_4^* V$.

From Theorem 3.3.11 we have the dimension of $\mathcal{T}((K_m)^{\wr 4})$ is 35. From our above discussion the ten non-isomorphic modules account for $5^2 + (1 + 1 + 1) + (1 + 1 + 1) + (1 + 1) + 1 = 34$ of the dimension. That leaves us with one non-isomorphic \mathcal{T} module of dimension 1 in the subconstituent $E_1^* V$ of multiplicity $m - 1$.

Lemma 4.4.7. *Pick any nonzero vector $\mathbf{v} \in E_1^*V$ which is orthogonal to all the irreducible modules $\{W_i^7\}_{2 \leq i \leq m^3(m-1)}$. Then \mathbf{v} spans a one-dimensional irreducible \mathcal{T} -module. The actions of $E_1^*A_0E_1^*$, $E_1^*A_1E_1^*$, $E_1^*A_2E_1^*$, $E_1^*A_3E_1^*$, $E_1^*A_4E_1^*$, $E_0^*A_1E_1^*$, $E_2^*A_1E_1^*$, $E_3^*A_1E_1^*$ and $E_4^*A_1E_1^*$ are nonzero on this element and rest are all zero.*

Proof. The module is one-dimensional is accounted by the discussion above the lemma. The module is orthogonal and non-isomorphic to all the irreducible modules $\{W_i^7\}_{2 \leq i \leq m^3(m-1)}$. The actions of $E_1^*A_0E_1^*$, $E_1^*A_1E_1^*$, $E_1^*A_2E_1^*$, $E_1^*A_3E_1^*$, $E_1^*A_4E_1^*$, $E_0^*A_1E_1^*$, $E_2^*A_1E_1^*$, $E_3^*A_1E_1^*$ and $E_4^*A_1E_1^*$ are nonzero on this element and rest are all zero. □

We will conclude the section by collecting all our non-isomorphic \mathcal{T} -modules and describing the Terwilliger algebra of $(K_m)^{l^4}$. The reasoning is similar to $K_m \wr K_m$.

Theorem 4.4.8. *Let $\mathcal{X} = (K_m)^{l^4}$ be a 4-class association scheme of order m^4 . Then dimension of \mathcal{T} is 35 and $\mathcal{T} \cong M_5(\mathbb{C}) \oplus \mathbb{C}^{\oplus 10}$.*

Proof. The dimension follows from Theorem 3.3.11. There are eleven non-isomorphic \mathcal{T} -modules in $(K_m)^{l^4}$ one of which is the primary module of dimension 5 and ten modules are each of dimension 1. Thus, $\mathcal{T} \cong M_5(\mathbb{C}) \oplus \mathbb{C}^{\oplus 10}$. □

4.5 The Terwilliger Algebra of $(K_m)^{\wr D}$ for $m \geq 3$

In this section we will develop a recursive method to describe the Terwilliger algebra of a D -class association scheme $(K_m)^{\wr(D)}$ from a $(D-1)$ -class scheme $(K_m)^{\wr(D-1)}$. Let K_m denote the one-class association scheme $([m], \{R_0, R_1\})$ where $[m] = \{1, 2, \dots, m\}$; i.e, let $A_1 = J - I$ be the associate matrix of the non-diagonal relation R_1 . Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote the D -class scheme $(K_m)^{\wr D}$ with $X = [m] \times [m] \times \dots \times [m] = \{(a_1, a_2, \dots, a_D) : a_i \in [m], \text{ for } i = 1, 2, \dots, D\}$

Let $(1, 1, \dots, 1) \in X$ be a fixed base vertex x of \mathcal{X} . Without loss of generality, we can arrange the vertices as follows.

- $R_0(x) = \{x\}$
- for $i = 1, 2, \dots, D-1$, $R_i(x) = \{(a_1, a_2, \dots, a_{D-i-1}, b, 1, 1, \dots, 1) : a_k \in [m] \text{ for } k \in \{1, 2, \dots, D-i-1\}, b \in [m] - \{1\}\}$.
- $R_D(x) = \{(a, 1, 1, \dots, 1) : a \in \{2, 3, \dots, m\}\}$

Next, we will look at the relation table of the D -class association scheme.

First let us look at the relation table of the $(D-1)$ -class association scheme $(K_m)^{\wr(D-1)}$.

$$\begin{pmatrix} 0 & \mathbf{s}_{D-1} & 2\mathbf{s}_{D-2} & \cdots & (D-2)\mathbf{s}_2 & (D-1)\mathbf{s}_1 \\ \mathbf{s}_{D-1}^t & T_{D-1} & J_{11} & \cdots & J_{1(D-3)} & J_{1(D-2)} \\ 2\mathbf{s}_{D-2}^t & J_{11}^t & T_{D-2} & \cdots & 2J_{2(D-4)} & 2J_{2(D-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (D-2)\mathbf{s}_2^t & J_{1(D-3)}^t & 2J_{2(D-4)}^t & \cdots & T_2 & (D-2)J_{(D-2)1} \\ (D-1)\mathbf{s}_1^t & J_{1(D-2)}^t & 2J_{2(D-3)}^t & \cdots & (D-2)J_{(D-2)1}^t & T_1 \end{pmatrix}$$

Here for $i \in \{2, 3, \dots, D-1\}$, \mathbf{s}_i are all-one column vectors of size $m^{i-1}(m-1)$, J_{ij} are all-one matrices, $T_1 = (D-1)(J_{m-1} - I_{m-1})$ and $T_i = I_m \otimes T_{i-1} + (D-i)(J_m - I_m) \otimes J_{m^{i-2}(m-1)}$ for $i \in \{2, 3, \dots, D-1\}$.

Now we will build the relation table for the D -class association scheme $(K_m)^{\wr D}$. In the diagonal blocks T_i make the following changes. 0 is kept same, 1 is replaced with 2, 2 is

replaced with 3, etc. $0 \rightarrow 0, 1 \rightarrow 2, \dots, i \rightarrow i + 1, \dots, (D - 1) \rightarrow D$. Let us name the new blocks U_i for $i \in \{1, 2, \dots, D - 1\}$. It is not hard to see that U_i are the diagonal blocks of the D -class association scheme. Let $U_D = I_m \otimes U_{D-1} + (J_m - I_m) \otimes J_{m^{D-2}(m-1)}$, \mathbf{s}_D is the all-one column matrix of size $m^{D-1}(m-1)$. Abusing notation and denoting all the all-one matrices in the relation table as J for all dimensions the relation table of the D -class association scheme is

$$\begin{pmatrix} 0 & \mathbf{s}_D & 2\mathbf{s}_{D-1} & 3\mathbf{s}_{D-2} & \cdots & (D-1)\mathbf{s}_2 & (D)\mathbf{s}_1 \\ \mathbf{s}_D^t & U_D & J & J & \cdots & J & J \\ 2\mathbf{s}_{D-1}^t & J & U_{D-1} & 2J & \cdots & 2J & 2J \\ 3\mathbf{s}_{D-2}^t & J & 2J & U_{D-2} & \cdots & 3J & 3J \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (D-1)\mathbf{s}_2^t & J & 2J & 3J & \cdots & U_2 & (D-1)J \\ (D)\mathbf{s}_1^t & J & 2J & 3J & \cdots & (D-1)J & U_1 \end{pmatrix}$$

In the next couple of paragraphs we will discuss the subconstituents of the D -class association scheme $(K_m)^{\wr D}$. Let $\mathbf{0}_i$ denote zero column vectors of size $m^{i-1}(m-1) - 1$ for $1 \leq i \leq D$. Any m^D dimensional column vectors can be divided into subparts 1, $m^{D-1}(m-1)$, $m^{D-2}(m-1)$, \dots , $m(m-1)$ and $(m-1)$ respectively so that any vector looks like $(\mathbf{p}^t, \mathbf{p}_D^t, \dots, \mathbf{p}_1^t)^t$ where \mathbf{p}_i is a $m^{i-1}(m-1) - 1$ dimensional column vector for $1 \leq i \leq D$.

It is easy to describe the primary module of the D -class association scheme $(K_m)^{\wr D}$ in the following way.

Theorem 4.5.1. *Let $\mathcal{X} = (K_m)^{\wr D}$ be a D -class association scheme of order m^D and let V denote the standard module. The vector $(1, \mathbf{0}_D^t, \dots, \mathbf{0}_1^t)^t$ and vectors $\mathbf{q}_i = (0, \mathbf{p}_D^t, \dots, \mathbf{p}_j^t, \dots, \mathbf{p}_1^t)^t$ for $1 \leq i \leq D$ such that*

$$\mathbf{p}_j = \begin{cases} \mathbf{s}_j & \text{if } i = j \\ \mathbf{0}_j & \text{if } i \neq j \end{cases}$$

generates the primary T -module.

Proof. Straightforward. □

The description of the Terwilliger algebra of a scheme involves describing the irreducible

modules that constitute the different subconstituents of the algebra. In an earlier section we have already studied all the subconstituents of the 3-class association scheme $(K_m)^{\wr 3}$. From the 3-class scheme we could describe all the irreducible modules for the 4-class scheme. From 4-class scheme we can describe all the irreducible modules of the 5-class $(K_m)^{\wr 5}$ scheme using the same method.

Let us consider the D -class association scheme $\mathcal{X} = (K_m)^{\wr D}$ of order m^D and let V denote the standard module. Finding the irreducible modules of the subconstituents E_i^*V for $2 \leq i \leq D$ and E_0^*V is more routine. E_1^*V needs to be treated differently than the other and we will come to that later.

Let the column vectors $(0, \mathbf{h}_j^t)^t$ generate the one dimensional modules of the subconstituent E_{i-1}^*V for the $(D-1)$ -class association scheme $(K_m)^{\wr (D-1)}$. Here $1 \leq j \leq \{m^{D-i}(m-1)-1\}$. If we add the column vector $\mathbf{0}_D^t$ right after 0 in the above vectors we land up with m^D dimensional vectors. For $1 \leq j \leq \{m^{D-i}(m-1)-1\}$ the vectors $(0, \mathbf{h}_j^t)^t$ and $(0, \mathbf{0}_D^t, \dots, \mathbf{s}_{D-i-1}^t, \dots, \mathbf{0}_1^t)^t$ span $\mathbb{C}^{\{m^{D-i}(m-1)\}}$. Also, $\langle (0, \mathbf{h}_j^t), (0, \mathbf{h}_k^t) \rangle = 0$ for $1 \leq j, k \leq \{m^{D-i}(m-1)-1\}$. For the scheme $(K_m)^{\wr (D-1)}$ since $(0, \mathbf{h}_j^t)^t$ generates a one dimensional module it is $E_i^*A_jE_h^*$ invariant for all $i, j, h \in \{0, 1, \dots, D-1\}$. For $1 \leq j \leq \{m^{D-i}(m-1)-1\}$ let $W_j^{\wr D}$ be the linear span of the vector $(0, \mathbf{0}_D^t, \mathbf{h}_j^t)^t$. The embedded structure of the $(D-1)$ -class association scheme $(K_m)^{\wr (D-1)}$ in the D -class scheme $(K_m)^{\wr D}$ ensures that for $1 \leq j \leq \{m^{D-i}(m-1)-1\}$, $(0, \mathbf{0}_D^t, \mathbf{h}_j^t)^t$ are $E_i^*A_jE_h^*$ invariant for all $i, j, h \in \{0, 1, \dots, D\}$. Note that now we are talking about the triple products of the D -class scheme $(K_m)^{\wr D}$. The vectors $(0, \mathbf{0}_D^t, \mathbf{h}_j^t)^t$ for $1 \leq j \leq \{m^{D-i}(m-1)-1\}$ generate all the one dimensional irreducible modules of the subconstituent E_i^*V for the D -class association scheme.

We have so far described the subconstituents $E_0^*V, E_2^*V, \dots, E_D^*V$ for the scheme $(K_m)^{\wr D}$. Now, $E_1^*V \cong \mathbb{C}^{m^{D-1}(m-1)}$. Observe that any $m^{D-1}(m-1)$ dimensional column vector can be partitioned into m equal parts each of dimension $m^{D-2}(m-1)$. Let each part be denoted by the index j where $j \in \{1, 2, \dots, m\}$ so that the $m^{D-1}(m-1)$ dimensional vector is of the form $(\mathbf{v}_1^t, \mathbf{v}_2^t, \dots, \mathbf{v}_m^t)^t$ where each $\mathbf{v}_j, j \in \{1, 2, \dots, m\}$ is a $m^{D-2}(m-1)$ dimensional column vector. For each $i \in \{2, 3, \dots, m^{D-2}(m-1)\}$ and $j \in \{1, 2, \dots, m\}$ let $\mathbf{d}_{ji} = (\mathbf{v}_1^t, \mathbf{v}_2^t, \dots, \mathbf{v}_j^t, \dots, \mathbf{v}_m^t)^t$

denote the $m^{D-1}(m-1)$ dimensional column vector such that $\mathbf{v}_l = \delta_{lj}\mathbf{h}_j$ where $(0, \mathbf{h}_j^t)$ generates the modules of E_1^*V for the $D-1$ scheme $(K_m)^{\wr(D-1)}$. It is easy to see that the vectors $(0, \mathbf{d}_{j_i}^t, \mathbf{0}_{D-1}^t, \dots, \mathbf{0}_1^t)^t$ generates one dimensional irreducible in the subconstituent E_1^*V for the scheme $(K_m)^{\wr D}$.

We have described only part of E_1^*V . For D -class scheme we have the following non-isomorphic \mathcal{T} -modules so far.

1. The primary module of dimension $D+1$.
2. $D-1$ non-isomorphic \mathcal{T} -modules of dimension 1 in E_1^*V .
3. $D-1$ non-isomorphic \mathcal{T} -modules of dimension 1 in E_2^*V .
4. $D-2$ non-isomorphic \mathcal{T} -modules of dimension 1 in E_3^*V
- \vdots
5. 2 non-isomorphic \mathcal{T} -modules of dimension 1 in E_{D-1}^*V .
6. 1 non-isomorphic \mathcal{T} -modules of dimension 1 in E_D^*V .

From Theorem 3.3.11 we have the dimension of $\mathcal{T}((K_m)^{\wr D})$ is $(D+1)^2 + \frac{1}{2}D(D+1)$. From our above discussion we have $(D+1)^2 + (D-1) + (D-1) + (D-2) + \dots + 2 + 1$ of the dimension. That leaves us with one non-isomorphic \mathcal{T} module of dimension 1 in the sub constituent E_1^*V of multiplicity $m-1$. Pick any nonzero vector $\mathbf{v} \in E_1^*V$ which is orthogonal to all the irreducible modules that are part of E_1^*V Then \mathbf{v} spans a one-dimensional irreducible \mathcal{T} -module.

In the discussion above we saw how we could build the irreducible modules of the scheme $(K_m)^{\wr D}$. We will conclude the section by collecting all our non-isomorphic \mathcal{T} -modules and describing the Terwilliger algebra of $(K_m)^{\wr D}$.

Theorem 4.5.2. *Let $\mathcal{X} = (K_m)^{\wr D}$ be a D -class association scheme of order m^D . Then the dimension of \mathcal{T} is $(D+1)^2 + \frac{1}{2}D(D+1)$ and $\mathcal{T} \cong M_{D+1}(\mathbb{C}) \oplus \mathbb{C}^{\oplus \frac{1}{2}D(D+1)}$.*

**CHAPTER 5. THE IRREDUCIBLE \mathcal{T} -MODULES OF WREATH
POWERS OF K_2**

In Chapter 3, while calculating the dimension of the Terwilliger algebra, it was found out that for the case $m = 2$ the number of nonzero triple products $E_i^* A_j E_h^*$ were fewer in number than in the general case where $m \geq 3$. Thus, the same method of approach does not work for this case. What is nice is that instead of just knowing the existence of vectors that generate the irreducible modules, we were actually able to get specific vectors that generated the irreducible modules.

Let $\mathcal{X} = (K_2)^{lD}$ be a D -class association scheme of order 2^D . Without loss of generality we label the 2^D elements of X by x_1, x_2, \dots, x_{2^D} . We fix x_1 as the base vertex and we will refer to it as x henceforth. As usual let $R_i(x) = \{y \in X : (x, y) \in R_i\}$. Then

1. $R_0(x) = \{x\}$
2. $R_1(x) = \{x_2, \dots, x_{2^{D-1}+1}\}$
3. $R_2(x) = \{x_{2^{D-1}+2}, \dots, x_{2^{D-1}+2^{D-2}}\}$
4. $R_D(x) = \{x_{2^D}\}$

Example 5.0.3. The relation table for a 4-class scheme $(K_2)^{l4}$ is the following

0	1	1	1	1	1	1	1	1	1	2	2	2	2	3	3	4
1	0	4	3	3	2	2	2	2	2	1	1	1	1	1	1	1
1	4	0	3	3	2	2	2	2	2	1	1	1	1	1	1	1
1	3	3	0	4	2	2	2	2	2	1	1	1	1	1	1	1
1	3	3	4	0	2	2	2	2	2	1	1	1	1	1	1	1
1	2	2	2	2	0	4	3	3	3	1	1	1	1	1	1	1
1	2	2	2	2	4	0	3	3	3	1	1	1	1	1	1	1
1	2	2	2	2	3	3	0	4	4	1	1	1	1	1	1	1
1	2	2	2	2	3	3	4	0	0	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1	1	0	4	3	3	2	2	2
2	1	1	1	1	1	1	1	1	1	4	0	3	3	2	2	2
2	1	1	1	1	1	1	1	1	1	3	3	0	4	2	2	2
2	1	1	1	1	1	1	1	1	1	3	3	4	0	2	2	2
3	1	1	1	1	1	1	1	1	1	2	2	2	2	0	4	3
3	1	1	1	1	1	1	1	1	1	2	2	2	2	4	0	3
4	1	1	1	1	1	1	1	1	1	2	2	2	2	3	3	0

We will describe vectors that will generate the irreducible \mathcal{T} -modules of $(K_2)^{\wr D}$.

5.1 The Vectors that Generate the Primary \mathcal{T} -module

Let δ denote the all-ones vector in the standard module. Then the vector space over \mathbb{C} spanned by $\{E_i^* \delta\}_{0 \leq i \leq D}$ is a thin irreducible \mathcal{T} -module of dimension $D + 1$ and is the primary module denoted as \mathcal{P} . Let $\eta_i = E_i^* \delta$ for $0 \leq i \leq D$. Hence $\{\eta_i\}_{0 \leq i \leq D}$ generates the primary module.

5.2 Construction of Some New Vectors

Let $\hat{\mathbf{x}}$ denote the column vector with 1 in the x -th position and 0 elsewhere. Let the new vectors be defined in the following way: For $l \in \{1, 2, \dots, D - 1\}$ define set of vectors

$\{\eta_i^l\}_{1 \leq i \leq 2^{D-l}-1}$ by

$$\eta_i^l = \sum_{k=0}^{2^{l-1}-1} \hat{\mathbf{x}}_{i+j+k} - \sum_{k=2^{l-1}}^{2^l-1} \hat{\mathbf{x}}_{i+j+k}$$

For each i , the corresponding values of j are successively

$$j = 1, 1 + 2^l - 1, 1 + 2(2^l - 1), 1 + 3(2^l - 1), \dots, 1 + (2^{D-l} - 2)(2^l - 1).$$

Thus,

1. $\{\eta_i^1\}_{1 \leq i \leq 2^{D-1}-1}$; $\eta_i^1 = \hat{\mathbf{x}}_{2i} - \hat{\mathbf{x}}_{2i+1}$
2. $\{\eta_i^2\}_{1 \leq i \leq 2^{D-2}-1}$; $\eta_i^2 = \hat{\mathbf{x}}_{i+j} + \hat{\mathbf{x}}_{i+j+1} - \hat{\mathbf{x}}_{i+j+2} - \hat{\mathbf{x}}_{i+j+3}$; $j = 1, 4, 7, \dots$
3. $\{\eta_i^3\}_{1 \leq i \leq 2^{D-3}-1}$; $\eta_i^3 = \sum_{k=0}^{2^{3-1}-1} \hat{\mathbf{x}}_{i+j+k} - \sum_{k=2^{3-1}}^{2^3-1} \hat{\mathbf{x}}_{i+j+k}$; $j = 1, 8, 15, \dots$
4. $\eta_1^{D-1} = \sum_{i=2}^{2^{D-2}+1} \hat{\mathbf{x}}_i - \sum_{i=2^{D-2}+2}^{2^{D-1}+1} \hat{\mathbf{x}}_i$

Example 5.2.1. The vectors for $(K_2)^4$ are

$$\begin{aligned} & (0, 1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^t, & (0, 0, 0, 1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^t, \\ & (0, 0, 0, 0, 0, 1, -1, 0, 0, 0, 0, 0, 0, 0, 0)^t, & (0, 0, 0, 0, 0, 0, 0, 1, -1, 0, 0, 0, 0, 0, 0)^t, \\ & (0, 0, 0, 0, 0, 0, 0, 0, 0, 1, -1, 0, 0, 0, 0)^t, & (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, -1, 0, 0)^t, \\ & (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, -1, 0)^t, & (0, 1, 1, -1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^t, \\ & (0, 0, 0, 0, 0, 1, 1, -1, -1, 0, 0, 0, 0, 0, 0)^t, & (0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, -1, -1, 0, 0)^t, \\ & (0, 1, 1, 1, 1, -1, -1, -1, -1, 0, 0, 0, 0, 0, 0)^t. \end{aligned}$$

5.3 Decomposition of the Standard Module into Irreducible \mathcal{T} -Modules

We will now describe the irreducible \mathcal{T} -modules, whose orthogonal direct sum forms the standard module V .

Lemma 5.3.1. *Let η_i^l be vectors as described in 5.2 Then $\langle \eta_i^j, \eta_l^k \rangle = 0$ unless $j = k$ and $i = l$.*

Proof. Proof follows from the construction of the vectors. □

Let $W_{\eta_i^l}$ be the linear span of η_i^l for all the vectors described in 5.2. We shall show that each of $W_{\eta_i^l}$ is an irreducible \mathcal{T} -module of dimension 1.

Lemma 5.3.2. *In the D -class association scheme $(K_2)^{\setminus D}$ of order 2^D , for $l \in \{1, 2, \dots, D-1\}$, define set of vectors $\{\eta_i^l\}_{1 \leq i \leq 2^{D-l}-1}$ by*

$$\eta_i^l = \sum_{k=0}^{2^{l-1}-1} \hat{\mathbf{x}}_{i+j+k} - \sum_{k=2^{l-1}}^{2^l-1} \hat{\mathbf{x}}_{i+j+k}.$$

For each i , the corresponding values of j are successively

$$j = 1, 1 + 2^l - 1, 1 + 2(2^l - 1), 1 + 3(2^l - 1), \dots, 1 + (2^{D-l} - 2)(2^l - 1).$$

Let $W_{\eta_i^l}$ be the linear span of η_i^l . Then $W_{\eta_i^l}$ is an irreducible \mathcal{T} -module of dimension 1.

Proof. To show that a nonzero subspace $W \subseteq \mathbb{C}$ is an irreducible \mathcal{T} -module we need to show that $BW \subseteq W$ for all $B \in \mathcal{T}$ and W contains no \mathcal{T} -modules other than 0 and W . To prove that $W_{\eta_i^l}$ is a \mathcal{T} -module we look at the action of the nonzero generators $E_i^* A_j E_h^*$ of the Terwilliger algebra on the vector η_i^l . We will consider three cases

1. The case when $l = 1$:

- (a) For $\{\eta_i^1\}_{1 \leq i \leq 2^{D-2}}$ by the construction of η_i^1 , the action of the triple products $E_i^* A_j E_h^*$ on η_i^1 can be nonzero only for $E_1^* A_0 E_1^*, E_1^* A_1 E_1^*, \dots, E_1^* A_D E_1^*$ and $E_i^* A_1 E_1^*$ for $2 \leq i \leq D$. Among them $E_1^* A_1 E_1^* = 0$. The generators $E_i^* A_j E_h^*$ of \mathcal{T} that act on η_i^1 in a nonzero manner are $E_1^* A_0 E_1^*$ and $E_1^* A_D E_1^*$ in the following manner:

$$E_1^* A_0 E_1^* \eta_i^1 = \eta_i^1, \quad E_1^* A_D E_1^* \eta_i^1 = -\eta_i^1.$$

- (b) For $\{\eta_i^1\}_{2^{D-2}+1 \leq i \leq 2^{D-2}+2^{D-3}}$ by the construction of η_i^1 , the action of the triple products $E_i^* A_h E_h^*$ on η_i^1 are nonzero only for $E_2^* A_0 E_2^*$ and $E_2^* A_D E_2^*$, and act in the following manner:

$$E_2^* A_0 E_2^* \eta_i^1 = \eta_i^1, \quad E_2^* A_D E_2^* \eta_i^1 = -\eta_i^1.$$

- (c) For every set of vectors corresponding to the different subconstituents will follow the same pattern. The last case is the following.

For $\eta_{2^{D-1}-1}^1$ the generators $E_i^* A_j E_h^*$ of \mathcal{T} that act on $\eta_{2^{D-1}-1}^1$ in a nonzero manner for

$$E_{D-1}^* A_0 E_{D-1}^* \eta_i^1 = \eta_i^1, \quad E_{D-1}^* A_D E_{D-1}^* \eta_i^1 = \eta_i^1.$$

2. The case when $l \in \{2, 3, \dots, D-2\}$:

- (a) For $\{\eta_i^l\}_{1 \leq i \leq 2^{D-(l+1)}}$ by the construction of η_i^l , the action of the triple products $E_i^* A_j E_h^*$ on η_i^l can be nonzero only for $E_1^* A_0 E_1^*, E_1^* A_1 E_1^*, \dots, E_1^* A_D E_1^*$. Among them $E_1^* A_1 E_1^* = 0$. The generators $E_i^* A_j E_h^*$ of \mathcal{T} that act on η_i^l in a nonzero manner are $E_1^* A_0 E_1^*, E_1^* A_D E_1^*, E_1^* A_{D-1} E_1^*, \dots, E_1^* A_{D-(l-1)} E_1^*$ in the following manner

$$E_1^* A_0 E_1^* \eta_i^l = \eta_i^l, \quad E_1^* A_D E_1^* \eta_i^l = \eta_i^l, \quad E_1^* A_{D-j} E_1^* \eta_i^l = 2^j \eta_i^l \text{ where } 1 \leq j \leq l-1.$$

- (b) For $\{\eta_i^l\}_{2^{D-(l+1)+1} \leq i \leq 2^{D-(l+1)+2^{D-(l+2)}}$ by the construction of η_i^l , the action of the triple products $E_i^* A_j E_h^*$ on η_i^l can be nonzero only for $E_2^* A_0 E_2^*, E_2^* A_1 E_2^*, \dots, E_2^* A_D E_2^*$. Among them $E_2^* A_1 E_2^* = 0$. The generators $E_i^* A_j E_h^*$ of \mathcal{T} that act on η_i^l in a nonzero manner are $E_2^* A_0 E_2^*, E_2^* A_D E_2^*, E_2^* A_{D-1} E_2^*, \dots, E_2^* A_{D-(l-1)} E_2^*$ in the following manner

$$E_2^* A_0 E_2^* \eta_i^l = \eta_i^l, \quad E_2^* A_D E_2^* \eta_i^l = \eta_i^l, \quad E_2^* A_{D-j} E_2^* \eta_i^l = 2^j \eta_i^l \text{ for } 1 \leq j \leq l-2,$$

and

$$E_2^* A_{D-(l-1)} E_2^* \eta_i^l = -2^{l-1} \eta_i^l.$$

- (c) Following a similar reasoning the last case will be as follows.

For $\eta_{2^{D-l}-1}^l$ the generators $E_i^* A_j E_h^*$ of \mathcal{T} that act on $\eta_{2^{D-l}-1}^l$ in a nonzero manner

are $E_{D-1}^* A_0 E_{D-1}^*$ and $E_{D-1}^* A_D E_{D-1}^*$ in the following manner

$$E_{D-l}^* A_0 E_{D-l}^* \eta_{2^{D-l-1}}^l = \eta_{2^{D-l-1}}^l, \quad E_{D-l}^* A_D E_{D-l}^* \eta_{2^{D-l-1}}^l = \eta_{2^{D-l-1}}^l,$$

$$E_{D-l}^* A_{D-j} E_{D-l}^* \eta_{2^{D-l-1}}^l = 2^j \eta_{2^{D-l-1}}^l \text{ for } 1 \leq j \leq l-2,$$

and

$$E_{D-l}^* A_{D-(l-1)} E_{D-l}^* \eta_{2^{D-l-1}}^l = -2^{l-1} \eta_{2^{D-l-1}}^l.$$

3. The case when $l = D - 1$:

For η_1^{D-1} by the construction of η_1^{D-1} , the action of the triple products $E_i^* A_j h E_h^*$ on η_1^{D-1} can be nonzero only for $E_1^* A_0 E_1^*, E_1^* A_1 E_1^*, \dots, E_1^* A_D E_1^*$. Among them $E_1^* A_1 E_1^* = 0$. The generators $E_i^* A_j E_h^*$ of \mathcal{T} that act on η_i^1 in a nonzero manner are $E_1^* A_0 E_1^*, E_1^* A_{D-1} E_1^*, \dots, E_1^* A_2 E_1^*$ in the following manner

$$E_1^* A_0 E_1^* \eta_1^{D-1} = \eta_1^{D-1}, \quad E_1^* A_{D-j} E_1^* \eta_1^{D-1} = 2^j \eta_1^{D-1} \text{ for } 1 \leq j \leq D-3,$$

and

$$E_1^* A_2 E_1^* \eta_1^{D-1} = -2^{D-2} \eta_1^{D-1}.$$

It is clear from the above cases that each of $W_{\eta_i^l}$ is a \mathcal{T} -module. They are irreducible since they are vector spaces generated by a single vector. This completes the proof. \square

Theorem 5.3.3. *Let $(K_2)^{lD}$ be a D -class association scheme of order 2^D . For $l \in \{1, 2, \dots, D-1\}$ define set of vectors $\{\eta_i^l\}_{1 \leq i \leq 2^{D-l-1}}$ by*

$$\eta_i^l = \sum_{k=0}^{2^{l-1}-1} \hat{\mathbf{x}}_{i+j+k} - \sum_{k=2^{l-1}}^{2^l-1} \hat{\mathbf{x}}_{i+j+k}.$$

For each i the corresponding values of j are successively

$$j = 1, 1 + 2^l - 1, 1 + 2(2^l - 1), 1 + 3(2^l - 1), \dots, 1 + (2^{D-l} - 2)(2^l - 1).$$

Let $W_{\eta_i^l}$ be the linear span of η_i^l . Let V be the standard module and \mathcal{P} be the primary module.

Then

$$V = \mathcal{P} \oplus \sum W_{\mathbf{v}}$$

where \mathbf{v} runs over all η_i^l defined above.

Proof. The \mathcal{T} -modules described are orthogonal follows from Lemma 5.3.1. By the construction of the modules it is straightforward that

$$V = \mathcal{P} \oplus \sum W_{\mathbf{v}}$$

where \mathbf{v} runs over all η_i^l . □

5.4 The Terwilliger Algebra of $(K_2)^{lD}$

Lemma 5.4.1. *Let $(K_2)^{lD}$ be a D -class association scheme of order 2^D . For $l \in \{1, 2, \dots, D-1\}$ define set of vectors $\{\eta_i^l\}_{1 \leq i \leq 2^{D-l-1}}$ by*

$$\eta_i^l = \sum_{k=0}^{2^{l-1}-1} \hat{\mathbf{x}}_{i+j+k} - \sum_{k=2^{l-1}}^{2^l-1} \hat{\mathbf{x}}_{i+j+k}.$$

For each i the corresponding values of j are successively

$$j = 1, 1 + 2^l - 1, 1 + 2(2^l - 1), 1 + 3(2^l - 1), \dots, 1 + (2^{D-l} - 2)(2^l - 1).$$

Let $W_{\eta_i^l}$ be the linear span of η_i^l . For $l \in \{1, 2, \dots, D-1\}$

- (a) $W_{\mathbf{v}}$ where $\mathbf{v} \in \{\eta_i^l\}_{1 \leq i \leq 2^{D-(l+1)}}$ are \mathcal{T} -isomorphic.
- (b) $W_{\mathbf{v}}$ where $\mathbf{v} \in \{\eta_i^l\}_{2^{D-(l+1)+1} \leq i \leq 2^{D-(l+1)+2^{D-(l+2)}}$ are \mathcal{T} -isomorphic.

Following above we finally have

- (c) $W_{\eta_{2^{D-l-3}}^l}$ and $W_{\eta_{2^{D-l-2}}^l}$ are \mathcal{T} -isomorphic.
- (d) Rest of $W_{\mathbf{v}}$ are not \mathcal{T} -isomorphic.

Proof. The proof is similar when we choose \mathcal{T} -modules from same groups as described in cases (a)-(c). We shall show that $W_{\eta_1^l}$ and $W_{\eta_2^l}$ are \mathcal{T} -isomorphic. Define an isomorphism

$$\sigma : W_{\eta_1^l} \rightarrow W_{\eta_2^l} \text{ by } \sigma(\eta_1^l) = \eta_2^l.$$

We need to show that $(\sigma B - B\sigma)W = 0$ for all $B \in \mathcal{T}$. Let us consider the action of nonzero $E_i^* A_j E_h^*$ on σ ; i.e, $E_1^* A_0 E_1^*$, $E_1^* A_D E_1^*$, $E_1^* A_{D-1} E_1^*$, \dots , $E_1^* A_{D-(l-1)} E_1^*$. Now

$$(\sigma E_1^* A_0 E_1^* - E_1^* A_0 E_1^* \sigma)W_{\eta_1^l} = (\sigma - E_1^* A_0 E_1^* \sigma)W_{\eta_1^l} = \eta_2^l - E_1^* A_0 E_1^*(W_{\eta_2^l}) = 0.$$

Similar kind of reasoning shows that for all $E_1^* A_0 E_1^*$, $E_1^* A_D E_1^*$, $E_1^* A_{D-1} E_1^*$, \dots , $E_1^* A_{D-(l-1)} E_1^*$, $(\sigma E_i^* A_j E_k^* - E_1^* A_0 E_1^* \sigma)W_{\eta_1^l} = 0$. Thus, $W_{\eta_1^l}$ and $W_{\eta_2^l}$ are \mathcal{T} -isomorphic.

Next we shall show that modules selected from different groups are not \mathcal{T} -isomorphic. In particular let us show that $W_{\eta_1^l}$ and $W_{\eta_{2^{D-(l+1)+1}}^l}$ are not \mathcal{T} -isomorphic.

Suppose we assume that there exists an isomorphism

$$\sigma : W_{\eta_1^l} \rightarrow W_{\eta_{2^{D-(l+1)+1}}^l} \text{ such that } (\sigma B - B\sigma)W_{\eta_1^l} = 0 \text{ for all } B \in \mathcal{T}.$$

Then for $E_1^* A_0 E_1^* \in \mathcal{T}$,

$$(E_1^* A_0 E_1^*)W_{\eta_1^l} = W_{\eta_1^l} \text{ and } (E_1^* A_0 E_1^*)W_{\eta_{2^{D-(l+1)+1}}^l} = 0.$$

Now,

$$[\sigma(E_1^* A_0 E_1^*) - (E_1^* A_0 E_1^*)\sigma]W_{\eta_1^l} = [\sigma - (E_1^* A_0 E_1^*)\sigma]W_{\eta_1^l} = W_{\eta_{2^{D-(l+1)+1}}^l} - 0 \neq 0$$

which is a contradiction to our assumption. Thus, $W_{\eta_1^l}$ and $W_{\eta_{2^{D-(l+1)+1}}^l}$ are not \mathcal{T} -isomorphic.

The other cases can be proved with a similar approach. \square

Theorem 5.4.2. *Let $\mathcal{X} = (K_2)^{\wr D}$ be a D -class association scheme of order 2^D .*

$$\mathcal{T} \cong M_{D+1}(\mathbb{C}) \oplus \mathbb{C}^{\oplus \frac{1}{2}D(D-1)}$$

where $M_k(\mathbb{C})$ denotes the full matrix algebra of $k \times k$ matrices over the complex. The dimension of \mathcal{T} is $(D+1)^2 + \frac{1}{2}D(D-1)$.

Proof.

$$\dim(\mathcal{T}) = (D+1)^2 + (D-1) + (D-1) + \cdots + 2 + 1 = (D+1)^2 + \frac{1}{2}(D-1)D.$$

□

CHAPTER 6. CONCLUDING REMARKS

In our attempt to describe the Terwilliger algebra of the D -class association scheme $(K_m)^{\wr D}$ our base was the Terwilliger algebra described by Tomiyama and Yamazaki [25] of a 2-class association scheme constructed from a strongly regular graph. Though our 3-class association scheme was neither strongly regular nor a P -polynomial scheme we were able to describe the Terwilliger algebra largely because of the fact that D -class association scheme $(K_m)^{\wr D}$ turned out to be triply regular and the structure of the $(D - 1)$ -class was so beautifully embedded in the D -class association scheme. We demonstrated further how we could extend the same method used to describe the Terwilliger algebra of the 3-class association scheme to the D -class association scheme $(K_m)^{\wr D}$.

In this chapter we will wrap up the dissertation by discussing some unanswered questions and some future research problems. Also, during our study we stumbled on the fact that the D -class association scheme $(K_m)^{\wr D}$ is formally self-dual. We will give a proof of that.

6.1 $(K_m)^{\wr D}$ is Formally Self-Dual

Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote the D -class scheme $(K_m)^{\wr D}$ with

$$X = [m] \times [m] \times \cdots \times [m] = \{(a_1, a_2, \dots, a_D) : a_i \in [m], \text{ for } i = 1, 2, \dots, D\}.$$

Let $(1, 1, \dots, 1) \in X$ be a fixed base vertex x of \mathcal{X} . Without loss of generality, we can arrange the vertices as follows.

- $R_0(x) = \{x\}$

- for $i = 1, 2, \dots, D - 1$,

$$R_i(x) = \{(a_1, a_2, \dots, a_{D-i-1}, b, 1, 1, \dots, 1) : a_k \in [m] \text{ for } k \in [D - i - 1], b \in [m] - \{1\}\}.$$

- $R_D(x) = \{(a, 1, 1, \dots, 1) : a \in \{2, 3, \dots, m\}\}$

Here we shall prove that the scheme $(K_m)^{iD}$ is formally self-dual. In other words we have to show that the first eigenmatrix is equal to the second eigenmatrix ($P = Q$). Repeated applications of Theorem 4.2 [19] gives us the first eigenmatrix of $(K_m)^{iD}$.

$$P = \begin{pmatrix} 1 & m^{D-1}(m-1) & m^{D-2}(m-1) & \cdots & m^2(m-1) & m(m-1) & m-1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & -1 \\ 1 & 0 & 0 & \cdots & 0 & -m & m-1 \\ 1 & 0 & 0 & \cdots & -m^2 & m(m-1) & m-1 \\ 1 & 0 & 0 & \cdots & m^2(m-1) & m(m-1) & m-1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & -m^{D-1} & m^{D-2}(m-1) & \cdots & m^2(m-1) & m(m-1) & m-1 \end{pmatrix}$$

Squaring the matrix P gives us $P^2 = I$ where I is an m^D dimensional identity matrix. But $PQ = I$ by page 60 [2]. Thus, $P = Q$ proving that the scheme $(K_m)^{iD}$ is formally self-dual.

6.2 Future Work

There is further work that is needed on the theme related to this dissertation. Here we state a few problems that are of our interest.

1. In this thesis we were able to describe the Terwilliger algebra of the D -class association scheme $(K_m)^{iD}$. The general scheme D -class $K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_D}$ turned out to be triply regular and we evaluated the dimension of the Terwilliger algebra \mathcal{T} . But we were unable to describe the irreducible \mathcal{T} -modules for the scheme. The method used for the case $(K_m)^{iD}$ did not work because the n_i 's were not equal. We could not naturally

generate vectors that generated the irreducible \mathcal{T} -modules. Paul Terwilliger pointed out that possibly all the nonprimary modules have dimension 1. We hope that in the near future we are able to describe the Terwilliger algebra of $K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_D}$ using some new tool.

In a slightly different direction, it will be interesting to look at some specific schemes obtained by taking the wreath power of two association schemes, such as the Hamming $H(2, q)$ instead of $H(1, q)$. We know that the Terwilliger algebra of a Hamming scheme can be described as symmetric d -tensors on the \mathcal{T} -algebra of $H(1, q)$ [15], although in general $H(d, q)$ is not realized as a product of $H(1, q)$. It would be interesting to see how the Terwilliger algebra changes when we take the wreath power of two Hamming schemes $H(d, q)$ for an arbitrary $d > 1$.

2. This problem was suggested by Paul Terwilliger as a future research problem. In this thesis we worked with an actual association scheme. As explained by Eric Egge [12] and introduction of [21] it is possible to define an “abstract version” of the \mathcal{T} -algebra using generators and relations. In all cases the concrete \mathcal{T} -algebra is a homomorphic image of the abstract \mathcal{T} -algebra, and in some cases they are isomorphic. In the case of $(K_m)^{\wr D}$ the entire structure of the \mathcal{T} -algebra is determined by the intersection numbers and Krein parameters, so it may be easy to see what is going on. Once all the vanishing intersection numbers and Krein parameters are worked out, we can obtain the defining relations for the algebra and we no longer need to consider the combinatorial structure further.

Terwilliger believes that for the association scheme in the thesis, the abstract \mathcal{T} -algebra and the concrete \mathcal{T} -algebra are isomorphic. It would be nice to work out the theory of the \mathcal{T} -algebra (basis, irreducible modules, dimension) from the generators/relations alone. For simplicity, assume that all the complete graphs in the wreath product have the same cardinality.

3. There are also other products besides the wreath product. Many examples of association schemes turn out to be products of smaller association schemes. The number of different

types of products in the theory of association schemes (wreathed, direct, quasidirect, semidirect) is significantly larger than the number of different types of products in group theory. In order to understand the nature of association schemes and their Terwilliger algebra the investigation of products is, therefore, much more important, useful, and intriguing than that in group theory. A comprehensive outcome of the investigations is an important goal in the theory of association schemes.

In any case, it is also an interesting problem to look at the direct power of $H(1, q)$. We study the wreath power first because the direct product of two association schemes has a lot more classes than the wreath product. Namely, the direct product of a d -class association scheme and a e -class association scheme is of class $de + d + e$ while the wreath product becomes $(d + e)$ -class association scheme. So a study of direct power requires a lot more work than that of wreath power. However, this is feasible and it may be worthy to look at it now as we know more about the schemes related to $H(1, q)$.

4. In [20], the irreducible \mathcal{T} -modules and \mathcal{T} -algebra has been investigated for the Doob schemes. The Doob schemes are the association schemes obtained by taking the direct product of copies of $H(2, 4)$ and copies of schemes coming from the Shrikhande graph. In this case, the direct product of these schemes preserves many properties of the original factor schemes. One important property that is remained as the same is P -polynomial property; that is all Doob schemes are also distance-regular. In terms of graphs, the Hamming $H(2, 4)$ and the Shrikhande graphs are the only distance-regular graphs whose direct product is also distance-regular. Nevertheless, the description of the \mathcal{T} -algebra of Doob schemes in terms of those of $H(2, 4)$ and Shrikhande scheme may shed a light in understanding how the \mathcal{T} -algebra of the product behaves when we study the \mathcal{T} -algebra of the direct product of $H(d, q)$ s with various d .

For the class of P -polynomial association schemes that come from distance-regular graphs, we observe that the wreath product hardly preserve the distance-regularity. That is, the wreath product of two distance-regular graphs is distance-regular only if the factors of the product are $H(1, q)$ or simple strongly regular graph. Although the distance reg-

ularity is not preserved in general, there are some characteristics preserved under the operation. It is our intention to examine what kind of characteristics of graphs can be preserved under each product operation as well as various decompositions by studying the Terwilliger algebra.

5. Finally we would like to mention what has been in our mind while we have been working on this theme. The known products provide us a way of constructing and decomposing association schemes in terms of ‘smaller’ and more ‘basic’ schemes. Nevertheless, there are still many schemes that we do not have such an interpretation (characterization) in terms of known product operations and decompositions. We realize that we need to develop more subtle concept of products or decomposition tools to explain the structure of such association schemes and relations to others. This requires us to figure out which schemes are the fundamental building blocks of general association schemes. In our study of \mathcal{T} -algebra of $H(1, q)$ and those of its wreath power, the order q does not play a critical role as far as the \mathcal{T} -algebra of the $(K_m)^{\wr D}$.

APPENDIX A. ASSOCIATION SCHEMES

In this chapter the basics of Association schemes have been discussed [2].

A.1 Commutative Association Schemes

Definition A.1.1. Let D denote a nonnegative integer. By a **commutative D -class association scheme**, we mean a sequence $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$, where X is a nonempty finite set, and where R_0, R_1, \dots, R_D are nonempty subsets of the cartesian product $X \times X$, such that

1. $R_0 = \{(x, x) : x \in X\}$
2. $X \times X = R_0 \cup R_1 \cup \dots \cup R_D$ (disjoint union),
3. For all integers i ($0 \leq i \leq D$), there exists $i' \in \{0, 1, \dots, D\}$ such that $R_i^t = R_{i'}$, where

$$R_i^t := \{(y, x) : (x, y) \in R_i\}$$

4. For all integers h, i, j ($0 \leq h, i, j \leq D$), and for all $x, y \in X$ such that $(x, y) \in R_h$, the scalar

$$p_{ij}^h := |\{z \in X : (x, z) \in R_i, (z, y) \in R_j\}|$$

depends only on h, i, j and not on x or y ,

5. $p_{ij}^h = p_{ji}^h$ ($0 \leq h, i, j \leq D$).

\mathcal{X} is said to be **symmetric** whenever

$$i' = i$$

for all i , ($0 \leq h, i, j \leq D$).

The elements of X are known as the **vertices** of \mathcal{X} , and D is known as the **diameter** of \mathcal{X} . The set R_i is known as the **i -th associate class** of \mathcal{X} , and the p_{ij}^h 's are known as the **intersection numbers** of \mathcal{X} . From now on, we abbreviate **scheme** for **commutative association scheme**.

Example A.1.2. Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter D , and define

$$R_i := \{(x, y) : x, y \in X, \delta(x, y) = i\}$$

($0 \leq i \leq D$), where δ denotes the distance function for Γ . Then $\mathcal{X}_\Gamma := (X, \{R_i\}_{0 \leq i \leq D})$ is a symmetric scheme.

A.2 The Bose-Mesner Algebra \mathcal{M}

Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme. For each integer i ($0 \leq i \leq D$), let A_i denote the matrix in $Mat_{|X|}(\mathbb{C})$ with xy entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } (x, y) \in R_i \\ 0 & \text{otherwise} \end{cases}$$

A_i is called the **i -th Associate matrix of \mathcal{X}** . For convenience, we set $A_{-1} = 0$ and $A_{D+1} = 0$.

Lemma A.2.1. *Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme, with associate matrices A_0, A_1, \dots, A_D .*

Then

1. A_0, A_1, \dots, A_D are linearly independent.
2. For all $x \in X$, and for all integers i ($0 \leq i \leq D$),

$$A_i \hat{x} = \sum_{y \in X, (y, x) \in R_i} \hat{y}$$

The definition of the association scheme is equivalent to the following four axioms:

1. $A_0 = I$
2. $A_0 + A_1 + \cdots + A_D = J$
3. $A_i^t = A_{i'}$ for some $i' \in \{0, 1, \dots, D\}$
4. $A_i A_j = \sum_{h=0}^D p_{ij}^h A_h$

where I and J are the $D \times D$ identity matrix and the all-one matrix, respectively and A^t denotes the transpose of the matrix A . If the scheme is **symmetric**, then $A_{i'} = A_i$ for all i . Commutativity of the scheme asserts that $p_{ij}^h = p_{ji}^h$, and thus $A_i A_j = A_j A_i$. If the scheme is commutative, the adjacency matrices A_0, A_1, \dots, A_D generate a $(d+1)$ -dimensional commutative algebra over the complex field. This algebra is called the **Bose-Mesner algebra** of the scheme.

Definition A.2.2. Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme. For $0 \leq i \leq D$ define

$$k_i := p_{ii}^0$$

Lemma A.2.3. Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme. Then (1)-(5) hold.

1. For all integers i ($0 \leq i \leq D$), and for all $x \in X$,

$$k_i = |\{z \in X : (x, z) \in R_i\}|$$

2. $k_0 = 1$.
3. $k_i > 0$ ($0 \leq i \leq D$).
4. $k_{i'} = k_i$ ($0 \leq i \leq D$).
5. $|X| = k_0 + k_1 + \cdots + k_D$.

Lemma A.2.4. Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme. Then the intersection numbers of \mathcal{X} satisfy (1)-(5).

1. $p_{0j}^h = \delta_{hj}$ ($0 \leq h, j \leq D$),
2. $p_{i0}^h = \delta_{hi}$ ($0 \leq h, j \leq D$),
3. $p_{ij}^0 = k_i \delta_{ij'}$ ($0 \leq h, j \leq D$),
4. $p_{ij}^h = p_{i'j'}^h$ ($0 \leq h, i, j \leq D$),
5. $\sum_{i=0}^D p_{ij}^h = k_j$ ($0 \leq h, j \leq D$).

A.3 The Primitive Idempotents

Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme. In this section we will discuss a second basis for the Bose-Mesner algebra \mathcal{M} .

Lemma A.3.1. *Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme.*

1. $A_i J = k_i J$ ($0 \leq i \leq D$),
2. $J^2 = |X|J$.

Lemma A.3.2. *Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme, with Bose-Mesner algebra \mathcal{M} . Then,*

1. $B^t \in \mathcal{M}$ for all $B \in \mathcal{M}$,
2. $\bar{B} \in \mathcal{M}$ for all $B \in \mathcal{M}$

Theorem A.3.3. *Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme, with Bose-Mesner algebra \mathcal{M} .*

Then there exists a basis E_0, E_1, \dots, E_D for \mathcal{M} such that

1. $E_0 = |X|^{-1}J$,
2. $I = E_0 + E_1 + \dots + E_D$,
3. $E_i E_j = \delta_{ij} E_i$ ($0 \leq i, j \leq D$).

Moreover, the set $\{E_0, E_1, \dots, E_D\} = \{E \in \mathcal{M} : E^2 = E, \dim(\mathcal{M}E) = 1\}$

E_i is called the i^{th} **primitive idempotent** of \mathcal{X} . E_0 is called the **trivial** idempotent.

Lemma A.3.4. Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$, with primitive idempotents E_0, E_1, \dots, E_D .

1. For all the integers i ($0 \leq i \leq D$), there exists $\hat{i} \in \{0, 1, \dots, D\}$ such that $E_i^t = E_{\hat{i}}$.
2. $\overline{E_i} = E_{\hat{i}}$ ($0 \leq i \leq D$).

Definition A.3.5. Let X denote any nonempty, finite set. Let $Mat_{|X|}(\mathbb{C})$ denote the \mathbb{C} -algebra of matrices with complex entries whose rows and columns are indexed by X . By the **standard module** for X , we mean the vector space $V = \mathbb{C}^{|X|}$ of column vectors whose coordinates are indexed by X .

Observe that $Mat_{|X|}(\mathbb{C})$ acts on V by left multiplication. We endow V with the hermitian inner product defined by $\langle u, v \rangle = u^t \bar{v}$; $(u, v) \in V$

Theorem A.3.6. Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$, with primitive idempotents E_0, E_1, \dots, E_D and $V := \mathbb{C}^{|X|}$.

1. $V = E_0V + E_1V + \dots + E_DV$ (orthogonal direct sum).
2. For all integers i ($0 \leq i \leq D$), E_i acts as the identity on E_iV , and vanishes on E_jV for $j \neq i$. In other words $E_i : V \rightarrow E_iV$ is the projection map.
3. Let \mathcal{M} denote the Bose-Mesner algebra of \mathcal{X} . then E_0V, E_1V, \dots, E_DV are precisely the maximal common eigenspaces of \mathcal{M} acting on V .

Lemma A.3.7. Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$, with primitive idempotents E_0, E_1, \dots, E_D and $V := \mathbb{C}^{|X|}$.

1. $E_0V = \text{Span}\{\eta\}$ where $\eta =$ all 1's vector in V .
2. $\overline{E_iV} = E_{\hat{i}}V$ ($0 \leq i \leq D$).

Lemma A.3.8. Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ with Bose-Mesner algebra \mathcal{M} . Then the following are equivalent.

1. \mathcal{X} is symmetric.
2. The elements of \mathcal{M} are all symmetric.
3. E_0, E_1, \dots, E_D are all symmetric.
4. $\hat{i} = i$ for all i ($0 \leq i \leq D$).
5. The entries of E_0, E_1, \dots, E_D are all real.

A.4 The Scalars $p_i(j), q_i(j)$.

Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme, with adjacency matrices A_0, A_1, \dots, A_D and primitive idempotents E_0, E_1, \dots, E_D . Since A_0, A_1, \dots, A_D and E_0, E_1, \dots, E_D are both bases for \mathcal{M} , there exists scalars $p_i(j)$ and $q_i(j) \in \mathbb{C}$ such that

$$A_i = \sum_{j=0}^D p_i(j) E_j$$

$$E_i = |X|^{-1} \sum_{j=0}^D q_i(j) A_j.$$

We refer to $p_i(j)$ as the **eigenvalues** of \mathcal{X} , and to the $q_i(j)$ as the **dual eigenvalues** of \mathcal{X} .

Definition A.4.1. Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme.

1. $P = P(\mathcal{X})$ will denote the $D + 1$ by $D + 1$ matrix with ij entry $p_j(i)$ ($0 \leq i, j \leq D$).
2. $Q = Q(\mathcal{X})$ will denote the $D + 1$ by $D + 1$ matrix with ij entry $q_j(i)$ ($0 \leq i, j \leq D$).

Lemma A.4.2. Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme. Then P and $|X|^{-1}Q$ are inverses.

Let \circ denote entrywise multiplication in $Mat_{|X|}(\mathbb{C})$.

Lemma A.4.3. Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme.

1. $A_i \circ A_j = \delta_{ij} A_i$
2. The Bose-Mesner algebra of \mathcal{X} is closed under \circ .
3. There exists $q_{ij}^h \in \mathbb{C}$ ($0 \leq h, i, j \leq D$) such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^D q_{ij}^h E_h$$

We call q_{ij}^h the **Krein parameters** of \mathcal{X} .

A.5 The Dual Bose-Mesner Algebra \mathcal{M}^*

Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme and fix a vertex $x \in X$. For each integer i ($0 \leq i \leq D$), let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $Mat_{|X|}(\mathbb{C})$ with yy entry

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } (x, y) \in R_i \\ 0 & \text{if } (x, y) \notin R_i \end{cases}$$

We refer to E_i^* as the i^{th} **dual idempotent of \mathcal{X} with respect to x** .

Next we shall mention some elementary facts about the dual idempotents.

Lemma A.5.1. *Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme and fix a vertex $x \in X$, and write $E_i^* = E_i^*(x)$ ($0 \leq i \leq D$).*

1. *For all $y \in X$, and all integers i ($0 \leq i \leq D$),*

$$(E_i^*)\hat{y} = \begin{cases} \hat{y} & \text{if } (x, y) \in R_i \\ 0 & \text{if } (x, y) \notin R_i \end{cases}$$

2. *Set $V = \mathbb{C}^{|X|}$. Then for all integers i ($0 \leq i \leq D$),*

$$E_i^*V = \text{Span}\{\hat{y} : y \in X, (x, y) \in R_i\}$$

We refer to E_i^*V as the i^{th} **subconstituent of \mathcal{X} with respect to x** .

Lemma A.5.2. *Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme and fix a vertex $x \in X$, and write $E_i^* = E_i^*(x)$ ($0 \leq i \leq D$). Then*

1. *$\text{rank}(E_i^*) = k_i$ ($0 \leq i \leq D$),*
2. *$\text{trace}(E_i^*) = k_i$ ($0 \leq i \leq D$),*
3. *$E_0^*, E_1^*, \dots, E_D^*$ are linearly independent.*

4. $I = E_0^* + E_1^* + \cdots + E_D^*$
5. $E_i^* E_j^* = \delta_{ij} E_i^*$ ($0 \leq i \leq D$).

Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme and fix a vertex $x \in X$, and write $E_i^* = E_i^*(x)$ ($0 \leq i \leq D$). By the above lemma, we find the matrices $E_0^*, E_1^*, \dots, E_D^*$ form a basis for a commutative subalgebra $\mathcal{M}^* = \mathcal{M}^*(x)$ of $\text{Mat}_{|X|}(\mathbb{C})$. We refer to \mathcal{M}^* as the **dual Bose-Mesner algebra of \mathcal{X} with respect to x** .

Theorem A.5.3. *Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme and fix a vertex $x \in X$, and write $E_i^* = E_i^*(x)$ ($0 \leq i \leq D$). Put $V := \mathbb{C}^{|X|}$.*

1. $V = E_0^*V + E_1^*V + \cdots + E_D^*V$ (orthogonal direct sum).
2. For all integers i ($0 \leq i \leq D$), E_i^* acts as the identity on E_i^*V , and vanishes on E_j^*V for $j \neq i$. In other words $E_i^* : V \rightarrow E_i^*V$ is the projection map.
3. Let \mathcal{M}^* denote the Bose-Mesner algebra of \mathcal{X} with respect to x . Then $E_0^*V, E_1^*V, \dots, E_D^*V$ are precisely the maximal common eigenspaces of \mathcal{M}^* acting on V .

A.6 The Dual Associate Matrices

Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme and fix a vertex $x \in X$. For each integer i ($0 \leq i \leq D$), let $A_i^* = A_i^*(x)$ denote the diagonal matrix in $\text{Mat}_{|X|}(\mathbb{C})$ with yy entry

$$(E_i^*)_{yy} = q_i(j) \text{ if } (x, y) \in R_j$$

Lemma A.6.1. *Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme and fix a vertex $x \in X$, and write $A_i^* = A_i^*(x)$ ($0 \leq i \leq D$). $\mathcal{M}^* = \mathcal{M}^*(x)$.*

1. $A_0^*, A_1^*, \dots, A_d^*$ is a basis for \mathcal{M}^* ,
2. $A_0^* = I$,
3. $A_0^* + A_1^* + \cdots, A_d^* = |X|E_0^*$,
4. $A_i^* A_j^* = \sum_{h=0}^D q_{ij}^h A_h^*$ ($0 \leq i, j \leq D$),

$$5. A_i^* A_j^* = A_j^* A_i^* \quad (0 \leq i, j \leq D).$$

Lemma A.6.2. *Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme and fix a vertex $x \in X$, and write*

$$A_i^* = A_i^*(x), \quad E_i^* = E_i^*(x) \quad (0 \leq i \leq D).$$

$$1. A_i^* = \sum_{j=0}^D q_i(j) E_j^* \quad (0 \leq i \leq D),$$

$$2. E_i^* = |X|^{-1} \sum_{j=0}^D p_i(j) A_j^* \quad (0 \leq i \leq D),$$

$$3. A_i^* E_j^* = q_i(j) E_j^* \quad (0 \leq i, j \leq D).$$

$q_i(j)$ is the eigenvalue of A_i^* associated with the eigenspace $E_j^* V$ ($0 \leq i, j \leq D$).

APPENDIX B. THE TERWILLIGER ALGEBRA

In this chapter the basics of the Terwilliger algebra has been discussed [24].

Definition B.0.3. Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme and fix a vertex $x \in X$, and let $\mathcal{T} = \mathcal{T}(x)$ denote the subalgebra of $\text{Mat}_{|X|}(\mathbb{C})$ generated by the Bose-Mesner algebra \mathcal{M} and the dual Bose-Mesner algebra $\mathcal{M}^* = \mathcal{M}^*(x)$. We call \mathcal{T} the **Terwilliger algebra of \mathcal{X} with respect to x** .

Lemma B.0.4. [21, Lemma 3.4.1] *Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme and fix a vertex $x \in X$, and let $\mathcal{T} = \mathcal{T}(x)$. Then, \mathcal{T} is semisimple.*

B.1 The Modules of \mathcal{T}

Definition B.1.1. Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme and fix a vertex $x \in X$, and let $\mathcal{T} = \mathcal{T}(x)$. By a **module** for \mathcal{T} , we mean a subspace $W \subseteq \mathbb{C}^X$ such that

$$BW \subseteq W \text{ for all } B \in \mathcal{T}.$$

Definition B.1.2. Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme and fix a vertex $x \in X$, and let $\mathcal{T} = \mathcal{T}(x)$. A \mathcal{T} -module is said to be **irreducible** whenever W is nonzero and W contains no other \mathcal{T} -modules other than 0 and W .

Definition B.1.3. Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme and fix a vertex $x \in X$, and let $\mathcal{T} = \mathcal{T}(x)$. Let W, W' denote \mathcal{T} -modules. By a **\mathcal{T} -isomorphism** from W to W' , we mean a vector space isomorphism $\sigma : W \rightarrow W'$ such that

$(\sigma B - B\sigma)W = 0$ for all $B \in \mathcal{T}$. The modules W, W' are said to be **\mathcal{T} -isomorphic** whenever there exists a \mathcal{T} -isomorphism from W to W' .

Lemma B.1.4. Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme and fix a vertex $x \in X$, and let $\mathcal{T} = \mathcal{T}(x)$.

1. $B^t \in \mathcal{T}$ for all $B \in \mathcal{T}$.
2. $\overline{B} \in \mathcal{T}$ for all $B \in \mathcal{T}$.

Lemma B.1.5. Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme and fix a vertex $x \in X$, let $\mathcal{T} = \mathcal{T}(x)$, and let W denote a nonzero \mathcal{T} -module. Then W is an orthogonal direct sum of irreducible \mathcal{T} -modules. (The sum may not be unique).

Definition B.1.6. Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme and fix a vertex $x \in X$, let $\mathcal{T} = \mathcal{T}(x)$, write $E_i^*(x) = E_i^*$. Let W denote an irreducible \mathcal{T} -module.

1. By the **diameter** of W , we mean the scalar

$$d := |\{i : 0 \leq i \leq D, E_i^*W \neq 0\}| - 1$$

2. By the **dual-diameter** of W , we mean the scalar

$$d^* := |\{i : 0 \leq i \leq D, E_iW \neq 0\}| - 1$$

Definition B.1.7. Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme and fix a vertex $x \in X$, let $\mathcal{T} = \mathcal{T}(x)$, write $E_i^*(x) = E_i^*$. Let W denote an irreducible \mathcal{T} -module.

1. W is said to be **thin** whenever

$$\dim E_i^*W \leq 1$$

2. W is said to be **dual-thin** whenever

$$\dim E_iW \leq 1$$

B.2 The Central Primitive Idempotents of \mathcal{T}

Definition B.2.1. Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme and fix a vertex $x \in X$, let $\mathcal{T} = \mathcal{T}(x)$. By the **center** of \mathcal{T} , we mean the subalgebra of \mathcal{T} given by

$$\text{Center}(\mathcal{T}) := \{B \in \mathcal{T} : BC = CB \quad \forall C \in \mathcal{T}\}$$

Lemma B.2.2. Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme and fix a vertex $x \in X$, let $\mathcal{T} = \mathcal{T}(x)$ and abbreviate $C = \text{Center}(\mathcal{T})$. Then there exists a set $\Phi = \Phi(x)$ and a basis $\{e_\lambda | \lambda \in \Phi\}$ for C such that

1. $I = \sum_{\lambda \in \Phi} e_\lambda$,
2. $e_\lambda e_\mu = \delta_{\lambda\mu} e_\lambda \quad (\forall \lambda \in \Phi, \forall \mu \in \Phi)$

We refer to e_λ as the **central primitive idempotents** of \mathcal{T} .

Let $V = \mathbb{C}^{|X|}$ denote the standard module. the standard module decomposes into irreducible \mathcal{T} -modules in a manner which reflects the structure of \mathcal{T} .

Lemma B.2.3. Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme and fix a vertex $x \in X$, let $\mathcal{T} = \mathcal{T}(x)$. Let $\{e_\lambda : \lambda \in \Phi\}$ be the central primitive idempotents of \mathcal{T} .

1. $V = \sum_{\lambda \in \Phi} e_\lambda V$ (Orthogonal direct sum). Moreover, $e_\lambda : V \rightarrow e_\lambda V$ is an orthogonal projection for all $\lambda \in \Phi$.
2. For each irreducible \mathcal{T} -module W , there is a unique $\lambda \in \Phi$ such that $W \subseteq e_\lambda V$. We refer to λ as the **type** of W .
3. Let W and W' denote irreducible \mathcal{T} -modules. Then W and W' are \mathcal{T} -isomorphic if and only if W and W' have the same type.
4. For all $\lambda \in \Phi$, $e_\lambda V$ can be decomposed as an orthogonal direct sum of irreducible \mathcal{T} -modules of type λ .

5. Referring to 4, the number of irreducible \mathcal{T} -modules in the decomposition is independent of the decomposition. We shall denote this number by $\text{mult}(e_\lambda)$ (or simply $\text{mult}(\lambda)$) and refer to it as the multiplicity (in V) of the irreducible \mathcal{T} -module of type λ .

B.3 The Primary (Trivial) Module

The following lemma is due to Terwilliger where he shows the existence of a thin irreducible \mathcal{T} -module.

Lemma B.3.1. [21, Lemma 3.6] *Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme and fix a vertex $x \in X$, let $\mathcal{T} = \mathcal{T}(x)$. Let δ denote the all 1's vector in the standard module V .*

Then the vector space over \mathbb{C} which is spanned by $\{E_i^ \delta\}_{0 \leq i \leq D}$ is a thin irreducible \mathcal{T} -module of dimension $D + 1$.*

This module is referred to as the **primary(trivial)** \mathcal{T} -module.

B.4 Some Triple Products in the Terwilliger Algebra

In this subsection we look at the triple products $E_i^* A_j E_h^*$ a little closely. We can view $E_i^* A_j E_h^*$ as a linear map from $V_h^* \rightarrow V_i^*$ such that $E_i^* A_j E_h^* \hat{y} = \sum_{z \in R_i(x) \cap R_j(y)} \hat{z}$ for each $\hat{y} \in V_h^*$. Terwilliger proved in [21] that

Proposition B.4.1. *Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme and fix a vertex $x \in X$, let $\mathcal{T} = \mathcal{T}(x)$. $E_i^* A_j E_k^* = 0$ if and only if $p_{ij}^k = 0$ for $(0 \leq i, j, k \leq D)$.*

Note that A_i and E_i^* can be written in terms of the triple products $E_i^* A_j E_h^*$ for $(0 \leq i \leq D)$. Thus, $E_i^* A_j E_h^*$ generates the \mathcal{T} -algebra. It is often easier to find the irreducible modules if we work with the triple products $E_i^* A_j E_h^*$ instead of A_i and E_i^* .

B.5 P -polynomial and Q -polynomial Schemes

In this section, two important classes of association schemes are discussed.

Lemma B.5.1. *Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme, with associate matrices A_0, A_1, \dots, A_D and the intersection numbers p_{ij}^h . Then the following are equivalent.*

1. For all integers $h, i, j (0 \leq h, i, j \leq d)$, $p_{ij}^h = 0$ if one of h, i, j exceeds the sum of the other two,

$p_{ij}^h \neq 0$ if one of h, i, j equals the sum of the other two.

2. For all integers $h, j (0 \leq h, j \leq D)$,

$p_{1j}^h = 0$ if $|h - j| > 1$,

$p_{1j}^h \neq 0$ if $|h - j| = 1$.

3. A_1 is symmetric, and for all integers $h, j (0 \leq h, j \leq D)$,

$p_{1j}^h = 0$ if $h > j + 1$,

$p_{1j}^h \neq 0$ if $h = j + 1$.

4. \mathcal{X} is symmetric, and there exists polynomials $f_0, f_1, \dots, f_D \in \mathbb{C}[\lambda]$ such that

$\deg f_i = i$ ($0 \leq i \leq D$),

$f_i(A_1) = A_i$ ($0 \leq i \leq D$)

If (1) – (4) hold \mathcal{X} is said to be a ***P*-polynomial**.

Lemma B.5.2. Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme, with associate matrices A_0, A_1, \dots, A_D and the intersection numbers q_{ij}^h . Then the following are equivalent.

1. For all integers $h, i, j (0 \leq h, i, j \leq d)$, $q_{ij}^h = 0$ if one of h, i, j exceeds the sum of the other two,

$q_{ij}^h \neq 0$ if one of h, i, j equals the sum of the other two.

2. For all integers $h, j (0 \leq h, j \leq D)$,

$q_{1j}^h = 0$ if $|h - j| > 1$,

$q_{1j}^h \neq 0$ if $|h - j| = 1$.

3. A_1^* is symmetric, and for all integers $h, j (0 \leq h, j \leq D)$,

$q_{1j}^h = 0$ if $h > j + 1$,

$q_{1j}^h \neq 0$ if $h = j + 1$.

4. \mathcal{X} is symmetric, and there exists polynomials $f_0, f_1, \dots, f_D \in \mathbb{C}[\lambda]$ such that

$$\deg f_i^* = i \quad (0 \leq i \leq D),$$

$$f_i^*(A_1^*) = A_i^* \quad (0 \leq i \leq D)$$

If (1) – (4) hold \mathcal{X} is said to be a **Q -polynomial**.

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