Analysis and control theory of some cochlear models

by

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DEDICATION

I would like to dedicate this thesis to my late brother, Joseph Chepkwony.

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CHAPTER 1. Introduction

The *cochlear* is the auditory organ located in the temporal bone (see figures 1.1 and 1.2). It is a cavity, resembling a snail's shell. The cochlear cavity is divided longitudinally into two chambers; *scala vestibuli* and *scala tympani*, by the cochlear partition. These chambers are joined by a hole situated at the apex of the cochlear, known as *helicotrema* (see figure 1.3). The cochlear is filled with fluid that is essentially incompressible and is sealed by two elastic membranes that cover *the oval* and *round windows*. The oval window is a membraneous opening that opens into the scala vestibuli whereas the round window opens into the scala tympani. The mechanically significant structures making up the cochlear partition include an elastic membrane called *basilar membrane* and the organ of *corti*.

External sound waves causes the ear drum to vibrate. These vibrations are transmitted to the inner ear by the three bones of the middle ear, the *malleus*, *incus* and *stapes* (also called the *hammer*, the *anvil* and the *stirrup*). The to- and- fro motion of the stapes against the oval window generates an oscillating pressure field in the cochlea fluid, which ultimately produces "traveling" waves along the basilar membrane. This wave is propagated by the combined movement of the basilar membrane and the fluid. Given that the cochlear fluid is incompressible, conservation of mass requires that the round window has to move in the opposite direction to that of the oval window, where the stapes transmits sound vibrations from the middle ear. The basilar membrane is shaped such that it is narrow and stiff at the basal end of the cochlear but gradually broadens and becomes less stiffer towards the apex (see figure 1.3). The waves therefore propagate quickly with long wavelength at base but slows down, become shorter and increase in amplitude as it travels down the cochlea. At some point, called the *best place* for the given input frequency, the membrane will vibrate with maximum

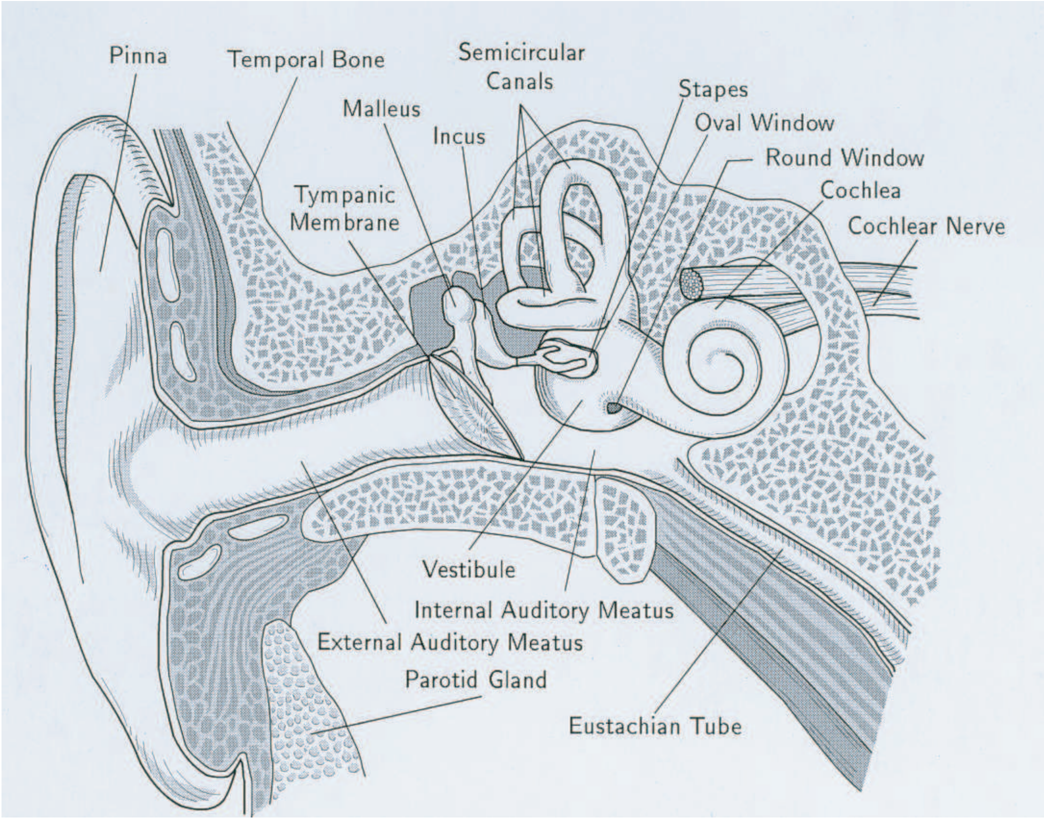

Figure 1.1 This is a schematic of an auditory, adapted from Richard G. Kessel/ Randy H. Kardon(44). It shows how the cochlear, in a coiled form, is embedded in the temporal bone. Most anatomists believe that the purpose of coiling is basically to minimize the packaging space.

amplitude but quickly dies off thereafter. The motion of basilar membrane stimulate the *inner hair cells*, located within the organ of corti. These cells act as sensors which transmit auditory information to the brain.

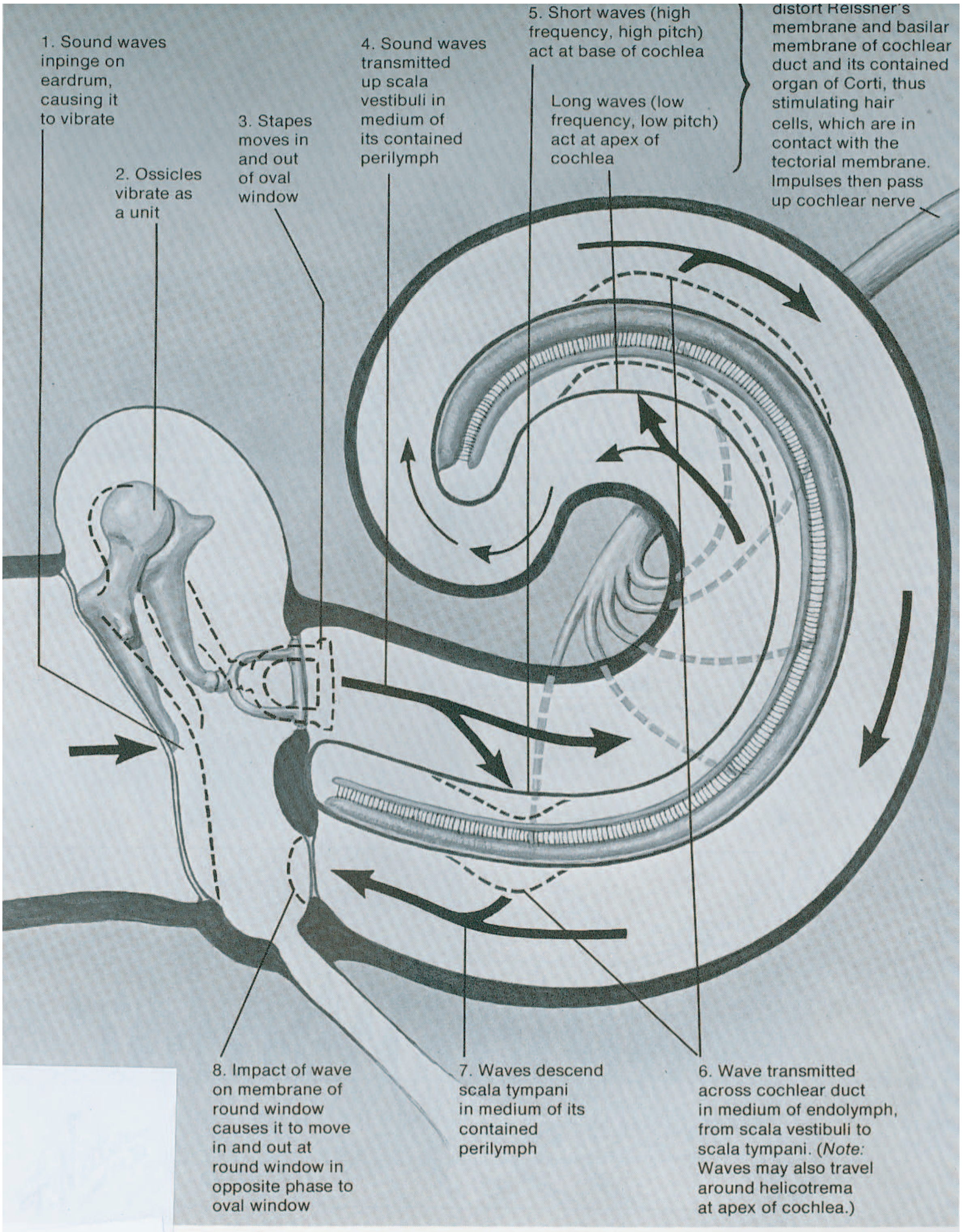

Figure 1.2 This is an extract of the cochlear slightly unwrapped, also adapted from Richard G. Kessel/ Randy H. Kardon(44), to show the two chambers- *scala tympani* and *scala vestibuli* with *basilar membrane* separating them. High frequency waves cause maximum amplitude at the base whereas the low frequency ones cause maximum amplitude near the apex.

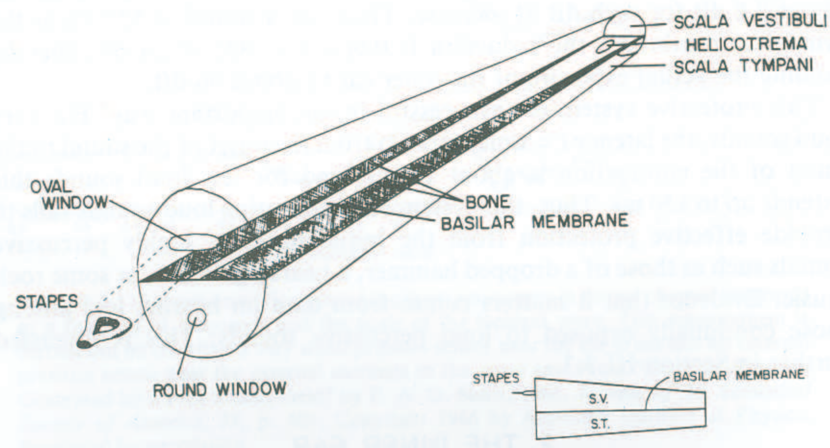

Figure 1.3 This is a schematic of a cochlear that is completely unwrapped, adapted from R. Duncan Luce(45). Its cross section is shown on the bottom right hand side. It is broader at the base and gradually narrows towards the apex.

1.1 History of Cochlear Research

The first anatomical study of the cochlea was conducted by Alfonso Corti (2) in 1851, for whom the organ of corti is named. This study arose interests of other anatomists, like Reissner, Deiters, Lowenberg, Hensen and Hasse, who helped shed light on more cochlea structures such as the Reissner membrane, Deiters, Hensens, cells etc. Perhaps, the first scientist to make a significant contribution towards understanding the complicated processes underlying the sound transduction by the ear was Herman Von Helmoltz (3). Helmholtz put forth an explanation of how mechanical coupling, provided by the middle ear, optimizes the transmission of energy from the air that sets the *tympanic membrane* (eardrum) into motion, to the fluid inside the cochlear. He is also the first to give a plausible explanation of how the cochlear functions. He viewed the basilar membrane as a continuum of vibrating strings with tension coefficients varying from high at the base (near the oval window) to low at the apex. The resulting model of the basilar membrane bore resemblance to a set of harmonic oscillators tuned to different frequencies. Consequently, the cochlear was thought of as a spectral analyzer providing a frequency -position map of sound fourier components. For a good number of years, Helmoltz's explanation was the accepted theory.

In 1928, Békésy (4) built a mechanical model of the cochlear involving a metal tube filled with water. Serving as a basilar membrane was a stretched membrane placed to run along the length of the tube. When the water was forced from one end, he observed a jut which moved like a wave along the membrane. He realized that by adjusting the tension of the membrane the position where the maximum amplitude of the wave occurred, varied along the membrane. Years later, Békésy (4) extracted the cochlear from various animals and also from human cadavers and carried out the same experiments. With the help of a microscope, he saw a "traveling" wave over the basilar membrane when the sound was introduced into the cochlear. This wave resembled the one he observed along the artificial membrane of his model. Békésy further noted that for a stimulus of a given frequency, the cochlear partition vibration formed a wave that swelled up in amplitude as it travels from the oval window, attains a maximum and then quickly dies out. It was observed that high frequency tones produced waves with maximum amplitudes occurring at the base of the cochlear whereas those of low frequencies occur at the apex. Because of his discovery of traveling wave phenomena in human cochlear, Von Békésy was awarded the Nobel prize in 1961.

Békésy's findings stimulated the development of more elaborate cochlear models which predict similar traveling waves, e.g Ranke (5) and Zwislocki (6). The main drawback of these models was that they predict a resonance profile, also called "tuning curve", which were too broad to explain the known frequency sensitivity of mammals. Later experiments in 1960s and 1970s (see Kiang (7), Rhode (8; 9)), using the Mössbauer technique, detected much sharper tuning curves as compared to those predicted by the passive models. Thus an action feedback mechanism was hypothesized (see Sellick, Patuzzi and Johnstone (10); Davies (11); Neely and Kim (32)). Peter Dallos (17), in 2000, cloned the integral membrane protein responsible for outer hair cell electromotility and named it *Prestin.* In 2002, Liberman *et al.*(19) showed that targeted deletion of the gene that encodes *Prestin*, revealed a hundred-fold loss in hearing sensitivity indicating that it is the fundamental component of the cochlear amplifier. Thus the outer hair cells provide active feedback which greatly sharpens the tuning curve.

Though many mathematical models of the cochlear have been proposed, [for example (20), (21), (22), (23), (24) and (25)], there remains much debate on how to include the sensing of the inner hair cells and the actuation of the outer hair cells in a cochlear model. Much of the on-going research is geared in this direction (e.g Xin *et al.*(31).)

The theoretical studies of the cochlear have generally been modeled as a straight fluid-filled duct, even though the cochlear's geometry is coiled- shaped. The common belief is that the coiling allows the cochlear to be packed into a small space. However, Manoussaki and Chadwick (39) have shown that coiling helps to lower the fluid impedance, particularly at the apex, where basilar membrane curvature is greatest.

1.2 Description of the main results

In this thesis we study controllability of two cochlear models- a model without longitudinal elasticity on the basilar membrane and one with longitudinal elasticity. The controls we consider act on a portion of the basilar membrane and/or the oval window. One possible practical motivation for this study is to lay a foundation for the design of better hearing aids and cochlear implants. In this thesis however, we restrict our focus to the controllability problem. In recent years there has been much interest in coupled fluid-elastic systems, motivated mainly by aerospace applications e.g fuselage-engine noise coupling (see Banks *et al* (28)). Controllability issues related to such structural-acoustic problems was subsequently studied by many researchers e.g Avalos (27), Lasiecka *et al.* (33), Lions and Zuazua (29), Micu and Zuazua (30). Controllability of elastic systems coupled to a potential fluid was studied by Hansen and Lyashenko (34) and Hansen (26). The cochlear models we study here also involves coupling of the motion of an elastic system (basilar membrane) to a potential fluid, however are more complex in a number of ways than the models considered in (34) and (26). In particular, the elastic model for the basilar membrane is of variable coefficients and involves a potential function not treated in (34) and (26).

Controllability results are given for the two models. In addition, well-posedness results are given. To the best of our knowledge, existence and uniqueness issues for the standard cochlear models have not been addressed. This could be because the studies have mainly been focused on understanding the traveling wave phenomena, where the effect of the fluid (and also the geometry of the cochlear) is eliminated by an approximation method (WKB (Wentzel-Kramers-Brillouin) approximation, "long wave approximations", "added mass" effects etc.).

CHAPTER 2. Derivation of Cochlear Models

2.1 Description of the cochlear model and its mechanics

We describe a two dimensional model and derive its dynamics. If the cochlear is unrolled, it takes the form shown in figure (1.3). For simplification purposes we consider the cross section corresponding to (1.3) to be like figure (2.1).

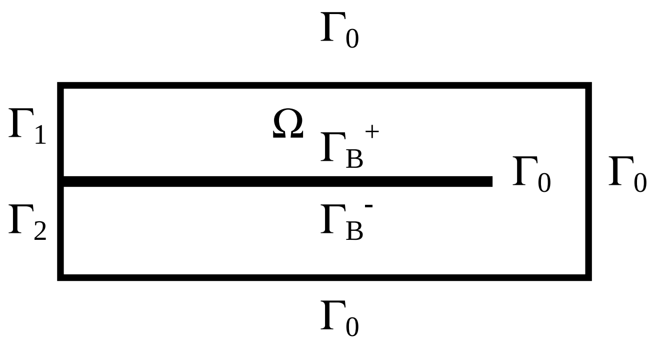

Figure 2.1 A simplified cross section of the unrolled cochlear.

The cavity, Ω, is filled with fluid. This fluid is assumed to be incompressible with density ρ. There are hard walls on the top, right and bottom sides of the model (all denoted by Γ_0), through which fluid cannot flow. In the center of the model is the basilar membrane, which is assumed to have stiffness and mass that vary with position a long the cochlea. The motion of stapes at the oval window (Γ_1) at the left side of the model drives the system. Because the fluid is incompressible, inward movement of the stapes at the oval window must result in a compensating outward movement at the round window, (Γ_2), so that movement of the fluid in upper and lower chambers is in opposite directions and pressure fluctuations about the initial pressure have opposite signs for corresponding points in the two chambers. The length dimension of the cochlear model runs from $x = 0$ to $x = L_B + \epsilon$ (where ϵ is the length dimension of the helicotrema) and the height dimensions $y = 0$ to $y = L_1$ (for the top chamber) and $y = -L_2$ to $y = 0$ lower chamber).

Remark 1. Of course the actual cochlear is spiral-shaped and more narrow at the apex than at the base. For us, the geometry in figure (2.1) is chosen for simplicity. Most of the results we obtain here also hold true under more realistic geometric assumptions.

2.2 Mathematical formulation of the dynamics of cochlear fluid

In general, the fluid velocity vector $\mathbf{v}$ at any point (x, y) will have x and y components $\mathbf{v}_x$ and $\mathbf{v}_y$ respectively. In terms of a velocity potential ψ, we have

$$\mathbf{v}_x = \frac{\partial \psi}{\partial x}, \quad \mathbf{v}_y = \frac{\partial \psi}{\partial y} \tag{2.1}$$

and therefore $\mathbf{v} = \nabla \psi$. For an incompressible fluid, there is no net flow into or out of any small region, so

$$\nabla \cdot \mathbf{v} = \frac{\partial \mathbf{v}_x}{\partial x} + \frac{\partial \mathbf{v}_y}{\partial y} = 0$$

or

$$\Delta \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0. \tag{2.2}$$

Thus the velocity potential ψ obeys Laplace's equation. The hard-wall boundary conditions at the top, right and bottom sides of the model,(all denoted by Γ_0), imply that there is no fluid flow in a direction normal to the boundary, i.e

$$\frac{\partial \psi}{\partial n} = 0 \quad \text{on} \quad \Gamma_0, \tag{2.3}$$

where n is outward normal direction.

On Γ_1 and Γ_2, the motion of the fluid is determined by the motion of the stapes. For simplicity, we assume that the displacement at the oval window $\eta(y, t)$ and the displacement of the round window $\xi(y, t)$ are constant with respect to y, that is ψ_n satisfies

$$\frac{\partial \psi}{\partial n} = \begin{cases} \eta_t(t) & \text{on} \quad \Gamma_1 \\ \xi_t(t) & \text{on} \quad \Gamma_2 \end{cases} \tag{2.4}$$

Along Γ_B

$$\frac{\partial \psi}{\partial n} = \begin{cases} -w_t, & \text{on} \quad \Gamma_B^+ \\ w_t & \text{on} \quad \Gamma_B^- \end{cases} \tag{2.5}$$

where w is the transverse displacement of Γ_B.

Since small vibrations are under consideration, we linearize by matching the velocities on the boundary of the fixed equilibrium domain Ω. Putting equations (2.2)- (2.5) together, we have

$$\left.\begin{array}{rcl} \Delta\psi &=& 0 \quad \text{in} \quad \Omega \\ \psi_n &=& 0 \quad \text{on} \quad \Gamma_0 \\ \psi_n &=& \psi_y \quad = -w_t \quad \text{on} \quad \Gamma_B^+ \\ \psi_n &=& \psi_y \quad = w_t \quad \text{on} \quad \Gamma_B^- \\ \psi_n &=& -\psi_x \quad = \eta_t \quad \text{on} \quad \Gamma_1 \\ \psi_n &=& -\psi_x \quad = \xi_t \quad \text{on} \quad \Gamma_2 \end{array}\right\}. \tag{2.6}$$

A necessary condition for the solvability of (2.6) is that the average of the Neumann data is zero. From the first equation of system (2.6), we have that

$$0 = \int_\Omega 1\Delta\psi = -\int_\Omega \nabla\psi \cdot \nabla 1 + \int_{\partial\Omega} 1\frac{\partial\psi}{\partial n} = \int_{\Gamma_1} \frac{\partial\psi}{\partial n} d\Gamma_1 + \int_{\Gamma_2} \frac{\partial\psi}{\partial n} d\Gamma_2,$$

where the part involving Γ_B cancels out. So

$$L_1\eta_t = -L_2\xi_t \Leftrightarrow \xi_t = -(L_1/L_2)\eta_t$$

and hence

$$\xi = -(L_1/L_2)\eta. \tag{2.7}$$

2.2.1 Dynamics of the system

The vibrations of the membranes on the oval and round windows are assumed to be modeled as spring-mass systems, $m_1\eta_{tt} + k_1\eta = F_1$ and $m_2\xi_{tt} + k_2\xi = F_2$ respectively. We assume that the basilar membrane has negligible longitudinal elasticity and hence modeling it as an infinite array of springs, taking the form:

$$m_0 w_{tt} + k_0 w = F_0,$$

is appropriate. So each portion of the basilar membrane has a unique frequency response to an input at the oval window. In all these equations m_i, k_i (which depend on x) and F_i are the mass densities of the corresponding springs, stiffness per unit length of the springs and forcing respectively. Now, to derive the equations of motion for the system, we first identify its kinetic and potential energies. The energy $\mathcal{E}(t)$ is the sum of the kinetic $K(t)$ and potential $P(t)$ energies where

$$K = \frac{1}{2}\int_{\Omega}(\rho_f|\nabla\psi|^2)d\Omega + \frac{1}{2}\int_0^{L_B} m_0 w_t^2 dx + \frac{1}{2}\int_0^{L_1} m_1\eta_t^2 dy + \frac{1}{2}\int_0^{L_2} m_2\xi_t^2 dy$$

and

$$P = \frac{1}{2}\int_0^{L_B} k_0 w^2 dx + \frac{1}{2}\int_0^{L_1} k_1\eta^2 dy + \frac{1}{2}\int_0^{L_2} k_2\xi^2 dy.$$

So the energy can be expressed in the form

$$\begin{aligned}
\mathcal{E}(t) &= \frac{1}{2}\int_{\Omega}(\rho_f|\nabla\psi|^2) + \frac{1}{2}\int_0^{L_B}(m_0 w_t^2 + k_0 w^2)dx + \frac{1}{2}\int_0^{L_1}(m_1\eta_t^2 + k_1\eta^2)dy \\
&+ \frac{1}{2}\int_0^{L_2}(m_2\xi_t^2 + k_2\xi^2)dy,
\end{aligned}$$

where k_0, k_1, k_2 are constants. The work term is given by

$$W(t) = \int_0^{L_B} F_0 w dx + \int_0^{L_1} F_1\eta dy + \int_0^{L_2} F_2\xi dy.$$

The equations of motion can be obtained from Hamilton's principle. That is, the first variation, with respect to a class of admissible variations, of the lagrangian
$\mathcal{L} = \int_0^T (K + W - P)dt$
is set to zero. Here W denotes the work done by the applied forces on the oval window, round window and the basilar membrane. The class of variations $\{\hat{w}, \hat{\psi}, \hat{\eta}, \hat{\xi}\}$ we consider includes those that satisfy (2.6), (2.7) and vanish near $t = 0$ and $t = T$.
The Lagrangian is

$$\begin{aligned}
\mathcal{L} &= \int_0^T \{\frac{1}{2}\int_{\Omega}\rho_f|\nabla\psi|^2 + \frac{1}{2}\int_0^{L_B}(m_0 {w_t}^2 - k_0 w^2)dx \\
&+\frac{1}{2}\int_0^{L_1}(m_1{\eta_t}^2 - k_1\eta^2)dy + \frac{1}{2}\int_0^{L_2}(m_2{\xi_t}^2 - k_2\xi^2)dy + \int_0^{L_B} F_0 w dx \\
&+ \int_0^{L_1} F_1\eta dy + \int_0^{L_2} F_2\xi dy\}dt,
\end{aligned}$$

where F_1 and F_2 are scalar forces acting on Γ_1 and Γ_2 respectively, whereas F_0 is a distributed force acting on the beam.

To obtain the equation of motion we set the first variation to zero i.e

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}(\mathcal{L}(w+\epsilon\hat{w}, \psi+\epsilon\hat{\psi}, \eta+\epsilon\hat{\eta}, \xi+\epsilon\hat{\xi}) - \mathcal{L}(w,\psi,\eta,\xi)) = 0.$$

This becomes

$$\begin{aligned} 0 &= \int_0^T \{\int_\Omega (\rho_f \nabla\psi\nabla\hat{\psi})d\Omega + \int_0^{L_B} (m_0 w_t \hat{w}_t - k_0 w\hat{w})dx \\ &\quad + \int_0^{L_1} (m_1\eta_t\hat{\eta}_t - k_1\eta\hat{\eta})dy + \int_0^{L_2} (m_2\xi_t\hat{\xi}_t - k_2\xi\hat{\xi})dy \\ &\quad + \int_0^{L_B} (F_0\hat{w})dx + \int_0^{L_1} (F_1\hat{\eta})dy + \int_0^{L_2} (F_2\hat{\xi})dy\}dt, \end{aligned}$$

By application of Green's formula, we have

$$\begin{aligned} 0 &= \int_0^T \{\int_\Omega -\rho_f\hat{\psi}\Delta\psi + \int_{\Gamma_B^+} (\rho_f\psi\hat{\psi}_n)dx + \int_{\Gamma_B^-} (\rho_f\psi\hat{\psi}_n)dx + \int_{\Gamma_1} (\rho_f\psi\hat{\psi}_n)dy \\ &\quad + \int_{\Gamma_2} (\rho_f\psi\hat{\psi}_n)dy + \int_0^{L_B} (m_0 w_t\hat{w}_t - k_0 w\hat{w} + F_0\hat{w})dx \\ &\quad + \int_0^{L_1} (m_1\eta_t\hat{\eta}_t - k_1\eta\hat{\eta} + F_1\hat{\eta})dy + \int_0^{L_2} (m_2\xi_t\hat{\xi}_t - k_2\xi\hat{\xi} + F_2\hat{\xi})dy\}dt. \end{aligned}$$

Since $\Delta\psi = 0$, the above equation reduces to

$$\begin{aligned} 0 &= \int_0^T \{\int_{\Gamma_B^+} -\rho_f\psi\hat{w}_t + \int_{\Gamma_B^-} \rho_f\psi\hat{w}_t + \int_{\Gamma_1} \rho_f\psi\hat{\eta}_t + \int_{\Gamma_2} \rho_f\psi\hat{\xi}_t \\ &\quad + \int_0^{L_B} \hat{w}(-m_0 w_{tt} - k_0 w + F_0)dx + \int_0^{L_1} \hat{\eta}(-m_1\eta_{tt} - k_1\eta + F_1)dy \\ &\quad + \int_0^{L_2} \hat{\xi}(-m_2\xi_{tt} - k_2\xi + F_2)dy\}dt. \end{aligned}$$

Integration by parts then gives

$$\begin{aligned} 0 &= \int_0^T \{\int_{\Gamma_B^+} \hat{w}\rho_f\psi_t dx - \int_{\Gamma_B^-} \hat{w}\rho_f\psi_t dx - \int_{\Gamma_1} \hat{\eta}\rho_f\psi_t dy \\ &\quad - \int_{\Gamma_2} \hat{\xi}\rho_f\psi_t dy + \int_0^{L_B} \hat{w}(-m_0 w_{tt} - k_0 w + F_0)dx \\ &\quad + \int_0^{L_1} \hat{\eta}(-m_1\eta_{tt} - k_1\eta + F_1)dy + \int_0^{L_2} \hat{\xi}(-m_2\xi_{tt} - k_2\xi + F_2)dy\}dt. \end{aligned}$$

Now factor out $\hat{w}, \hat{\eta}$ and $\hat{\xi}$ so that

$$\begin{aligned} 0 &= \int_0^T \{ \int_{\Gamma_B^+} \hat{w}(-m_0 w_{tt} - k_0 w - [\rho_f \psi_t]_{\Gamma_B} + F_0) dx \\ &+ \int_0^{L_1} \hat{\eta}(-m_1 \eta_{tt} - k_1 \eta + F_1 \rho_f \psi_t) dy \\ &+ \int_0^{L_2} \hat{\xi}(-m_2 \xi_{tt} - k_2 \xi + F_2 \rho_f \psi_t) dy \} dt. \end{aligned}$$

Using the relationship between ξ and η on one hand and $\hat{\xi}$ and $\hat{\eta}$ on the other, we have

$$\begin{aligned} 0 &= \int_0^T \{ \int_{\Gamma_B^+} \hat{w}(-m_0 w_{tt} - k_0 w - [\rho_f \psi_t]_{\Gamma_B} + F_0) \\ &+ \hat{\eta}[\int_0^{L_1} (-m_1 \eta_{tt} - k_1 \eta + F_1 - \rho_f \psi_t) dy \\ &+ \int_0^{L_2} (-\frac{L_1^2}{L_2^2} m_2 \eta_{tt} - \frac{L_1^2}{L_2^2} k_2 \eta - \frac{L_1}{L_2} F_2 + \frac{L_1}{L_2} \rho_f \Psi_t) dy] \} dt. \end{aligned}$$

This then reduces to

$$\begin{aligned} 0 &= \int_0^T \{ \int_{\Gamma_B^+} \hat{w}(-m_0 w_{tt} - k_0 w - [\rho_f \psi_t]_{\Gamma_B} + F_0) \\ &+ \hat{\eta}[-L_1 m_1 \eta_{tt} - L_1 k_1 \eta + L_1 F_1 - \int_0^{L_1} (\rho_f \psi_t) dy \\ &+ -\frac{L_1^2}{L_2} m_2 \eta_{tt} - \frac{L_1^2}{L_2} k_2 \eta - L_1 F_2 + \frac{L_1}{L_2} \int_0^{L_2} \rho_f \psi_t dy] \} dt. \end{aligned}$$

Hence

$$\begin{aligned} 0 &= \int_0^T \{ \int_{\Gamma_B^+} \hat{w}(-m_0 w_{tt} - k_0 w - [\rho_f \psi_t]_{\Gamma_B} + F_0) \\ &+ \hat{\eta}[(-L_1 m_1 - \frac{L_1^2}{L_2} m_2) \eta_{tt} + (-L_1 k_1 - \frac{L_1^2}{L_2} k_2) \eta + L_1 (F_1 - F_2) \\ &+ L_1 (\frac{1}{L_2} \int_0^{L_2} (\rho_f \psi_t) dy - \frac{1}{L_1} \int_0^{L_1} (\rho_f \psi_t) dy)] \} dt. \end{aligned}$$

So in strong form we have

$$\left. \begin{aligned} & m_0 w_{tt} + k_0 w + [\rho_f \psi_t]_{\Gamma_B} = F_0 \\ & (\operatorname{Avg}_{\Gamma_1}(\rho_f \psi_t) - \operatorname{Avg}_{\Gamma_2}(\rho_f \psi_t)) + \hat{m_1} \eta_{tt} + \hat{k_1} \eta = \hat{F_1} \end{aligned} \right\} \tag{2.8}$$

with ψ satisfying

$$\left.\begin{aligned}&\Delta\psi = 0 \text{ in } \ \Omega\\ &\psi_n = 0 \text{ on } \Gamma_0\\ &\psi_n = \psi_y = -w_t \text{ on } \Gamma_B^+\\ &\psi_n = \psi_y = w_t \text{ on } \Gamma_B^-\\ &\psi_n = -\psi_x = \eta_t \text{ on } \Gamma_1\\ &\psi_n = -\psi_x = -\tfrac{L_1}{L_2}\eta_t \text{ on } \Gamma_2\end{aligned}\right\} \tag{2.9}$$

and where

$$\left.\begin{aligned}&[\rho_f\psi_t]_{\Gamma_B} = \rho_f\psi_t|_{\Gamma_B^-} - \rho_f\psi_t|_{\Gamma_B^+}\\ &\hat{m_1} = (m_1 + \tfrac{L_1}{L_2}m_2)\\ &\hat{k_1} = (k_1 + \tfrac{L_1}{L_2}k_2)\\ &\hat{F_1} = (F_1 - F_2)\\ &\operatorname{Avg}_{\Gamma_1}(\rho_f\psi_t) - \operatorname{Avg}_{\Gamma_2}(\rho_f\psi_t) = \tfrac{1}{L_1}\textstyle\int_0^{L_1}(\rho_f\psi_t)dy - \tfrac{1}{L_2}\int_0^{L_2}(\rho_f\psi_t)dy.\end{aligned}\right\} \tag{2.10}$$

Initial conditions are of the form

$$(w, w_t, \eta, \eta_t)|_{t=0} = (w^0, w^1, \eta^0, \eta^1). \tag{2.11}$$

The natural energy space E for the system is

$$E = (w, w_t, \eta, \eta_t, \psi) \in L^2(\Gamma_B) \times L^2(\Gamma_B) \times \mathbf{R}(\Gamma_1) \times \mathbf{R}(\Gamma_1) \times H^1(\Omega). \tag{2.12}$$

Remark 2. Actually, Γ_B is flexible and its position is determined by the solution of (2.8)–(2.9) and (2.11). Hence the system (2.8)–(2.9) and (2.11) is a free boundary problem. On the other hand, motions are typically on the nano-scale, well within the linear range. Consequently, here, as is typical in cochlea models, in (2.9), Γ_B will be assumed to be fixed in its equilibrium position.

2.3 Analysis of the fluid part of the model

Here we begin the analysis of model (2.9). Split (2.9) in the form :

$$\left.\begin{array}{l}\Delta\psi = 0 \text{ in } \Omega \\ \psi_n = 0 \text{ on } \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \\ \psi_n = -\psi_x = -g \text{ on } \Gamma_B^+ \\ \psi_n = -\psi_x = g \text{ on } \Gamma_B^-\end{array}\right\} \tag{2.13}$$

and

$$\left.\begin{array}{l}\Delta\psi = 0 \text{ in } \Omega \\ \psi_n = 0 \text{ on } \Gamma_0 \cup \Gamma_B^+ \cup \Gamma_B^- \\ \psi_n = -\psi_x = f \text{ on } \Gamma_1 \\ \psi_n = -\psi_x = -\frac{L_1}{L_2} f \text{ on } \Gamma_2\end{array}\right\}. \tag{2.14}$$

Since Γ (here $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_B$) is Lipschitz we may consider variational solutions. Define

$$a(\phi, \psi) = \int_\Omega \nabla\phi\nabla\psi d\Omega \tag{2.15}$$

and

$$(u, v)_\Gamma = \int_\Gamma uv d\Gamma. \tag{2.16}$$

Let $\tau : H^1(\Omega) \to H^{\frac{1}{2}}(\Gamma)$ denote the trace operator $\tau\phi = \phi|_\Gamma$. Multiply the first equation of (2.13) by ϕ and apply Green's formula to it, so that

$$0 = \int_\Omega \phi\Delta\psi = -\int_\Omega \nabla\phi\nabla\psi d\Omega + \int_\Gamma \phi\frac{\partial\psi}{\partial n} d\Gamma \quad \forall\phi \in H^1(\Omega).$$

This reduces to

$$\int_\Omega \nabla\phi\nabla\psi d\Omega = \int_\Gamma \phi\frac{\partial\psi}{\partial n} d\Gamma. \tag{2.17}$$

By using the notation defined above, we have

$$a(\phi, \psi) = (\tilde{g}, \tau\phi)_\Gamma, \tag{2.18}$$

where

$$\tilde{g} = \begin{cases} 0 & \text{on } \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \\ -g & \text{on } \Gamma_B^+ \\ g & \text{on } \Gamma_B^- \end{cases}. \tag{2.19}$$

Now define

$$\tilde{H}^s(U) = \{v \in H^s(U) : \int_U v dU = 0\}, \tag{2.20}$$

where $s \in [0,1]$ and $U =$ either Ω or Γ. The dual of $\tilde{H}^s(U)$ with respect to $\tilde{H}^0(U)$ (which is identified to $(\tilde{H}^0(U))'$) is $\tilde{H}^{-s}(U)$. We claim that a is sesquilinear, conjugate symmetric and continuous on $\tilde{H}^1(\Omega) \times \tilde{H}^1(\Omega)$. To see this

$$\begin{aligned} a(\Phi, c_1\Psi_1 + c_2\Psi_2) &= \int_\Omega \nabla\Phi\overline{\nabla(c_1\Psi_1 + c_2\Psi)}d\Omega, \quad (c_1, c_2 \in \mathcal{C}) \\ &= \int_\Omega \overline{c_1}\nabla\Phi\overline{\nabla\Psi} + \overline{c_2}\nabla\Phi\overline{\nabla\Psi}d\Omega \\ &= \overline{c_1}a(\Phi,\Psi) + \overline{c_2}a(\Phi,\Psi), \quad \text{(thus } a \text{ is sequilinear).} \end{aligned}$$

It is clear that a is conjugate symmetric. Finally,

$$\begin{aligned} |a(\Phi,\Psi)| &= |\int_\Omega \nabla\Phi\nabla\Psi d\Omega| \\ &\leq \|\nabla\Phi\|_{L^2(\Omega)}\|\nabla\Psi\|_{L^2(\Omega)}, \quad \text{(by Schwarz inequality)} \\ &\leq \|\Phi\|_{H^1(\Omega)}\|\Psi\|_{H^1(\Omega)}, \end{aligned}$$

where $\|\Phi\|_{H^1(\Omega)} = (\int_\Omega(\|\nabla\Phi\|^2 + \|\Phi\|^2)d\Omega)^{\frac{1}{2}}$.

Thus a is continuous. Furthermore, a is coercive on $\tilde{H}^1(\Omega)$ i.e for some $\delta > 0$

$$a(u,u) \geq \delta \parallel u \parallel^2_{H^1(\Omega)} \quad \forall \quad u \in \tilde{H}^1(\Omega). \tag{2.21}$$

We note that $\tilde{H}^1(\Omega)$ is densely and compactly embedded in $\tilde{H}^0(\Omega)$. So by Lax- Milgram theorem $\exists$ a unique solution ϕ to the problem

$$\begin{cases} \phi \in \tilde{H}^1(\Omega) \\ a(\phi,\psi) = l(\psi) \ \forall\psi \in \tilde{H}^1(\Omega) \end{cases} \tag{2.22}$$

where l is a given element of $\tilde{H}^{-1}(\Omega)$. By the trace theorem, if $\tilde{g} \in L^2(\Gamma)$ the form $l_{\tilde{g}(\psi)} = (\tilde{g}, \tau\psi)_\Gamma$ defines an element of $\tilde{H}^{-1}(\Omega)$.

So given any element $\tilde{g} \in \tilde{H}^{-\frac{1}{2}}(\Gamma)$ there is a unique solution $\phi \in \tilde{H}^1(\Omega)$ to the problem

$$a(\phi,\psi) = (\tilde{g}, \tau\psi)_\Gamma \ \forall\psi \in \tilde{H}^1(\Omega).$$

Let G denote the solution operator to (2.22) i.e $\phi = G\tilde{g}$ if and only if

$$\begin{cases} \phi \in \tilde{H}^1(\Omega) \\ a(\phi,\psi) = (\tilde{g},\tau\psi)_\Gamma \quad \forall \psi \in \tilde{H}^1(\Omega) \end{cases}$$

$G : \tilde{H}^{-\frac{1}{2}}(\Gamma) \mapsto \tilde{H}^1(\Omega)$ is continuous. Now define $\Lambda_0\tilde{g} = \tau G\tilde{g}$. Λ_0 is the Neumann to Dirichlet map. By continuity of G and τ

$$\Lambda_0 : \tilde{H}^{-\frac{1}{2}}(\Gamma) \mapsto \tilde{H}^{\frac{1}{2}}(\Gamma) \text{ continuously.}$$

Note that

$$< \Lambda_0\tilde{g}, \tilde{f} >= a(G\tilde{g}, G\tilde{f}) =< \tilde{g}, \Lambda_0\tilde{f} > \ \forall \tilde{f}, \tilde{g} \in \tilde{H}^{-\frac{1}{2}}(\Gamma).$$

So Λ_0 is self-adjoint. Nečas (40) showed that τG remains continuous from $\tilde{H}^0(\Gamma)$ to $\tilde{H}^1(\Gamma)$. It therefore follows that Λ_0 remains continuous as a mapping from $\tilde{H}^0(\Gamma) \mapsto \tilde{H}^1(\Gamma)$ i.e

$$\|\Lambda_0\tilde{g}\|_{H^1(\Gamma)} \leq C \parallel \tilde{g} \parallel_{L^2(\Gamma)} \forall \tilde{g} \in \tilde{H}^0(\Gamma) \ .$$

Now define

$$\Lambda g = \Lambda_0\tilde{g} \mid_{\Gamma_B^-} - \Lambda_0\tilde{g} \mid_{\Gamma_B^+} .$$

Proposition 1. *The operator* $\Lambda : L^2(0, L_B) \mapsto H^1(0, L_B)$ *is continuous and satisfies for some* $C > 0$

$$\parallel \Lambda\tilde{g} \parallel_{H^1(0,L_B)} \leq C \parallel \tilde{g} \parallel_{L^2(0,L_B)} .$$

Furthermore, Λ *is a positive and self-adjoint operator on* $L^2(0, L_B)$*.*

Proof:

$$\begin{aligned} \parallel \Lambda g \parallel_{H^1(0,L_B)} &= \parallel \Lambda_0\tilde{g} \mid_{\Gamma_B^-} - \Lambda_0\tilde{g} \mid_{\Gamma_B^+} \| \leq \parallel \Lambda_0\tilde{g} \parallel_{H^1(0,L_B)} + \parallel \Lambda_0\tilde{g} \parallel_{H^1(0,L_B)} \\ &\leq C \parallel \tilde{g} \parallel_{H^1(0,L_B)}, \end{aligned}$$

since Λ_0 is continuous.

If $f, g \in L^2(0, L_B)$,

$$\begin{aligned}
< \Lambda f, g > &= \int_\Gamma (\Lambda_0 \tilde{f} \mid_{\Gamma_B^-} - \Lambda_0 \tilde{f} \mid_{\Gamma_B^+}) g dl = \int_{\Gamma_B^-} (\Lambda_0 \tilde{f}) \tilde{g} dl + \int_{\Gamma_B^+} (\Lambda_0 \tilde{f}) \tilde{g} dl \\
&= \int_\Gamma \tilde{g} \Lambda_0 \tilde{f} dl, \quad \text{since } \Lambda_0 \text{ is self-adjoint,} \\
\int_\Gamma \tilde{g} \Lambda_0 \tilde{f} dl &= \int_\Gamma \tilde{f} \Lambda_0 \tilde{g} dl \\
&= \int_{\Gamma_B^-} f \Lambda_0 \tilde{g} dl + \int_{\Gamma_B^+} -f \Lambda_0 \tilde{g} dl \\
&= \int_\Gamma f (\Lambda_0 \tilde{g} \mid_{\Gamma_B^-} - \Lambda_0 \tilde{g} \mid_{\Gamma_B^+}) dl \\
&= < f, \Lambda g > .
\end{aligned}$$

So Λ is self-adjoint. To show that Λ is positive, we use the fact that a satisfies (2.21). So

$$\int_0^{L_B} (\Lambda f) \tilde{f} = a(G\tilde{f}, G\tilde{f}) \geq \delta \|G\tilde{f}\|^2_{H^1}(\Omega) > 0, \quad \forall f \in L^2(0, L_B), f \neq 0. \tag{2.23}$$

This implies that Λ is positive. This completes the proof.

Next consider

$$\begin{aligned}
\Delta \psi &= 0 \text{ on } \Omega \\
\psi_n &= \tilde{\eta} \text{ on } \Gamma,
\end{aligned}$$

where

$$\tilde{\eta} = \begin{cases} 0 & \text{on } \Gamma_0 \cup \Gamma_B^+ \cup \Gamma_B^- \\ \eta & \text{on } \Gamma_1 \\ -\frac{L_1}{L_2}\eta & \text{on } \Gamma_2 \end{cases},$$

η being a constant.

Define $H\eta = L_1(\text{Avg}_{\Gamma_1} \Lambda_0 \tilde{\eta} - \text{Avg}_{\Gamma_2} \Lambda_0 \tilde{\eta})$.

Here, $\text{Avg}_{\Gamma_i} \Lambda \tilde{\eta} = \int_0^{L_i} \Lambda_0 \tilde{\eta} dl \quad i = 1, 2.$

Proposition 2. *The operator $H : \mathbf{R} \mapsto H^1(0, L_1)$ is self-adjoint and positive.*

Proof:

$$\begin{aligned}
< \mathrm{H}\eta, v > &= < L_1(\mathrm{Avg}_{\Gamma_1}\Lambda_0\tilde{\eta} - \ \mathrm{Avg}_{\Gamma_2}\Lambda_0\tilde{\eta}), v > \\
&= L_1((\frac{1}{L_1}\int_{\Gamma_1}\Lambda_0\tilde{\eta})v - (\frac{1}{L_2}\int_{\Gamma_2}\Lambda_0\tilde{\eta})v) \\
&= L_1(\frac{1}{L_1}\int_{\Gamma_1} v\mid_{\Gamma_1}\Lambda_0\tilde{\eta} - \frac{1}{L_2}\int_{\Gamma_2} v\mid_{\Gamma_2}\Lambda_0\tilde{\eta}) \\
&= L_1(\frac{1}{L_1}\int_{\Gamma_1}\tilde{v}\Lambda_0\tilde{\eta} - \frac{1}{L_2}\int_{\Gamma_2} -\frac{L_2}{L_1}\tilde{v}\Lambda_0\tilde{\eta}) \\
&= L_1(\frac{1}{L_1}\int_{\Gamma_1}\tilde{v}\Lambda_0\tilde{\eta} + \frac{1}{L_1}\int_{\Gamma_2}\tilde{v}\Lambda_0\tilde{\eta}) \\
&= L_1(\frac{1}{L_1}(\int_{\Gamma_1}\tilde{v}\Lambda_0\tilde{\eta} + \int_{\Gamma_2}\tilde{v}\Lambda_0\tilde{\eta})) \\
&= \int_{\Gamma}\tilde{v}\Lambda_0\tilde{\eta}, \qquad \text{since } \Lambda_0 \text{ is self-adjoint} \qquad (2.24) \\
&= \int_{\Gamma}\tilde{\eta}\Lambda_0\tilde{v} \\
&= \int_{\Gamma_1}\eta\Lambda_0\tilde{v} + \int_{\Gamma_2} -\frac{L_1}{L_2}\eta\Lambda_0\tilde{v} \\
&= L_1(\frac{1}{L_1}\int_{\Gamma_1}\eta\Lambda_0\tilde{v} - \frac{1}{L_2}\int_{\Gamma_2}\eta\Lambda_0\tilde{v}) \\
&= < \eta, L_1(\ \mathrm{Avg}_{\Gamma_1}\Lambda_0\tilde{v} - \ \mathrm{Avg}_{\Gamma_2}\Lambda_0\tilde{v}) > \\
&= < \eta, Hv >
\end{aligned}$$

where $\tilde{v}$ is defined in the same way as $\tilde{\eta}$. Thus H is self-adjoint. Positivity of H follows from positivity of Λ_0 and (2.24). This completes the proof.

Next, let S be defined as:

$$S\beta = (\Lambda_0\tilde{\beta}\mid_{\Gamma_B^-} - \Lambda_0\tilde{\beta}\mid_{\Gamma_B^+}).$$

where

$$\tilde{\beta} = \begin{cases} 0 & \text{on } \Gamma_0 \cup \Gamma_B^+ \cup \Gamma_B^- \\ \beta & \text{on } \Gamma_1 \\ -\frac{L_1}{L_2}\beta & \text{on } \Gamma_2 \end{cases}$$

and β is a constant. Further, define

$$T\alpha = L_1(\ \mathrm{Avg}_{\Gamma_1}\Lambda_0\tilde{\alpha}|_{\Gamma_1} - \ \mathrm{Avg}_{\Gamma_2}\Lambda_0\tilde{\alpha}|\Gamma_2),$$

$\alpha \in L^2(\Gamma_B^+ \cup \Gamma_B^-)$ and $\tilde{\alpha}$ is defined the same way as $\tilde{g}$ was.

Proposition 3. $S = T^*$

Proof:

$$
\begin{aligned}
< \alpha, S\beta > &= \int_\Gamma \alpha(\Lambda_0\tilde{\beta} \mid_{\Gamma_B^-} - \Lambda_0\tilde{\beta} \mid_{\Gamma_B^+}) d\Gamma \\
&= \int_{\Gamma_B^-} \tilde{\alpha}(\Lambda_0\tilde{\beta}) \mid_{\Gamma_B^-} - \int_{\Gamma_B^+} -\tilde{\alpha}(\Lambda_0\tilde{\beta}) \mid_{\Gamma_B^+} \\
&= \int_{\Gamma_B^-} \tilde{\alpha}(\Lambda_0\tilde{\beta}) \mid_{\Gamma_B^-} + \int_{\Gamma_B^+} \tilde{\alpha}(\Lambda_0\tilde{\beta}) \mid_{\Gamma_B^+} \\
&= \int_\Gamma \tilde{\alpha}\Lambda_0\tilde{\beta} \\
&= \int_\Gamma \tilde{\beta}\Lambda_0\tilde{\alpha}, \quad \text{since } \Lambda_0 \text{ is self-adjoint}
\end{aligned}
$$

Now consider

$$
\begin{aligned}
< T\alpha, \beta > &= L_1(\ \mathrm{Avg}_{\Gamma_1}\Lambda_0\tilde{\alpha} - \mathrm{Avg}_{\Gamma_2}\Lambda_0\tilde{\alpha})\beta \\
&= L_1(\frac{1}{L_1}\int_{\Gamma_1} \tilde{\beta}\Lambda_0\tilde{\alpha} - \frac{1}{L_2}\int_{\Gamma_2} -\frac{L_2}{L_1}\tilde{\beta}\Lambda_0\tilde{\alpha}) \\
&= L_1(\frac{1}{L_1}\int_{\Gamma_1} \tilde{\beta}\Lambda_0\tilde{\alpha} + \frac{1}{L_1}\int_{\Gamma_2} \tilde{\beta}\Lambda_0\tilde{\alpha}) \\
&= \int_{\Gamma_1} \tilde{\beta}\Lambda_0\tilde{\alpha} + \int_{\Gamma_2} \tilde{\beta}\Lambda_0\tilde{\alpha} \\
&= \int_\Gamma \tilde{\beta}\Lambda_0\tilde{\alpha}
\end{aligned}
$$

Hence $< \alpha, S\beta > = < T\alpha, \beta >$ and therefore $S = T^*$. This completes the proof.

With these operators at our disposal, we can rewrite the system of differential equations (2.8)-(2.9) as:

$$\left.\begin{array}{ll}(m_0+\rho_f\Lambda)w_{tt}+\rho_f S\eta_{tt}+k_0 w=F_0 & \text{on } (0,L_B)\times\mathbf{R}^+\\ (\hat{m}_1+\rho_f H)\eta_{tt}+\rho_f T w_{tt}+\hat{k}_1\eta=\hat{F}_1 & \text{on } \mathbf{R}\times\mathbf{R}^+\end{array}\right\}. \tag{2.25}$$

Given initial data

$$w(x,0)=w^0, \quad w_t(x,0)=w^1, \quad \eta(0)=\eta^0, \quad \eta_t(0)=\eta^1, \tag{2.26}$$

the model is complete as an initial value problem.

2.4 A cochlear model with longitudinal membrane elasticity

In this section we discuss the derivation of the cochlear model that has longitudinal membrane elasticity. The basilar membrane should have some longitudinal membrane elasticity (though small) since some of its portion has connected membrane. The inclusion of this term has also been considered by other researchers e.g Jack Xin, (43), included the term for reasons of numerical stability for his model.

The derivation is similar to the first model except that here the potential energy and work term have extra terms i.e the energy term

$$\frac{1}{2}\int_0^{L_B}\beta(x)w_x^2 dx.$$

Here $\beta(x)$ describes the elasticity of the basilar membrane. So potential energy takes the form

$$P(t)=\frac{1}{2}\{\int_0^{L_B}(k_0w^2+\beta(x)w_x^2)dx+\int_0^{L_1}k_1\eta^2dy+\int_0^{L_2}k_2\xi^2dy\}.$$

Two boundary conditions are needed. It is natural to assume that $w(0,t)=0$ at the base and at the helicotrema end we assume an imposition of a transverse control $f(t)$. Hence the additional work term is

$$w(L_B,t)f(t).$$

Therefore, the Lagrangian in this case is

$$L=\int_0^T\{\frac{1}{2}\int_\Omega\rho|\nabla\Psi|^2+\frac{1}{2}\int_0^{L_B}(m_0w_t^2-\beta(x)w_x^2-k_0w^2)dx+f(t)+$$
$$\frac{1}{2}\int_0^{L_1}(m_1\eta_t^2-k_1\eta^2)dy+\frac{1}{2}\int_0^{L_2}(m_2\xi_t^2-k_2\xi^2)dy\}dt.$$

The class of variations $\{\hat{w}, \hat{\Psi}, \hat{\eta}, \hat{\xi}\}$ we consider are those that vanish at $t = 0$ and $T = 0$. Furthermore, $\hat{w} \in H^1(0, L_B)$ such that $\hat{w}(0) = 0$.

By running through the same procedure as for the first model, we end up with the system

$$m_0 w_{tt} - (\beta(x)w_x)_x + k_0 w + [\rho\Psi_t]_{\Gamma_B} = 0 \quad \text{on} \quad \Gamma_B \times \mathbf{R}^+ \tag{2.27}$$

$$\hat{m}_1\eta_{tt} + \hat{k}_1\eta + Avg_{\Gamma_1}(\rho\Psi_t) - Avg_{\Gamma_2}(\rho\Psi_t) = 0 \quad \text{on } (\Gamma_1 \cup \Gamma_2) \times \mathbf{R}^+ \tag{2.28}$$

$$\Delta\Psi = 0 \quad \text{in } \Omega \times \mathbf{R}^+ \tag{2.29}$$

$$\frac{\partial\Psi}{\partial\nu} = \begin{cases} 0 & \text{on } \Gamma_0 \times \mathbf{R}^+ \\ -w_t & \text{on } \Gamma_B^+ \times \mathbf{R}^+ \\ w_t & \text{on } \Gamma_B^- \times \mathbf{R}^+ \\ \eta_t & \text{on } \Gamma_1 \times \mathbf{R}^+ \\ -\frac{L_1}{L_2}\eta_t & \text{on } \Gamma_2 \times \mathbf{R}^+ \end{cases}, \tag{2.30}$$

with the boundary conditions

$$w_x(L_B, 0) = f, \quad w(0, t) = 0, \tag{2.31}$$

where f is a control input at the right end. In the equations (2.27)–(2.28)
$Avg_{\Gamma_i}\rho\Psi_t = \frac{1}{L_i}\int_0^{L_i} \rho\Psi_t dy \quad i = 1, 2;$
$\hat{m}_1 = L_1(m_1 + \frac{L_1}{L_2}m_2), \quad \hat{k}_1 = L_1(k_1 + \frac{L_1}{L_2}k_2).$
The initial conditions are of the form

$$(w, w_t, \eta, \eta_t)|_{t=0} = (w^0, w^1, \eta^0, \eta^1). \tag{2.32}$$

On introducing the operators defined in section (2.3), the above system becomes

$$\left.\begin{array}{l} (m_0 + \rho\Lambda)w_{tt} + \rho S\eta_{tt} - (\beta(x)w_x)_x + k_0 w = 0 \quad \text{on } \Gamma_B \times \mathbf{R}^+ \\ (\hat{m}_1 + \rho H)\eta_{tt} + \rho T w_{tt} + \hat{k}_1\eta = 0, \quad \text{on} \quad (\Gamma_1 \cup \Gamma_2) \times \mathbf{R}^+ \end{array}\right\}, \tag{2.33}$$

with boundary conditions (2.31) and initial conditions (2.32).
Thus the system (2.33) with boundary conditions (2.31) and initial conditions (2.32) describes the dynamics of a cochlear model with longitudinal elasticity.

CHAPTER 3. Existence and Uniqueness of solutions of the cochlear models

3.1 Existence and uniqueness of solutions to the cochlear model without longitudinal elasticity

In this section, we will discuss the existence and uniqueness of solutions of (2.8)–(2.9)and (2.11). The matrix form of system (2.25) together with initial conditions (2.26) takes the form

$$\begin{pmatrix} m_0 & 0 \\ 0 & \tilde{m_1} \end{pmatrix} \begin{pmatrix} w_{tt} \\ \eta_{tt} \end{pmatrix} + \rho_f \begin{pmatrix} \Lambda & S \\ T & H \end{pmatrix} \begin{pmatrix} w_{tt} \\ \eta_{tt} \end{pmatrix} + \begin{pmatrix} k_0 & 0 \\ 0 & \tilde{k_1} \end{pmatrix} \begin{pmatrix} w \\ \eta \end{pmatrix} = \begin{pmatrix} F_0 \\ \tilde{F_1} \end{pmatrix}, \quad (3.1)$$

with initial conditions

$$\begin{pmatrix} w(.,0) \\ \eta(0) \end{pmatrix} = \begin{pmatrix} w^0 \\ \eta^0 \end{pmatrix}, \qquad \begin{pmatrix} w_t(.,0) \\ \eta_t(0) \end{pmatrix} = \begin{pmatrix} w^1 \\ \eta^1 \end{pmatrix}, \quad (3.2)$$

and where

$$\begin{pmatrix} \Lambda & S \\ T & H \end{pmatrix}$$

is compact and self-adjoint on $L^2(0, L_B) \times \mathbf{R}$. This form is more convenient when studying the well-posedness of (2.8)–(2.9)and (2.11).

We will consider three approaches.

3.1.1 The fixed point approach

By writing

$$M = \begin{pmatrix} m_0 & 0 \\ 0 & \tilde{m_1} \end{pmatrix} \quad (3.3)$$

$$R = \begin{pmatrix} \Lambda & S \\ T & H \end{pmatrix} \tag{3.4}$$

$$K = \begin{pmatrix} k_0 & 0 \\ 0 & \tilde{k_1} \end{pmatrix} \tag{3.5}$$

and

$$\overline{V} = \begin{pmatrix} w \\ \eta \end{pmatrix}, \quad \rho_f = \rho, \quad \overline{F} = \begin{pmatrix} F_0 \\ \tilde{F_1} \end{pmatrix}$$

equation (3.1)–(3.2) take the form

$$(M + \rho R)\overline{V}_{tt} + K\overline{V} = \overline{F} \quad \text{with} \quad V(0) = V^0, \quad V_t(0) = V^1. \tag{3.6}$$

Theorem 1. *: Let*

$$Q = \begin{pmatrix} \frac{\sqrt{k_0}\sin\sqrt{\frac{k_0}{m_0}}t}{m_0^{3/2}} & 0 \\ 0 & \frac{\sqrt{k_1}\sin\sqrt{\frac{k_1}{m_1}}t}{m_1^2} \end{pmatrix},$$

where $m_0(x) \geq \delta > 0$. *Also let* $\mathbf{X}_\infty = L^\infty[(0,\tau);\mathbf{X}]$, *where* $\mathbf{X} = L^2(0, L_B) \times \mathbf{R}$ *and* M *be as defined by (3.3),* R *as defined by (3.4). Then for* $\rho < \frac{1}{\|R\|_{\mathbf{X}_\infty}(\|M^{-1}\|+\|Q\|_{L^1[(0,\tau);\mathbf{X}]})}$, *the system (2.8)-(2.9) has a unique solution in the class* $C([0,\tau];\mathbf{X})$. *Furthermore, the solution can be computed by fixed point iteration.*

Proof: Split the solution $\overline{V}$ so that $\overline{V} = \overline{U} + \overline{W}$ and consider the two equations

$$(M + \rho R)\overline{U}_{tt} + K\overline{U} = \overline{F} \quad \text{with} \quad \overline{U}(0) = 0, \quad \overline{U}_t(0) = 0 \tag{3.7}$$

$$(M + \rho R)\overline{W}_{tt} + K\overline{W} = 0 \quad \text{with} \quad \overline{W(0)} = V^0, \quad \overline{W}_t(0) = V^1 \tag{3.8}$$

We first look at (3.7) in the form

$$M\overline{U}_{tt} + K\overline{U} = \overline{F} - \rho R\overline{U}_{tt} \quad \text{with} \quad \overline{U}(0) = 0, \quad \overline{U}_t(0) = 0. \tag{3.9}$$

Define G_{00} as a solution operator i .e $G_{00} = (MD_{tt} + KI)^{-1}$ so that $G_{00}\overline{F} = y$ means that

$$M\overline{y}_{tt} + K\overline{y} = \overline{F} \quad \text{with} \quad \overline{y}(0) = 0, \quad \overline{y}_t(0) = 0.$$

Take the initial guess as $U_0 = 0$. Then

$$\begin{aligned} U_1 &= G_{00}(\overline{F} - \rho R\overline{U}_{0tt}) = G_{00}\overline{F} \\ U_2 &= G_{00}(\overline{F} - \rho R\overline{U}_{1tt}) \\ &\vdots \\ U_{k+1} &= G_{00}(\overline{F} - \rho R\overline{U}_{ktt}). \end{aligned}$$

This defines a fixed point iteration, with fixed point equation,

$$U = G_{00}(\overline{F} - \rho R\overline{U}_{tt}).$$

So we can write this as

$$U = T_1 U; \quad \text{where} \quad T_1 U = G_{00}(\overline{F} - \rho R\overline{U}). \tag{3.10}$$

Define

$$\mathbf{X}_\infty = L^\infty[(0,\tau);\mathbf{X}], \quad \text{where} \quad \mathbf{X} = L^2(0, L_B) \times \mathbf{R}.$$

Now let $T_1 : \mathbf{X}_\infty \to \mathbf{X}_\infty$ (we show in (3.14) that $G_{00}D_t^2$ is bounded on its domain in $\mathbf{X}$, hence T_1 extends as a bounded operator to all of $\mathbf{X}$) and consider

$$\begin{aligned} \|T_1 U - T_1 V\| &= \|G_{00}\overline{F} - \rho G_{00} R U_{tt} - G_{00}\overline{F} + \rho G_{00} R V_{tt}\|_{\mathbf{X}_\infty} \\ &= \|\rho G_{00} R V_{tt} - \rho G_{00} R U_{tt}\|_{\mathbf{X}_\infty} \\ &= \rho \|R G_{00}(V_{tt} - U_{tt})\|_{\mathbf{X}_\infty}. \end{aligned} \tag{3.11}$$

Let $\overline{Z} = \overline{U} - \overline{V}$ and $Z(0) = Z_t(0) = 0$ then $G_{00}\overline{Z}_{tt} = Y$ means

$$MY_{tt} + KY = Z_{tt} \quad \text{with} \quad Y(0) = 0, Y_t(0) = 0.$$

Take the Laplace transform so that

$$Ms^2\hat{Y} + K\hat{Y} = s^2\hat{Z},$$

where $\hat{Y}$ and $\hat{Z}$ are the Laplace transforms of Y and Z respectively. Solving for $\hat{Y}$ gives

$$\begin{aligned}
\hat{Y} &= (Ms^2+K)^{-1}s^2\hat{Z} \\
&= (Ms^2+K)^{-1}(Is^2+KM^{-1}-KM^{-1})\hat{Z} \\
&= [(s^2+KM^{-1})M]^{-1}(Is^2+KM^{-1}-KM^{-1})\hat{Z} \\
&= M^{-1}(s^2+KM^{-1})^{-1}[s^2+KM^{-1}-KM^{-1}]\hat{Z} \\
&= M^{-1}(I-(S^2+KM^{-1})^{-1}KM^{-1})\hat{Z} \\
&= (M^{-1}-M^{-1}(s^2+KM^{-1})^{-1}KM^{-1})\hat{Z} \qquad (3.12)
\end{aligned}$$

In matrix form this becomes

$$\begin{pmatrix} m_0^{-1} & 0 \\ 0 & \tilde{m}_1^{-1} \end{pmatrix} - \begin{pmatrix} m_0^{-1} & 0 \\ 0 & \tilde{m}_1^{-1} \end{pmatrix} \begin{pmatrix} s^2+k_0m_0^{-1} & 0 \\ 0 & s^2+\tilde{k}_1\tilde{m}_1^{-1} \end{pmatrix}^{-1} \begin{pmatrix} k_0m_0^{-1} & 0 \\ 0 & \tilde{k}_1\tilde{m}_1^{-1} \end{pmatrix} \hat{Z}$$

This then simplifies to

$$\hat{Y} = \left(\begin{pmatrix} {m_0}^{-1} & 0 \\ 0 & {\tilde{m}_1}^{-1} \end{pmatrix} - \begin{pmatrix} \frac{k_0m_0^{-1}}{m_0s^2+k_0} & 0 \\ 0 & \frac{\tilde{k}_1\tilde{m}_1}{m_1s^2+k_1} \end{pmatrix} \right) \hat{Z}$$

By taking the inverse Laplace transform we have

$$Y = M^{-1}Z - Q * Z \qquad (3.13)$$

where

$$M^{-1} = \begin{pmatrix} m_0^{-1} & 0 \\ 0 & \tilde{m}_1^{-1} \end{pmatrix}$$

and

$$Q = \begin{pmatrix} \frac{\sqrt{k_0}\sin\sqrt{\frac{k_0}{m_0}}t}{m_0^{3/2}} & 0 \\ 0 & \frac{\sqrt{k_1}\sin\sqrt{\frac{k_1}{m_1}}t}{m_1^2} \end{pmatrix}.$$

From (3.13), we have that

$$\|Y\|_{\mathbf{X}_\infty} \leq \|M^{-1}\|\|Z\|_{\mathbf{X}_\infty} + \|Q\|_{L^1[(0,\tau);\mathbf{X}]}\|Z\|_{\mathbf{X}_\infty}. \qquad (3.14)$$

From equation (3.11), we have that

$$\|T_1U - T_1V\| \leq \rho\|R\|(\|M^{-1}\| + \|Q\|_{L^1[(0,\tau);\mathbf{X}]})\|Z\|_{\mathbf{X}_\infty}. \qquad (3.15)$$

So T_1 will be a contraction mapping if

$$\rho < \frac{1}{\|R\|_{\mathbf{X}_\infty}(\|M^{-1}\| + \|Q\|_{L^1[(0,\tau);\mathbf{X}]})}. \tag{3.16}$$

Now consider the non-homogenous part

$$M\overline{W}_{tt} + K\overline{W} = -\rho R\overline{W}_{tt}; \qquad \overline{W}(0) = V^0, \quad \overline{W}_t(0) = V^1. \tag{3.17}$$

Let H be the solution of

$$M\overline{Z}_{tt} + K\overline{Z} = 0; \qquad \overline{Z}(0) = V^0, \quad \overline{Z}_t(0) = V^1. \tag{3.18}$$

So, if we let $W_0 = 0$ to be the initial guess for the solution of (3.17), then

$$\begin{aligned} \overline{W}_1 &= H + G_{00}(-\rho R\overline{W}_{0tt}) \\ \overline{W}_2 &= H + G_{00}(-\rho R\overline{W}_{1tt}) \\ &\vdots \\ \overline{W}_{k+1} &= H + G_{00}(-\rho R\overline{W}_{ktt}), \end{aligned} \tag{3.19}$$

this defines a fixed point iteration with fixed point equation

$$\overline{W} = H + G_{00}(-\rho R\overline{W}_{tt}). \tag{3.20}$$

Define

$$T_2\overline{W} = H - \rho G_{00}(R\overline{W}_{tt}). \tag{3.21}$$

Then

$$\|T_2\overline{W} - T_2\overline{U}\|_{\mathbf{X}_\infty} = \rho\|G_{00}R(\overline{W}_{tt} - \overline{U}_{tt})\|_{\mathbf{X}_\infty} = \rho\|RG_{00}\overline{Z}_{tt}\|_{\mathbf{X}_\infty}. \tag{3.22}$$

where $Z = W - U$. Recall that $G_{00}\overline{Z}_{tt} = V$ implies that

$$M\overline{V}_{tt} + K\overline{V} = \overline{Z}_{tt}; \quad \overline{V}(0) = \overline{V}_t(0) = 0. \tag{3.23}$$

Taking the Laplace transform on both sides gives

$$Ms^2\hat{V} + K\hat{V} = s^2\hat{Z} - sZ^0 - Z^1, \tag{3.24}$$

but note that $Z(0) = Z_t(0) = 0$ since U, W satisfy (3.23) and H satisfies (3.18).Thus, we have

$$Ms^2\hat{V} + K\hat{V} = s^2\hat{Z}. \tag{3.25}$$

So by repeating the same arguments as before we have that T_2 is a contraction mapping if

$$\rho \leq \frac{1}{\|R\|_{\mathbf{X}_\infty}(\|M^{-1}\| + \|Q\|_{L^1[(0,\tau);\mathbf{X}]})}. \tag{3.26}$$

Combining the two cases, we find that $\|T(U-V)\|_{\mathbf{X}_\infty} = \|(T_1+T_2)(U-V)\|_{\mathbf{X}_\infty} \leq \tilde{\rho}\|U - V\|_{\mathbf{X}_\infty}, \quad 0 < \tilde{\rho} < 1 \quad$ and where $\quad \tilde{\rho} \leq \frac{1}{2}\frac{1}{\|R\|_{\mathbf{X}_\infty}(\|M^{-1}\|+\|Q\|_{L^1[(0,\tau);\mathbf{X}]})}$.

To show uniqueness, we assume U_1 and U_2 satisfy (3.20), then

$$\|(U_1 - U_2)\|_{\mathbf{X}_\infty} = \|T(U_1 - U_2)\|_{\mathbf{X}_\infty} \leq \tilde{\rho}\|(U_1 - U_2)\|_{\mathbf{X}_\infty}, \quad 0 < \tilde{\rho} < 1,$$

which only holds if $U_1 - U_2 = 0$ i.e $\quad U_1 = U_2$.

Remark 3. The fixed point solution is actually in $C([0,\tau];\mathbf{X})$. This will follow from the semigroup approach in Section (3.1.3).

Remark 4. If $m_0(x), k_0(x)$, initial data are continuous and ρ is sufficiently small, then by the same argument, we obtain a fixed point with $\mathbf{X} = C[0, L_B] \times \mathbf{R}$. Thus, under these conditions we know that the solutions are spatially continuous.

3.1.2 Spectral theory approach

Consider the differential equation

$$(M + \rho R)\overline{V}_{tt} + K\overline{V} = \overline{F}; \quad V(0) = V^0, \quad V_t(0) = V^1.$$

We can rewrite this as:

$$M\overline{V_{tt}} + \rho R\overline{V_{tt}} + K\overline{V} = \overline{F}; \quad V(0) = V^0, \quad V_t(0) = V^1. \tag{3.27}$$

Let us define the sesquilinear forms

$$a(u,v) = \int_0^{L_B} k_0 u_1 \tilde{v}_1 dx + \hat{k}_1 u_2 \tilde{v}_2,$$

and

$$b(u,v) = \int_0^{L_B} (m_0 u_1 \tilde{v}_1 + \rho(\Lambda u_1)\tilde{v}_1 + S u_2 \tilde{v}_1)dx + \hat{m}_1 u_2 \tilde{v}_2 + T u_1 \tilde{v}_2 + H u_2 \tilde{v}_2.$$

Then a and b are conjugate-symmetric and continuous on $(L^2(0,1) \times \mathbf{R})^2 \quad (= \mathbf{X}^2)$. Furthermore since R is nonnegative,

$$b(u,u) \geq \|u\|^2_{\mathbf{X}}.$$

By the Lax-Milgram theorem there exists an isomorphism $B : \mathbf{X} \to \mathbf{X}$ and a continuous mapping $A : \mathbf{X} \to \mathbf{X}'$ (note that $\mathbf{X} = \mathbf{X}'$) such that

$$b(V,\hat{V}) = (BV,\hat{V})_{\mathbf{X}\times\mathbf{X}}, \quad \forall V,\hat{V} \in \mathbf{X}.$$

$$a(V,\hat{V}) = (AV,\hat{V})_{\mathbf{X}\times\mathbf{X}}, \quad \forall V,\hat{V} \in \mathbf{X}$$

We can write (3.27) in the operational form as

$$\frac{d^2}{dt^2} B\overline{V} + A\overline{V} = \overline{F}. \tag{3.28}$$

We claim that $B^{-1}A$ is self-adjoint relative to $b(.,.)$ for we have that

$$\begin{aligned} b(B^{-1}Au, v) &= (BB^{-1}Au, v) \\ &= (Au, v) \\ &= (u, Av), \quad \text{since } A \text{ is conjugate-symmetric we have} \\ &= (BB^{-1}u, Av) \\ &= b(B^{-1}u, Av), \quad \text{since B is conjugate symmetric so is its inverse, so} \\ &= b(u, B^{-1}Av) \end{aligned}$$

Let $\tilde{A} = B^{-1}A$.

The principal assertion of the spectral theorem is that every bounded normal operator $\tilde{A}$ on a Hilbert space induces (in a canonical way) a resolution E of the identity on the Borel subsets of its spectrum $\sigma(\tilde{A})$ and that $\tilde{A}$ can be reconstructed from E.

Now split equation (3.28), into

$$(i) \quad \overline{V}_{tt} + \tilde{A}\overline{V} = 0; \quad \overline{V}(0) = \overline{V}^0, \quad \overline{V}_t(0) = 0$$

$$(ii) \quad \overline{V}_{tt} + \tilde{A}\overline{V} = 0; \quad \overline{V}(0) = 0, \quad \overline{V}_t(0) = \overline{V}^1$$

$$(iii) \quad \overline{V}_{tt} + \tilde{A}\overline{V} = \tilde{\overline{F}}; \quad \overline{V}(0) = 0, \quad \overline{V}_t(0) = 0, \quad \text{where} \quad \tilde{\overline{F}} = B^{-1}\overline{F}.$$

Let us first consider (i) :

$$\overline{V}_{tt} + \tilde{A}\overline{V} = 0, \tag{3.29}$$

$$\text{with initial conditions} \quad \overline{V}(0) = \overline{V}^0, \quad \overline{V}_t(0) = 0.$$

This system, formally, has a solution of the form $\overline{V} = (\cos A_1 t)\overline{V}^0$, where$\tilde{A} = A_1^2$ (note that since $\tilde{A}$ is nonnegative, there is a unique nonnegative operator A_1 such that $\tilde{A} = A_1^2$). Let E be the resolution of the identity for $\tilde{A}$ then by the spectral theorem,

$$\cos A_1 t = \int_{\sigma(\tilde{A})} \cos\sqrt{\lambda} t E(d\lambda), \tag{3.30}$$

therefore

$$\overline{V} = (\cos A_1 t)\overline{V}^0 = \int_{\sigma(\tilde{A})} E(d\lambda)(\cos\sqrt{\lambda} t)\overline{V}^0.$$

Now, we verify that, $\overline{V}$, defined this way satisfies (3.29). We first check the identity for $\overline{V}_{tt}$ and then that for $\tilde{A}\overline{V}$.

$$\overline{V}_{tt} = -\int_{\sigma(\tilde{A})} E(d\lambda)\lambda(\cos\sqrt{\lambda} t)\overline{V}^0,$$

due to the boundedness of the spectrum of $\tilde{A}$, differentiation with respect to time is continuous and hence we can put the derivative thorough the integral sign. On the other hand , $\tilde{A}\overline{V}$ is given by the spectral theorem as:

$$\tilde{A}\overline{V} = \int_{\sigma(\tilde{A})} E(d\lambda)\lambda(\cos\sqrt{\lambda} t)\overline{V}^0.$$

Then since $\int_{\sigma(\tilde{A})} E d\lambda = I$, equation (3.29) holds.

Next consider (ii), which has the formal solution $\overline{V} = A_1^{-1}(\sin A_1 t)\overline{V}^1$.

$$\begin{aligned}
\overline{V} &= \int_{\sigma(\tilde{A})} E(d\lambda)\frac{1}{\sqrt{\lambda}}(\sin\sqrt{\lambda} t)\overline{V}^1 \\
\overline{V}_{tt} &= -\int_{\sigma(\tilde{A})} E(d\lambda)\sqrt{\lambda}(\sin\sqrt{\lambda} t)\overline{V}^1 \qquad (3.31)\\
\tilde{A}\overline{V} &= \int_{\sigma(\tilde{A})} E(d\lambda)\sqrt{\lambda}(\sin\sqrt{\lambda} t)\overline{V}^1. \qquad (3.32)
\end{aligned}$$

By adding equations (3.31) and (3.32) system (*ii*) holds.

Finally consider (*iii*).

Take the Laplace transform on both sides of (*iii*) so that

$$s^2\hat{\overline{V}} + \tilde{A}\hat{\overline{V}} = \hat{\overline{F}}(s)$$

and hence

$$\hat{\overline{V}} = (s^2 I + \tilde{A})^{-1}\hat{\overline{F}}(s).$$

Take inverse Laplace's transform to get

$$\overline{V} = \overline{F} * (A_1^{-1} \sin A_1 t),$$

which can be expressed as

$$\overline{V} = \int_{\sigma(\tilde{A})} E(d\lambda)\frac{\overline{F}}{\sqrt{\lambda}} * (\sin\sqrt{\lambda}t).$$

V_{tt} and $\tilde{A}\overline{V}$ will take the form,

$$\begin{aligned}
\overline{V}_{tt} &= -\int_{\sigma(\tilde{A})} E(d\lambda)\sqrt{\lambda}\overline{F} * \sin\sqrt{\lambda}t &\quad (3.33)\\
\tilde{A}\overline{V} &= \int_{\sigma(\tilde{A})} E(d\lambda)\lambda[\frac{1}{\sqrt{\lambda}}\overline{F} * (\sin\sqrt{\lambda}t)] \\
&= \int_{\sigma(\tilde{A})} E(d\lambda)\sqrt{\lambda}\overline{F} * \sin\sqrt{\lambda}t). &\quad (3.34)
\end{aligned}$$

So on adding (3.33) and (3.34), (*iii*) is satisfied. Therefore solution of (3.27) can be expressed as

$$\overline{V}(t) = \int_{\sigma(\tilde{A})} E(d\lambda)[\cos(\sqrt{\lambda}t)\overline{V}^0 + \frac{1}{\sqrt{\lambda}}\sin(\sqrt{\lambda}t)\overline{V}^1 + \sqrt{\lambda}\overline{F} * \sin(\sqrt{\lambda}t)]. \quad (3.35)$$

Remark 5. Equation (3.35) shows that V is differentiable in time to the extent that $\overline{F}$ is.

3.1.3 The Semigroup theory approach

First, we review basic definitions of semigroup theory. An elaborate treatment of this branch of mathematics can be found, among others, in (37).

3.1.3.1 Preliminaries

Let E be a Banach space and let $\mathcal{L}(E)$ be the set of all linear bounded operators from E to E, endowed with the operator norm $||.||_{\mathcal{L}(E)}$, defined by

$$||U||_{\mathcal{L}(E)} = \sup_{||x|| \leq 1} ||Ux||$$

for each $U \in \mathcal{L}(E)$, $\mathcal{L}(E)$ is a Banach space.

Definition 1. A family, $\{\mathcal{T}(t); t \geq 0\}$ in $\mathcal{L}(E)$ is a semigroup of linear operators on E, or simply semigroup if:

1. $\mathcal{T}(0) = 1$
2. $\mathcal{T}(t+s) = \mathcal{T}(t)\mathcal{T}(s)$ for each $t, s \geq 0$.

If in addition, it satisfies the continuity condition at t=0,

$$\lim_{t \downarrow 0} \mathcal{T}(t) = I,$$

in the norm topology of $\mathcal{L}(E)$, the semigroup is called uniformly continuous.

Remark 6. We say that the semigroup is strongly continuous, or a C_0 semigroup, if and only if

$$t \mapsto \mathcal{T}(t)x$$

is continuous for all $x \in E$ and $t \in [0, \infty)$. Here, $[0, \infty)$ has the usual topology and E has the norm topology.

Definition 2. The infinitesimal generator, or generator of a semigroup of linear operators $\{\mathcal{T}(t); t \geq 0\}$ is the operator $A : D(A) \subseteq E \mapsto E$ defined by

$$D(A) = \{x \in E : \exists \lim_{t \downarrow 0} \frac{\mathcal{T}(t)x - x}{t}, \}$$

and

$$Ax = \lim_{t \downarrow 0} \frac{\mathcal{T}(t)x - x}{t}.$$

Equivalently, we say that A generates $\{\mathcal{T}(t); t \geq 0\}$.

Remark 7. If $A : D(A) \subseteq E \mapsto E$ is the infinitesimal generator of a semigroup of linear operators then $D(A)$ is a vector subspace of E and A is a possibly unbounded linear operator.

Remark 8. If $\{\mathcal{T}(t); t \geq 0\}$ is a uniformly continuous semigroup of linear operators then, for each $t \geq 0$, $\mathcal{T}(t)$ is invertible.

Definition 3. A family of operators $\{G(t); t \in \mathbf{R}\}$ in $\mathcal{L}(E)$ is called a group of linear operators on E if $G(t+s) = G(t)G(s)$ for each $t, s \in \mathbf{R}$. If in addition $\lim_{t\downarrow 0} G(t) = I$ in the norm topology of $\mathcal{L}(E)$, the group is called uniformly continuous.

Remark 9. If $\{\mathcal{T}(t); t \geq 0\}$ is a uniformly continuous semigroup of linear operators then it can be extended to a uniformly continuous group of linear operators.

Remark 10. If $\{\mathcal{T}(t); t \geq 0\}$ is a uniformly continuous semigroup of linear operators then the mapping $t \mapsto \mathcal{T}(t)$ is continuous from $[0, \infty)$ to $\mathcal{L}(E)$ endowed with operator norm. $\square$

3.1.3.2 Semigroup formulation

The first order form of (3.6) can be written as

$$\mathcal{B}v_t = \mathcal{A}v + \mathcal{C}F, \tag{3.36}$$

where

$$v = \begin{pmatrix} V \\ V_t \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} 0 \\ I \end{pmatrix}.$$

and

$$F = \begin{pmatrix} F_0 \\ \hat{F}_1 \end{pmatrix}.$$

Define the sesquilinear form e on $(L_2(\Gamma_B) \times \mathbf{R})^2$ by

$$e\left(\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}\right) = a\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) + b\left(\begin{pmatrix} u_3 \\ u_4 \end{pmatrix}, \begin{pmatrix} v_3 \\ v_4 \end{pmatrix}\right).$$

$\mathcal{B}^{-1}\mathcal{A}$, denoted by $\tilde{\mathcal{A}}$, is skew-adjoint with respect to e since

$$< \left(\begin{pmatrix} I & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right) >_e$$

$$=< \left(\begin{pmatrix} v_1 \\ -B^{-1}(Au_1) \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right) >_e = a(v_1, u_2) + b(-B^{-1}(Au_1), v_2)$$

$$= (Av_1, u_2) + (-Au_1, v_2).$$

Let us now calculate

$$- < \left(\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right) >_e$$

$$= - < \left(\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} v_2 \\ -B^{-1}(Au_2) \end{pmatrix} \right) >_e$$

$$= -a((u_1, v_2) + b(v_1, -B^{-1}(Au_2)))$$

$$= -((Au_1, v_2) + (Bv_1, -B^{-1}(Au_2)))$$

$$= -((Au_1, v_2) + (v_1, -Au_2))$$

$$= -((Au_1, v_2) + (-Av_1, u_2)).$$

Therefore $\tilde{\mathcal{A}}$ is skew-adjoint. Hence by Stone's theorem (Pazy (37)), $\tilde{\mathcal{A}}$ is an infinitesimal generator of a C_0 group, $\{e^{\tilde{\mathcal{A}}t}\}_{t\in\mathbf{R}}$ of unitary operators. Since $\tilde{A}$ is bounded, the semigroup is uniformly continuous.

Theorem 2. *$\tilde{\mathcal{A}}$ is an infinitesimal generator of a uniformly continuous group of unitary operators on $\mathbf{X}^2 = (L_2(\Gamma_B) \times \mathbf{R})^2$. Given the control $F \in L_2(0, \tau; X)$, V^0, $V^1 \in \mathbf{X}$ the solution V of initial value problem*

$$(M + \rho R)V_{tt} + KV = F \quad \textit{with} \quad V(0) = V^0, \quad V_t(0) = V^1,$$

satisfies

$$V \in C([0, \tau]; \mathbf{X}).$$

Furthermore if $F \in C^k(\mathbf{R}; \mathbf{X})$ then $V \in C^{k+2}(\mathbf{R}; \mathbf{X})$. $\square$

Since $\tilde{A}$ is bounded, the following are immediate.

Corollary 1. *(i) Solutions of homogeneous problem are* $C^{\infty}(\mathbf{R};\mathbf{X})$.

(ii) If $F \in C^k([0,\tau];\mathbf{X})$ *then* $V \in C^{k+2}([0,\tau];\mathbf{X})$.

Remark 11. The solution of (2.8)-(2.9) (in form (3.36)), can be represented as

$$v(t) = \exp(\tilde{\mathcal{A}}t)v(0) + \int_0^t \exp\tilde{\mathcal{A}}(t-s)\begin{pmatrix} 0 \\ \mathcal{B}^{-1}F(s)\end{pmatrix} ds, \tag{3.37}$$

where

$$\exp(\tilde{\mathcal{A}}t) = \begin{pmatrix} \cos A_1 t & A_1^{-1}\sin A_1 t \\ -A_1 \sin A_1 t & \cos A_1 t \end{pmatrix}.$$

Corollary 2. *The mapping* $t \mapsto \exp\tilde{\mathcal{A}}t$ *is differentiable in norm and*

$$\frac{d}{dt}(\exp(\tilde{\mathcal{A}}t)) = \tilde{\mathcal{A}}\exp(\tilde{\mathcal{A}}t).$$

3.2 Existence and uniqueness results to the cochlear model with longitudinal elasticity

In this section we study existence and uniqueness of solutions to the cochlea model with longitudinally elastic basilar membrane. Recall the equations governing this system are given by

$$(m_0 + \rho\Lambda)w_{tt} + \rho S\eta_{tt} - (\beta(x)w_x)_x + k_0 w = F_0 \quad \text{on } \Gamma_B \times \mathbf{R}^+ \tag{3.38}$$

$$(\hat{m}_1 + \rho H)\eta_{tt} + \rho T w_{tt} + \hat{k}_1\eta = \hat{F}_1, \quad \text{on } (\Gamma_1 \cup \Gamma_2) \times \mathbf{R}^+ \tag{3.39}$$

$$w_x(L_B, 0) = f, \quad w(0,t) = 0 \tag{3.40}$$

$$\{w, w_t, \eta, \eta_t\}|_{t=0} = \{w^0, w^1, \eta^0, \eta^1\}. \tag{3.41}$$

Let us define

$$H^1_*(\Gamma_B) = \{f \in H^1(\Gamma_B) : f(0) = 0\}. \tag{3.42}$$

The natural energy space E_1 for the system is

$$E_1 = \{(w, \eta, w_t, \eta_t, \Psi)\} \in H^1_*(\Gamma_B) \times \mathbf{R} \times L^2(\Gamma_B) \times \mathbf{R} \times H^1(\Omega). \tag{3.43}$$

3.2.1 Semigroup approach

We first consider the homogenous problem:

$$(m_0+\rho\Lambda)w_{tt}+\rho S\eta_{tt}-(\beta(x)w_x)_x+k_0w = 0 \quad \text{on } \Gamma_B\times\mathbf{R}^+ \tag{3.44}$$

$$(\hat{m}_1+\rho H)\eta_{tt}+\rho Tw_{tt}+\hat{k}_1\eta = 0, \quad \text{on } (\Gamma_1\cup\Gamma_2)\times\mathbf{R}^+ \tag{3.45}$$

with homogenous boundary conditions:

$$w(0)=0, \quad w_x(L_B)=0 \tag{3.46}$$

and initial conditions (3.41). Define the sesquilinear forms

$$b\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = \int_0^{L_B}(m_0u_1\tilde{v}_1+\rho(\Lambda u_1)\tilde{v}_1+\rho(Su_2)\tilde{v}_1)dx+m_1u_2\tilde{v}_2+\rho Tu_1\tilde{v}_2+\rho Hu_2\tilde{v}_2 \tag{3.47}$$

and

$$a\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = \int_0^{L_B}(\beta(x)u_{1_x}\tilde{v}_{1_x}+k_0u_1\tilde{v}_1)dx+k_1u_2\tilde{v}_2. \tag{3.48}$$

In the variational formulation of (3.44)–(3.46), together with (3.41), we seek a function $V=\begin{pmatrix} w \\ \eta \end{pmatrix} \in C([0,\infty);\mathcal{V}_*)\bigcap C^1([0,\infty);\mathbf{X})$ satisfying

$$\frac{d}{dt}b(V_t,\hat{V})+a(V,\hat{V})=0 \quad \forall \hat{V}\in\mathcal{V}_*, \tag{3.49}$$

where $\mathcal{V}_*=\{\begin{pmatrix} f \\ g \end{pmatrix}\in H^1_*(0,L_B)\times\mathbf{R} : f_x(L_B)=0\}$ and $\mathbf{X}=\{\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\in L^2(0,L_B)\times\mathbf{R}\}$. Equation (3.49) holds in the sense of distributions on $[0,\infty)$. Since $m_0, m_1, \rho, S, T, H, \Lambda$ are positive and that Λ, T, S, H are continuous and self-adjoint (see Propositions (1)–(3), the form b is conjugate-symmetric and continuous on $\mathbf{X}\times\mathbf{X}$ and that

$$b(\underline{v},\underline{v})\geq\|\underline{v}\|^2_{\mathcal{H}}, \tag{3.50}$$

where $\underline{v}=\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. Similarly, a is symmetric and continuous on $\mathcal{V}_*\times\mathcal{V}_*$ and that

$$a(\underline{v},\underline{v})\geq\|\underline{v}\|^2_{H^1(0,L_B)}, \quad \forall \quad \underline{v}\in\mathcal{V}_*. \tag{3.51}$$

Now we make the identification $\mathbf{X} = \mathbf{X}'$, so that we have the dense and continuous embeddings

$$\mathcal{V}_* \subset \mathbf{X} \subset \mathcal{V}'_*. \tag{3.52}$$

By the Lax-Milgram Theorem there exist an isomorphism $B : \mathbf{X} \longrightarrow \mathbf{X}$ and a continuous mapping $A : \mathcal{V}_* \longrightarrow \mathcal{V}'_*$ such that

$$\begin{aligned} b(V, \hat{V}) &= (BV, \hat{V}), \quad \forall \quad V, \hat{V} \in \mathbf{X} \\ a(V, \hat{V}) &= < AV, \hat{V} >, \quad \forall \quad V, \hat{V} \in \mathcal{V}_*. \end{aligned}$$

We rewrite (3.50) in an operational form as

$$\frac{d}{dt} BV_t + AV = 0 \quad \text{in } \mathcal{V}'_*. \tag{3.53}$$

The first order form of (3.53) can be rewritten

$$\mathcal{B}\mathbf{V_t} = \mathcal{A}\mathbf{V}, \tag{3.54}$$

where

$$\mathbf{V} = \begin{pmatrix} V \\ V_t \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}.$$

We set $\mathcal{D}(A) = \{ \begin{pmatrix} f_1 \\ g_1 \end{pmatrix} \in (H^2(0, L_B) \bigcap H^1_*(0, L_B) \times \mathbf{R}) : f_{1_x}(L_B) = 0 \}$. Now define

$$\mathcal{D}(\mathcal{A}) = \mathcal{D}(A) \times \mathcal{V}_* \tag{3.55}$$

and note that $\mathcal{A} : \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{V}_* \times \mathbf{X}$ is continuous and $\mathcal{B} : \mathcal{V}_* \times \mathbf{X} \longrightarrow \mathcal{V}_* \times \mathbf{X}$ is an isomorphism. Hence, $\mathcal{B}^{-1}\mathcal{A} : \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{V}_* \times \mathbf{X}$ is continuous. In particular, $\mathcal{B}^{-1}\mathcal{A}$ is densely defined as an operator on $\mathcal{V}_* \times \mathbf{X}$. Define the sesquilinear form e on $\mathcal{V}_* \times \mathbf{X}$ by

$$e\left(\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \right) = a\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) + b\left(\begin{pmatrix} u_3 \\ u_4 \end{pmatrix}, \begin{pmatrix} v_3 \\ v_4 \end{pmatrix} \right). \tag{3.56}$$

Since

$$\mathcal{B}^{-1}\mathcal{A}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \mathcal{B}^{-1}\begin{pmatrix} u_2 \\ u_4 \\ -A\begin{pmatrix} u_3 \\ u_4 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} u_3 \\ u_4 \\ -B^{-1}A\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{pmatrix},$$

then for all $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \in \mathcal{D}(\mathcal{A})$, we have

$$\begin{aligned} e\left(\mathcal{B}^{-1}\mathcal{A}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}\right) &= e\left(\begin{pmatrix} u_3 \\ u_4 \\ -B^{-1}A\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}\right) \\ &= a\left(\begin{pmatrix} u_3 \\ u_4 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) + b\left(-B^{-1}A\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_3 \\ v_4 \end{pmatrix}\right) \\ &= a\left(\begin{pmatrix} u_3 \\ u_4 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) - \left(A\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_3 \\ v_4 \end{pmatrix}\right) \\ &= a\left(\begin{pmatrix} u_3 \\ u_4 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) - a\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_3 \\ v_4 \end{pmatrix}\right) \end{aligned}$$

$$= -(a\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_3 \\ v_4 \end{pmatrix}\right) - a\left(\begin{pmatrix} u_3 \\ u_4 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right)),$$

using the fact that a is symmetric, we have

$$= -(a\left(\begin{pmatrix} v_3 \\ v_4 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) - \left(A\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} u_3 \\ u_4 \end{pmatrix}\right))$$

On back tracking,

$$= -(a\left(\begin{pmatrix} v_3 \\ v_4 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) + b\left(-B^{-1}\left(A\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right), \begin{pmatrix} u_3 \\ u_4 \end{pmatrix}\right))$$

$$= -e\left(\begin{pmatrix} u_3 \\ u_4 \\ -B^{-1}A\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}\right)$$

$$= -e\left(\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}, \begin{pmatrix} v_3 \\ v_4 \\ -B^{-1}\left(A\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) \end{pmatrix}\right)$$

$$= -e\left(\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}, \mathcal{B}^{-1}\mathcal{A}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}\right).$$

By denoting $\mathcal{B}^{-1}\mathcal{A}$ by $\mathcal{A}_1$, the above calculations show that $\mathcal{A}_1$ is skew-symmetric with respect to $e(.,.)$ and $\mathcal{D}(\mathcal{A}_1) = \mathcal{D}(\mathcal{A}_1^*)$. Therefore by Stone's theorem, $\mathcal{A}_1$ is the infinitesimal generator of a strongly continuous group of unitary operators with respect to the energy inner product $e(.,.)$ on the finite energy space $\mathcal{V}_* \times \mathbf{X}$. We can summarize the above in the following proposition.

Proposition 4. *$\mathcal{A}_1$ is the infinitesimal generator of a strongly continuous group $\{\mathcal{T}\}_{t\in\mathbf{R}}$ on $\mathcal{V}_* \times \mathbf{X}$. Hence, given the initial conditions $(w_0, \eta_0, w_1, \eta_1) \in H^1_*(\Gamma_B) \times \mathbf{R} \times L^2(\Gamma_B) \times \mathbf{R} = \mathcal{V}_* \times \mathbf{X}$, the system (3.44)–(3.46), together with initial conditions (3.41)have a unique solution $V = \{w, \eta\}$ with*

$$V \in C([0,\infty); \mathcal{V}_* \cap C^1([0,\infty); \mathbf{X}).$$

Furthermore, for each $t > 0$,

$$a(V(t), V(t)) + b(V_t(t), V_t(t)) = a(V(0), V(0)) + b(V_t(0), V_t(0)). \qquad \square$$

3.2.2 Weak Solutions

Let the dual space of $\mathcal{V}_* \times \mathbf{X}$, relative to $b(.,.)$, be denoted by $\mathbf{X} \times \mathcal{V}_*'$. Since $\mathcal{A}_1 : \mathcal{V}_* \longrightarrow \mathbf{X}$ is bijective, when $\mathcal{V}_*$ is provided with the graph norm $\|\{u, v\}\| = \|\mathcal{A}_1\{u, v\}\|_e$, $\mathcal{A}_1$ becomes an isomorphism also. We can consider an extension of $\mathcal{A}_1$, denoted by $\hat{\mathcal{A}}_1$, from $\mathbf{X}$ to $(\mathcal{D}(\mathcal{A}_1^*))' = (\mathcal{D}(\mathcal{A}_1))' = \mathcal{V}_*'$ as follows:

$$\left(\hat{\mathcal{A}}_1 \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}\right)_e = \left(\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}, -\hat{\mathcal{A}}_1 \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}\right)_e, \quad \forall \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \in \mathcal{V}_* \times \mathbf{X}. \tag{3.57}$$

The dual $(\mathcal{D}(\mathcal{A}_1))'$ of $\mathcal{D}(\mathcal{A}_1)$ is $\mathbf{X} \times \mathbf{R} \times H^{-1}(\Gamma_B)$, where $\mathbf{R} \times H^{-1}$ is the dual space to $H^2(\Gamma_B) \cap H^1_*(\Gamma_B) \times \mathbf{R}$ relative to $b(.,.)$. The extended operator can be shown to be the generator of a strongly continuous semigroup of unitary operators isomorphic to the original one. Hence we can replace $\hat{\mathcal{A}}_1$ by $\mathcal{A}$ with impunity.

Corollary 3. *The semigroup defined in Proposition 4 extends continuously to a strongly continuous, unitary group on the space $\mathcal{V}_*'$. Therefore, given the initial data $\{y^0, v^0, y^1, v^1\} \in \mathcal{V}_*'$, there is uniquely defined solution to (3.44)–(3.46) and initial conditions (3.41) which satisfies*

$$\{y, v\} \in C((-\infty, \infty); \mathbf{X}) \cap C^1((-\infty, \infty); \mathcal{V}_*').$$

In chapter 5 we study controllability of cochlear model with longitudinal elasticity but with no coupling with the oval window. So, the system that we will consider reduces to

$$(m_0 + \rho\Lambda)w_{tt} - (\beta(x)w_x)_x + k_0 w = 0 \quad \text{on } \Gamma_B \times \mathbf{R}, \tag{3.58}$$

with boundary conditions

$$w(0, t) = 0, \quad w(L_B, t) = f \tag{3.59}$$

and initial conditions

$$w(x, 0) = w^0, \quad w_t(x, 0) = w^1. \tag{3.60}$$

Here, the semigroup treatment is similar to the one above. However, the relevant space in this case is $H_0^1(0, L_B) \times L^2(0, L_B)$. It turns out that there is a strongly continuous group $\{\mathcal{T}_1(t)\}_{t \in \mathbf{R}}$ on $H_0^1(0, L_B) \times L^2(0, L_B)$ so that the following results hold:

Proposition 5. *Given the initial conditions* $w^0, w^1 \in H_0^1(\Gamma_B) \times L^2(0, L_B)$, *equation (3.58) with homogeneous boundary conditions has a unique solution*

$$w \in C([0, \infty); H_0^1(\Gamma_B)) \cap C^1([0, \infty); L^2(\Gamma_B)).$$

Proposition 6. *Given the initial data* $\{w^0, w^1\} \in H^{-1}(0, L_B) \times L^2(0, L_B)$, *there is uniquely defined solution, to equation (3.58) with homogeneous boundary conditions and initial conditions (3.59), which satisfies*

$$w \in C((-\infty, \infty); L^2(0, L_B)) \cap C^1((-\infty, \infty); H^{-1}(0, L_B)). \tag{3.61}$$

CHAPTER 4. Controllability of Cochlear Model without longitudinal elasticity

In this chapter we study the controllability of the cochlear model without longitudinal elasticity. We will first review the preliminaries approximate controllability. Two cases will be studied: a case where a control acts on the whole basilar membrane and one where control acts on a small portion.

4.1 Preliminaries on approximate controllability

We highlight a few definitions from controllability theory. A comprehensive theory can be found in (42), (1) etc. Consider the system

$$\xi = A\xi + Bv \qquad \xi(0) = \xi_0 \tag{4.1}$$

where A is infinitesimal generator of a strongly continuous semigroup $\mathcal{T}(t)$ on a Hilbert space $\mathcal{X}$, B is a bounded operator from a control space $\mathcal{U}$ to $\mathcal{X}$. In equation (4.1), $\mathcal{X}$ is referred to as the state space, $\xi(t)$ is the state of the system and $\xi(0)$ is the initial condition.

Definition 4. A mild solution in $[0, T]$ for equation (4.1) is defined as

$$\xi(t) = \mathcal{T}(t)\xi(0) + \int_0^t (\mathcal{T}(t-s)Bv(s)ds, \quad \forall t \in [0, T], \text{ and } Bv \in L^2([0, T]; \mathcal{X}) \tag{4.2}$$

Definition 5. The controllability map of (4.1) on $[0, \tau]$ (for a finite $\tau > 0$) is the bounded map $H_\tau : L_2(0, \tau; \mathcal{U}) \to \mathcal{X}$ defined by

$$H_\tau v = \int_0^\tau T(\tau - s)Bv(s)ds.$$

Remark 12. In many cases e.g boundary control, H_τ is unbounded. However, this can often be controlled by considering weaker solutions and enlarging the definition of $\mathcal{X}$.

Definition 6. The system (4.1) is approximately controllable on $[0,\tau]$ (for a finite $\tau > 0$) if for $\varepsilon > 0$, it is possible to steer from the origin to a distance ε from any element of $\mathcal{X}$ in a finite time τ, say

$$\overline{ranH_\tau} = \mathcal{X}. \tag{4.3}$$

To show (4.3) one usually tries to show that H_τ^* is one to one i.e

$$B^*T(t)^*\xi = 0 \quad \text{on} \quad [0,\tau] \Rightarrow \xi = 0. \tag{4.4}$$

4.1.1 Approximate controllability of the cochlear model without longitudinal elasticity

We now consider the distributed control problem

$$v_t = A_*v + B_*F \tag{4.5}$$

$$v(x,0) = v^0, \quad v_t(x,0) = v^1. \tag{4.6}$$

Here, $A_* = \begin{pmatrix} 0 & I \\ -B^{-1}A & 0 \end{pmatrix}$, $F = \begin{pmatrix} F_0 \\ \hat{F}_1 \end{pmatrix}$ and $B_* = \begin{pmatrix} 0 \\ B^{-1} \end{pmatrix}$, where A and B are defined as in (3.53)whereas $\hat{F}_1$ is as in (2.10). Hence the controllability map is

$$H_\tau F = \int_0^\tau \exp A_*(\tau - s) \begin{pmatrix} 0 \\ B^{-1}F(s) \end{pmatrix} ds. \tag{4.7}$$

By the change of variables $t = \tau - s$ we have

$$H_\tau F = \int_0^\tau \exp\left(A_*t\right) \begin{pmatrix} 0 \\ B^{-1}\tilde{F}(t) \end{pmatrix} dt, \tag{4.8}$$

where $\tilde{F}(t) = F(\tau - t)$. We first note that

$$\begin{aligned}
\langle H_\tau \tilde{F}, Z\rangle_\varepsilon &= \langle \int_0^\tau \exp(A_* t) \begin{pmatrix} 0 \\ B^{-1}\tilde{F}(t) \end{pmatrix} dt, Z\rangle_\varepsilon \\
&= \int_0^\tau \langle \exp(A_* t) \begin{pmatrix} 0 \\ B^{-1}\tilde{F}(t) \end{pmatrix}, Z\rangle dt \\
&= \int_0^\tau \langle \begin{pmatrix} 0 \\ B^{-1}\tilde{F}(t) \end{pmatrix}, \exp(-A_* t) Z\rangle dt
\end{aligned}$$

$$\begin{aligned}
&= \langle \begin{pmatrix} 0 \\ B^{-1}\tilde{F}(t) \end{pmatrix}, \int_0^\tau \exp(-A_* t) Z dt\rangle_\varepsilon \\
&= \langle \begin{pmatrix} 0 \\ B^{-1}\tilde{F}(t) \end{pmatrix}, \int_0^\tau \exp(-A_* t) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} dt\rangle_\varepsilon \\
&= b\Big(B^{-1}\tilde{F}(t), \left(\exp(-A_* t) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right)_{2comp}\Big) \\
&= \Big(\tilde{F}(t), \left(\exp(-A_* t) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right)_{2comp}\Big)
\end{aligned} \tag{4.9}$$

where "2 comp" means $2nd$ component(i.e the velocity component). So,

$$H_\tau^* \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \left(\exp(-A_* t) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right)_{2comp} \tag{4.10}$$

We now show the following result

Theorem 3. *Given* $\{w^0, w^1\} \in (L^2(0, L_B))^2$, $\{\eta^0, \eta^1\} \in \mathcal{R}\times\mathcal{R}$ $F_0 \in L^2(0, \tau; X)$ *and* $F_1 \in \mathbf{R}$ *then (2.8), which is equivalent to (4.5)–(4.6), is approximately controllable on* $[0, \tau]$.

Proof:

We consider the system

$$\left.\begin{array}{l} Z_t = -A_* Z, \quad t \in [0,\tau] \\ Z(\tau) = \begin{pmatrix} -z_1 \\ z_2 \end{pmatrix} \end{array}\right\}. \tag{4.11}$$

The solution of this system is $Z = \exp(-A_*(t-\tau))Z(\tau)$. Considered as a backward in time problem, it becomes

$$\left.\begin{array}{l} Z_t = -A_* Z, \quad t \in [0,\tau] \\ Z(0) = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \end{array}\right\}. \tag{4.12}$$

It's solution is $Z(t) = \exp(-A_* t)Z(0)$. Now, to show approximate controllability we need to show that the operator H_τ^* is one to one, i.e

$$\exp(-A_* t)Z(0) = 0,$$

implies that

$$Z(0) = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0, \quad \Longrightarrow z_1 = z_2 = 0. \tag{4.13}$$

For our case

$$\left(\exp(-A_* t)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right)_2 \equiv 0 \quad \text{on} \quad [0,\tau],$$

means the velocity component $V_t = 0$. From the differential

$$(M + \rho R)V_{tt} + KV = 0,$$

we have that $V_{tt} = 0$ and therefore $KV = 0$. Since K is invertible, $V \equiv 0$. Hence

$$\begin{pmatrix} V \\ V_t \end{pmatrix}(0) = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0.$$

Hence the system is approximately controllable on $[0,\tau]$.

The following theorem examines approximate controllability when a small portion, ω, of the beam and Γ_1 are controlled.

Theorem 4. *Given* $\{w^0, w^1\} \in (L^2(0,L_B))^2$, $\{\eta^0,\eta^1\} \in \mathcal{R}\times\mathcal{R}$, $F_0 \in L^2(0,\tau;\chi_\omega)$ *and* $F_1 \in \mathbf{R}$ *then (2.8), which is equivalent to (4.5)–(4.6) is approximately controllable on* $[0,\tau]$.

Proof:

Consider the dual "observed problem"

$$\left.\begin{array}{l} Z_t = \mathcal{A}Z, \quad t \in [0,\tau] \\ Z(0) = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\ y = Z_2(t)\big|_\omega \end{array}\right\}.$$

Need to show:

$$y = 0 \text{ on } (0,T) \Longrightarrow z_1 = z_2 = 0.$$

Now consider observed system in the original form:

$$\begin{aligned} m_0 w_{tt} + \rho\Lambda w_{tt} + \rho S\eta_{tt} + k_0 w &= 0 \quad \text{on } (0,L_B)\times[0,T] \\ m_1\eta_{tt} + \rho T w_{tt} + \rho H\eta_{tt} + k_1\eta &= 0 \quad \text{on } [0,T] \\ w_t &= 0 \quad \text{on } \omega\times[0,T] \\ \eta_t &= 0 \quad \text{on } (0,T). \end{aligned}$$

since we can freely differentiate in time, also we have $w_{tt} = 0$ a.e. on $\omega\times[0,T]$ and $\eta_{tt} \equiv 0 \quad a.e$ on $[0,\tau]$. In particular,

$$\begin{aligned} \rho\Lambda w_{tt} + k_0 w &= 0 \quad \text{on} \quad \omega\times[0,T] \\ \rho T w_{tt} + k_1\eta &= 0 \quad \text{on} \quad [0,T]. \end{aligned}$$

Differentiating these equations with respect to t , we have

$$\begin{aligned} \rho\Lambda w_{ttt} + k_0 w_t &= 0 \quad a.e \quad \text{on} \quad \omega\times[0,T] \\ T w_{ttt} + k_1\eta_t &= 0 \quad a.e \quad \text{on} \quad [0,T]. \end{aligned}$$

Thus,

$$\begin{aligned} \rho \Lambda w_{ttt} &= 0 \quad a.e \quad \text{on} \quad \omega \times [0, T] \\ T w_{ttt} &= 0 \quad a.e \quad \text{on} \quad [0, T]. \end{aligned}$$

Recall that $\Lambda f = y$ means

$$\left.\begin{aligned} &\Delta \psi = 0 \quad \text{in } \Omega \\ &\psi|_{\Gamma_B^+} - \psi|_{\Gamma_B^-} = y \quad \text{on } \Gamma_B \\ &\partial_n \psi = \begin{cases} f & \text{on} \quad \Gamma_B^+ \\ -f & \text{on} \quad \Gamma_B^- \end{cases} \\ &\partial_n \psi = 0 \quad \text{on} \quad \Gamma_1 \cup \Gamma_0 \cup \Gamma_2 \end{aligned}\right\} \tag{4.14}$$

For our case, $\Lambda w_{ttt} = 0$ means

$$\left.\begin{aligned} &\Delta \psi = 0 \quad \text{in } \mathcal{D} \\ &\psi|_{\chi_\omega(\Gamma_B^+)} - \psi|_{\chi_\omega(\Gamma_B^-)} = 0 \\ &\partial_n \psi = 0 \quad \text{on} \quad \omega \end{aligned}\right\} \tag{4.15}$$

We now consider symmetric subset, $\mathcal{D}$ containing ω. Define $\phi(x, y) = \frac{\psi(x,y) - \psi(x,-y)}{2}$. Then

$$\left.\begin{aligned} &\Delta \phi = 0 \quad \text{in} \quad \mathcal{D} \\ &\phi(x, \tfrac{h}{2}) = 0 \quad \text{i.e on} \quad \omega \times [0, \tau] \\ &\frac{\partial \phi(x,y)}{\partial y} = \frac{\psi_y(x, \frac{h}{2}) + \psi_y(x, \frac{-h}{2})}{2} = 0 \quad \text{on} \quad \omega \times [0, \tau] \\ &\phi_n = 0 \quad \text{on} \quad \Gamma_0 \end{aligned}\right\} \tag{4.16}$$

It is easy to show, by separation of variables, that $\phi \equiv 0$ in $\mathcal{D}$. By continuation of solution, $\phi \equiv 0$ in Ω. This implies that (4.16) holds when ω is substituted by Γ_B. Furthermore, $\Lambda w_{ttt} \equiv 0$ on Γ_B and $w_{ttt} \equiv 0$ on Γ_B. Therefore, the original system of equations reduces to

$$k_0 w_t = 0 \quad \text{on} \quad \Gamma_B \times [0, \tau]. \tag{4.17}$$

So, $w_t \equiv 0$ on Γ_B and $\eta_t \equiv 0$ on $[0, \tau]$. So from the first theorem, $w \equiv 0$ in $\Gamma_B \times [0, \tau]$ and $\eta \equiv 0$ in $[0, \tau]$. Hence the system is approximately controllable on $[0, \tau]$.

CHAPTER 5. Controllability of Cochlear Model with longitudinal elasticity

In this chapter controllability of the cochlear model with longitudinal elasticity is considered. For simplification purposes, we consider the controllability of the basilar membrane without coupling with the oval window. First, we change the variables to get an equivalent system to which the method of multipliers would suitably be used. Let us recall the system in old variables with a control applied on one end of the basilar membrane.
The non-homogeneous problem in old co-ordinates is given by

$$(m_0 + \rho\Lambda)w_{tt} - (\beta(x)w_x)_x + k_0 w = 0 \quad \text{on } \Gamma_B \times \mathbf{R} \tag{5.1}$$

$$w(0,t) = 0, \quad w(L_B,t) = f \quad \text{on } \mathbf{R} \tag{5.2}$$

$$w(x,0) = w^0, \quad w_t(x,0) = w^1 \quad \text{on } \Gamma_B, \tag{5.3}$$

whereas homogeneous problem is

$$(m_0 + \rho\Lambda)w_{tt} - (\beta(x)w_x)_x + k_0 w = 0 \quad \text{on } \Gamma_B \times \mathbf{R} \tag{5.4}$$

$$w(0,t) = 0, \quad w(L_B,t) = 0 \quad \text{on } \mathbf{R} \tag{5.5}$$

$$w(x,0) = w^0, \quad w_t(x,0) = w^1 \quad \text{on } \Gamma_B. \tag{5.6}$$

For every $(w^0, w^1) \in H_0^1(\Gamma_B) \times L^2(\Gamma_B)$ problem (5.1)–(5.3) has a unique solution whose energy

$$E := \frac{1}{2}\int_{\Gamma_B} m_0|w_t|^2 + \rho w_t \Lambda w_t + \beta|w_x|^2 + k_0|w|^2 dx, \tag{5.7}$$

is conserved.

Change of variables: We change the variables as follows (see (41).)

$$\begin{aligned}
s &= \int_0^x m_0(\tau)^{\frac{1}{2}}\beta(\tau)^{-\frac{1}{2}}d\tau \\
\tilde{\Omega} &= (0, L_B), \quad 0 = s(0), \quad \tilde{L} = s(L_B) \\
b(s) &= m_0(x)^{-1}\frac{d}{dx}\sqrt{m_0(x)\beta(x)}|_{x=x(s)} \\
\tilde{u}(s,t) &= \exp(\int_0^s \frac{b(r)}{2}dr)u(x(s),t)
\end{aligned}$$

Therefore,

$$\begin{aligned}
u(x(s),t) &= \exp(-\int_0^s \frac{b(r)}{2}dr)\tilde{u}(s,t), \text{ let } H(s) = \exp(-\int_0^s \frac{b(r)}{2}dr), \text{ then} \\
u_x &= -\frac{b(s)}{2}\sqrt{\frac{m_0(x)}{\beta(x)}}H(s)\tilde{u}(s,t) + H(s)\tilde{u}_s\sqrt{\frac{m_0(x)}{\beta(x)}} \\
&= (\tilde{u}_s - \frac{b(s)}{2}\tilde{u})H(s)\sqrt{\frac{m_0(x)}{\beta(x)}}.
\end{aligned}$$

So

$$\begin{aligned}
\beta(x)u_x &= \beta(x)(\tilde{u}_s - \frac{b(s)}{2}\tilde{u})H(s)\sqrt{\frac{m_0(x)}{\beta(x)}} \\
&= \sqrt{\beta(x)m_0(x)}(\tilde{u}_s - \frac{b(s)}{2}\tilde{u})H(s), \quad \text{therefore,} \\
(\beta(x)u_x)_x &= \frac{d}{dx}(\sqrt{\beta(x)m_0(x)}(\tilde{u}_s - \frac{b(s)}{2}\tilde{u})H(s)) \\
&+ \sqrt{\beta(x)m_0(x)}(\tilde{u}_{ss}\sqrt{\frac{m_0(x)}{\beta(x)}} - \frac{b'(s)}{2}\sqrt{\frac{m_0(x)}{\beta(x)}}\tilde{u} - \frac{b(s)}{2}\sqrt{\frac{m_0(x)}{\beta(x)}}\tilde{u}_s)H(s) \\
&+ \sqrt{\beta(x)m_0(x)}(\tilde{u}_s - \frac{b(s)}{2}\tilde{u})(-\frac{b(s)}{2}\sqrt{\beta(x)m_0(x)}H(s) \\
&= b(s)m_0(x)(\tilde{u}_s - \frac{b(s)}{2}\tilde{u})H(s) + m_0(x)(\tilde{u}_{ss} - \frac{b'(s)}{2}\tilde{u} - \frac{b(s)}{2}\tilde{u}_s)H(s) \\
&- \frac{b(s)}{2}m_0(x)(\tilde{u}_s - \frac{b(s)}{2}\tilde{u})H(s) \\
&= [b(s)m_0(x)(\tilde{u}_s - \frac{b(s)}{2}\tilde{u}) + m_0(x)\tilde{u}_{ss} - \frac{m_0(x)b'(s)}{2}\tilde{u} \\
&- m_0(x)\frac{b(s)}{2}\tilde{u}_s - \frac{b(s)}{2}m_0(x)\tilde{u}_s + (\frac{b(s)}{2})^2 m_0(x)\tilde{u}]H(s) \\
&= m_0(x)[\tilde{u}_{ss} - (\frac{b^2(s)}{4}\tilde{u} + \frac{b'(s)}{2})\tilde{u}]H(s)
\end{aligned}$$

and

$$u_{tt} = H(s)\tilde{u}_{tt}$$

Therefore, equation (5.1) becomes

$$m_0(x)H(s)\{\tilde{u}_{tt} + \frac{\rho}{H(s)m_0(x)}\Lambda H(s)\tilde{u}_{tt} - \tilde{u}_{ss} + q\tilde{u}\} = 0,$$

where $q = \frac{b^2(s)}{4} + \frac{b'(s)}{2} + \frac{k_0}{m_0}$.

The non-homogeneous system in the new variables:

$$m_0(s)H(s)\tilde{u}_{tt} + \rho\Lambda H(s)\tilde{u}_{tt} - m_0(s)H(s)\tilde{u}_{ss} + m_0(s)H(s)q\tilde{u} = 0 \tag{5.8}$$

$$\tilde{u}(0,t) = 0, \quad \tilde{u}(\tilde{L},t) = f \tag{5.9}$$

$$\tilde{u}(s,0) = \tilde{u}^0, \quad \tilde{u}_t(s,0) = \tilde{u}^1, \tag{5.10}$$

where f is a control applied at the left of the BM (*basilar membrane*). Let $\mathcal{H}$ and $\mathcal{V}$ be complex Hilbert spaces defined as

$$\mathcal{H} := \{v \in L^2(0,\tilde{L})\}, \quad \mathcal{V} := \{v \in \mathcal{H}^1(0,\tilde{L})\},$$

with their respective norms given by

$$||v||_{\mathcal{H}} = (\int_0^{\tilde{L}} |v|^2 dx)^{\frac{1}{2}}, \quad ||v||_{\mathcal{V}} = (\int_0^{\tilde{L}} |v_x|^2 + q|v|^2 dx)^{\frac{1}{2}}.$$

5.1 The homogeneous system in new variables:

The homogeneous system in the new variables is given by

$$\sqrt{m_0(s)\beta}H^2(s)\{y_{tt} + \frac{\rho}{m_0 H^2}\Lambda H(s)y_{tt} - y_{ss} + qy\} = 0 \tag{5.11}$$

$$y(0,t) = y(\tilde{L},t) = 0 \tag{5.12}$$

$$y(s,0) = y^0, \quad y_t(s,0) = y^1. \tag{5.13}$$

Energy in the new variables:

For every $(y^0, y^1) \in H_0^1(0,\tilde{L}) \times L^2(0,\tilde{L})$ this problem has a unique solution whose energy

$$E := \frac{1}{2}\int_0^{\tilde{L}} \sqrt{m_0\beta}H^2(|y_t|^2 + \frac{\rho}{m_0 H^2}Hy_t\Lambda Hy_t + |y_s|^2 + q|y|^2)ds, \tag{5.14}$$

is conserved.

Details of calculations

$$E := \frac{1}{2}\int_0^{\tilde{L}} \left(m_0 H^2|y_t|^2 + \rho H y_t \Lambda H y_t + \beta(x(s))\{(y_s - \frac{b(s)}{2}y)H\sqrt{\frac{m_0}{\beta(x(s))}}\}^2 + k_0 H^2|y|^2\right\}\sqrt{\frac{\beta(x(s))}{m_0}}ds \tag{5.15}$$

Now simplify

$$\begin{aligned}
&\int_0^{\tilde{L}} \beta(x(s))\{(y_s - \frac{b(s)}{2}y)H\sqrt{\frac{m_0}{\beta(x(s))}}\}^2\sqrt{\frac{\beta(x(s))}{m_0}}ds \\
&= \int_0^{\tilde{L}} \beta(x(s))(y_s - \frac{b(s)}{2}y)^2 H^2 \frac{m_0}{\beta(x(s))}\sqrt{\frac{\beta(x(s))}{m_0}}ds \\
&= \int_0^{\tilde{L}} m_0 H^2(y_s^2 - b(s)yy_s + \frac{b^2(s)}{4}y^2)\sqrt{\frac{\beta(x(s))}{m_0}}ds \\
&= \int_0^{\tilde{L}} \sqrt{m_0\beta(x(s))}H^2(y_s^2 - b(s)yy_s + \frac{b^2(s)}{4}y^2)ds,
\end{aligned}$$

and consider

$$\begin{aligned}
&\int_0^{\tilde{L}} \sqrt{m_0\beta(x(s))}H^2 b(s)yy_s ds = \frac{1}{2}\int_0^{\tilde{L}} \sqrt{m_0\beta(x(s))}H^2 b(s)\frac{d}{ds}y^2 ds \\
&= \frac{1}{2}\sqrt{m_0\beta(x(s))}H^2 b(s)y^2|_0^{\tilde{L}} - \frac{1}{2}\int_0^{\tilde{L}} \frac{d}{ds}(\sqrt{m_0\beta(x(s))}H^2 b(s))y^2 ds \\
&= -\frac{1}{2}\int_0^{\tilde{L}} \sqrt{m_0\beta(x(s))}H^2 b(s)' y^2 ds.
\end{aligned}$$

$$\begin{aligned}
&\int_0^{\tilde{L}} \beta(x(s))\{(y_s - \frac{b(s)}{2}y)H\sqrt{\frac{m_0}{\beta(x(s))}}\}^2\sqrt{\frac{\beta(x(s))}{m_0}}ds \\
&= \frac{1}{2}\int_0^{\tilde{L}} \sqrt{m_0\beta(x(s))}H^2(y_s^2 + (\frac{b^2(s)}{4} + \frac{b'(s)}{2})y^2)ds.
\end{aligned}$$

Hence rewriting equation (5.15), we get equation (5.14).

Remark 13. In equation (5.11), denote

$$A_q y = -y_{ss} + qy, y \in \mathcal{V} \cap H^2(0, \tilde{L}).$$

Then A_q is compact and self-adjoint. Hence by spectral theorem, there exists a sequence $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \cdots$ of distinct eigenvalues of A_q and a sequence $V_1, V_2, V_3, \cdots$ of subspaces of $\mathcal{V}$

such that

$$|\lambda_k| \to \infty,$$

$$A_q v = \lambda_k v, \quad \forall v \in V_k, \quad \forall k \geq 1$$

$$\dim V_k < \infty \quad \forall k \geq 1,$$

$$V_k \perp V_l \quad \text{in } V \text{ if } k \neq l$$

and the vector space $\mathbb{V}$ generated by $\cup V_k$ is dense in $\mathcal{V}$.

Remark 14. From the regularity theory, given $q \in L^\infty(0, \tilde{L})$ and any $g \in L^2(0, \tilde{L})$ the solution $v \in \mathcal{V}$ of the problem

$$-v_{ss} + qv = g \qquad \text{in} \quad (0, \tilde{L}) \times \mathbf{R}$$

$$v(0,t) = v(L,t) = 0 \quad \text{in} \quad \mathbf{R}$$

belongs to $H^2(0, \tilde{L})$ and that we have the estimate

$$||v||_{H^2(0,\tilde{L})} \leq c||g||_{L^2(0,\tilde{L})}$$

with a constant c independent of the choice of g. It therefore follows that the eigenfunctions of the operator A_q belong to $H^2(0, \tilde{L})$.

5.2 The Hidden Regularity:

Theorem 5. *Let $T > 0$ and y be the solution to the homogeneous system (5.11)–(5.13) with $\{y^0, y^1\} \in \mathcal{V} \times \mathcal{H}$. Then there is a constant $c = c(T) > 0$ such that*

$$\int_0^T y_s^2(\tilde{L}, t)dt \leq cE(0). \tag{5.16}$$

Proof:

Multiply equation (5.11) by Hsy_s and integrate by parts:

$$0 = \int_0^T \int_0^{\tilde{L}} \sqrt{m_0\beta} H^2 s y_s (y_{tt} + \rho m_0^{-1} H^{-1} \Lambda H y_{tt} - y_{ss} + q(s)y) ds dt. \tag{5.17}$$

Term 1:

$$\int_0^T \int_0^{\tilde{L}} \sqrt{m_0\beta} H^2 s y_s (y_{tt}) ds dt$$

$$= \int_0^{\tilde{L}} \sqrt{m_0\beta} H^2 s y_s y_t |_0^T ds - \int_0^T \int_0^{\tilde{L}} \frac{d}{dt} (\sqrt{m_0\beta} H^2 s y_s) y_t ds dt. \tag{5.18}$$

Let $X_0 = \int_0^{\tilde{L}} \sqrt{m_0\beta} H^2 s y_s y_t |_0^T ds$. Then equation (5.18) becomes

$$\begin{aligned}
X_0 \quad - & \quad \int_0^T \int_0^{\tilde{L}} \sqrt{m_0\beta} H^2 s y_{st} y_t ds dt = X_0 - \frac{1}{2} \int_0^T \int_0^{\tilde{L}} \sqrt{m_0\beta} H^2 s \frac{d}{ds} (y_t)^2 ds dt \\
= & \quad X_0 - \frac{1}{2} \int_0^T \sqrt{m_0\beta} H^2 s y_t |_0^{\tilde{L}} dt + \frac{1}{2} \int_0^T \int_0^{\tilde{L}} \frac{d}{ds} (\sqrt{m_0\beta} H^2 s) y_t^2 ds dt \\
= & \quad X_0 + \frac{1}{2} \int_0^T \int_0^{\tilde{L}} \frac{d}{ds} (\sqrt{m_0\beta} H^2 s) y_t^2 ds dt \\
= & \quad X_0 + \frac{1}{2} \int_0^T \int_0^{\tilde{L}} (\sqrt{m_0\beta} b(s) H^2 s - \sqrt{m_0\beta} b(s) H^2 s + \sqrt{m_0\beta} H^2) y_t^2 ds dt \\
= & \quad X_0 + \frac{1}{2} \int_0^T \int_0^{\tilde{L}} \sqrt{m_0\beta} H^2 y_t^2 ds dt.
\end{aligned}$$

Next we consider

Term 2: $\int_0^T \int_0^{\tilde{L}} m_0 \beta H^2 s y_s (\rho m_0^{-1} H^{-1} \Lambda H y_{tt}) \sqrt{\frac{\beta}{m_0}} ds dt.$

Which, on formally integrating by parts, becomes

$$\begin{aligned}
& X_1 - \int_0^T \int_0^{\tilde{L}} \rho \frac{d}{dt} (s \sqrt{\frac{\beta}{m_0}} H y_s) \Lambda H y_t ds dt \\
& = X_1 - \int_0^T \int_0^{\tilde{L}} \rho (s \sqrt{\frac{\beta}{m_0}} H y_{st}) \Lambda H y_t ds dt
\end{aligned}$$

where $X_1 = \int_0^{\tilde{L}} \rho s \sqrt{\frac{\beta}{m_0}} H y_s \Lambda H y_t ds |_0^T$.

But,

$$\begin{aligned}
\int_0^T \int_0^{\tilde{L}} \rho (s \sqrt{\frac{\beta}{m_0}} H y_{st}) \Lambda H y_t ds dt = \int_0^T \int_0^{\tilde{L}} \frac{1}{2} \rho (s \sqrt{\frac{\beta}{m_0}} H [\frac{d}{ds} (y_t \Lambda H y_t) + \frac{1}{2} \frac{d}{ds} (y_t) \Lambda H y_t \\
- \frac{1}{2} y_t \frac{d}{ds} (\Lambda H y_t)] ds dt
\end{aligned}$$

where

$$\begin{aligned}
\frac{1}{2} \int_0^T \int_0^{\tilde{L}} \rho (s \sqrt{\frac{\beta}{m_0}} H \frac{d}{ds} (y_t \Lambda H y_t) ds dt = \frac{1}{2} \int_0^T \rho s \sqrt{\frac{\beta}{m_0}} H y_t \Lambda H y_t |_0^{\tilde{L}} dt \\
- \frac{1}{2} \int_0^T \int_0^{\tilde{L}} \rho \frac{d}{ds} (s \sqrt{\frac{\beta}{m_0}} H) y_t \Lambda H y_t) ds dt
\end{aligned}$$

and

$$\frac{1}{2}\int_0^T\int_0^{\tilde{L}}\rho(s\sqrt{\frac{\beta}{m_0}}(H\frac{d}{ds}y_t\Lambda Hy_tdsdt - Hy_t\frac{d}{ds}(\Lambda Hy_t))dsdt = \int_0^T\int_0^{\tilde{L}}\rho y_tMy_tdsdt.$$

Here $M\Psi$ is defined as

$$M\Psi = s\sqrt{\frac{\beta}{m_0}}H(\Lambda H\frac{d}{ds}\Psi - \frac{d}{ds}\Lambda H\Psi).$$

Therefore **Term 2** reduces to

$$X_1 + \frac{1}{2}\int_0^T\int_0^{\tilde{L}}\frac{d}{ds}(\rho s\sqrt{\frac{\beta}{m_0}}H)y_t\Lambda Hy_tdsdt + \frac{1}{2}\int_0^T\int_0^{\tilde{L}}\rho y_tMy_tdsdt.$$

Next we consider

Term 3: $= \int_0^T\int_0^{\tilde{L}}\sqrt{m_0\beta}H^2sy_s(-y_{ss}+qy)dsdt$, where

$$\begin{aligned}
-\int_0^T\int_0^{\tilde{L}}\sqrt{m_0\beta}H^2sy_sy_{ss} &= -\frac{1}{2}\int_0^T\int_0^{\tilde{L}}\sqrt{m_0\beta}H^2s\frac{d}{ds}y_s^2dsdt\\
&= -\frac{1}{2}\int_0^T s\sqrt{m_0\beta}H^2y_s^2dt|_0^{\tilde{L}} + \frac{1}{2}\int_0^T\int_0^{\tilde{L}}\frac{d}{ds}(s\sqrt{m_0\beta}H^2)y_s^2dsdt\\
= -\frac{1}{2}\int_0^T s\sqrt{m_0\beta}H^2y_s^2dt|_0^{\tilde{L}} &+ \frac{1}{2}\int_0^T\int_0^{\tilde{L}}(\sqrt{m_0\beta}H^2 + s\sqrt{m_0\beta}b(s)H^2 - s\sqrt{m_0\beta}b(s)H^2)dsdt\\
&= -\frac{\tilde{L}}{2}\int_0^T\sqrt{m_0\beta}H^2y_s^2dt|_0^{\tilde{L}} + \frac{1}{2}\int_0^T\int_0^{\tilde{L}}\sqrt{m_0\beta}H^2y_s^2dsdt
\end{aligned}$$

By putting together all these expressions, equation (5.17) then becomes

$$\begin{aligned}
0 = X^0 + \frac{1}{2}\int_0^T\int_0^{\tilde{L}}(\sqrt{m_0\beta}H^2y_t^2 &+ \frac{1}{2}\frac{d}{ds}(\rho s\sqrt{\frac{\beta}{m_0}}H)y_t\Lambda Hy_t + \frac{1}{2}\rho y_tMy_t + \frac{1}{2}\sqrt{m_0\beta}H^2y_s^2\\
&+\frac{1}{2}(\sqrt{m_0\beta}H^2qysy_sdsdt - \frac{1}{2}\int_0^T\tilde{L}(\sqrt{m_0\beta})|_{s=\tilde{L}}H^2(\tilde{L})y_s^2(\tilde{L},t)dt,
\end{aligned}$$

where $X^0 = X_0 + X_1 = \int_0^{\tilde{L}}(\sqrt{m_0\beta}H^2sy_sy_t + \rho s\sqrt{\frac{\beta}{m_0}}Hy_s\Lambda Hy_t)ds|_0^T$.

Therefore,

$$\begin{aligned}
\int_0^T\tilde{L}(\sqrt{m_0\beta})|_{s=\tilde{L}}H^2(\tilde{L})y_s^2(\tilde{L},t)dt &= 2X^0 + \int_0^T\int_0^{\tilde{L}}(\sqrt{m_0\beta}H^2y_t^2 + \frac{d}{ds}(\rho s\sqrt{\frac{\beta}{m_0}}H)y_t\Lambda Hy_t\\
&\quad+\rho y_tMy_t + \sqrt{m_0\beta}H^2y_s^2 + 2(\sqrt{m_0\beta}H^2qysy_sdsdt\\
&= 2X^0 + \int_0^T\int_0^{\tilde{L}}\sqrt{m_0\beta}H^2(y_t^2 + y_s^2 + \frac{\rho}{m_0H^2}Hy_t\Lambda Hy_t) + s\rho\frac{d}{ds}(\sqrt{\frac{\beta}{m_0}}H)y_t\Lambda Hy_t\\
&\quad+\rho y_tMy_t + 2\sqrt{m_0\beta}H^2qysy_sdsdt \qquad (5.19)
\end{aligned}$$

We now estimate X^0 by first considering

$$X^0(t) = \int_0^{\tilde{L}} (\sqrt{m_0\beta}H^2 s y_s y_t + \rho s\sqrt{\frac{\beta}{m_0}} H s y_s \Lambda H y_t) ds,$$

so that

$$\begin{aligned}
|X^0(t)| &= |\int_0^{\tilde{L}} (\sqrt{m_0\beta}H^2 s y_s y_t + \rho s\sqrt{\frac{\beta}{m_0}} H s y_s \Lambda H y_t) ds| \\
&\leq \int_0^{\tilde{L}} |\sqrt{m_0\beta}H^2 s y_s y_t| + \rho |s\sqrt{\frac{\beta}{m_0}} H s y_s \Lambda H y_t)| ds \\
&\leq \tilde{L}\int_0^{\tilde{L}} \sqrt{m_0\beta}H^2 |y_s y_t| + \rho\sqrt{\frac{\beta}{m_0}} |H y_s \Lambda H y_t| ds
\end{aligned}$$

For ρ small enough, $\rho(y_s^2 + y_t^2)$ dominates $\rho H y_s \Lambda H y_t$ so that

$$\begin{aligned}
|X^0(t)| \leq \tilde{L}\int_0^{\tilde{L}} & \sqrt{m_0\beta}H^2 (y_s^2 + y_t^2 + \frac{\rho}{m_0 H^2}(y_s^2 + y_t^2)) ds \\
&\leq \tilde{L}\int_0^{\tilde{L}} \sqrt{m_0\beta}H^2 (y_s^2 + y_t^2 + \rho c_0 (y_s^2 + y_t^2))
\end{aligned}$$

$$\leq \tilde{L}\int_0^{\tilde{L}} \sqrt{m_0\beta}H^2 (y_s^2 + y_t^2 + \rho c_0 (y_s^2 + y_t^2)) ds \leq 2\tilde{L} + 2\rho c_0 E(0). \tag{5.20}$$

Next we estimate $y_t M y_t$. We note that for $y_t \in L^2(0, \tilde{L})$,

$$\begin{aligned}
\int_0^{\tilde{L}} y_t M y_t ds &= \int_0^{\tilde{L}} s\sqrt{\frac{\beta}{m_0}} H y_t (\Lambda H y_{st} - \frac{d}{ds}\Lambda H y_t) ds \\
&= \int_0^{\tilde{L}} (s\sqrt{\frac{\beta}{m_0}} H y_t \Lambda H y_{st} - s\sqrt{\frac{\beta}{m_0}} H y_t \frac{d}{ds}\Lambda H y_t) ds \\
&= \int_0^{\tilde{L}} (s\sqrt{\frac{\beta}{m_0}} H y_{st} \Lambda H y_t - s\sqrt{\frac{\beta}{m_0}} H y_t \frac{d}{ds}\Lambda H y_t) ds \\
= s\sqrt{\frac{\beta}{m_0}} H y_t \Lambda H y_{st}|_0^{\tilde{L}} - \int_0^{\tilde{L}} & y_t \frac{d}{ds}(s\sqrt{\frac{\beta}{m_0}} H y_t \Lambda H y_t) - s\sqrt{\frac{\beta}{m_0}} H y_t \frac{d}{ds}\Lambda H y_t) ds \\
= -\int_0^{\tilde{L}} s\sqrt{\frac{\beta}{m_0}} H y_t \frac{d}{ds}\Lambda H y_t - s\sqrt{\frac{\beta}{m_0}} & H y_t \frac{d}{ds}\Lambda H y_t - \int_0^{\tilde{L}} y_t \frac{d}{ds}(s\sqrt{\frac{\beta}{m_0}} H)\Lambda H y_t \\
&= -2\int_0^{\tilde{L}} s\sqrt{\frac{\beta}{m_0}} H y_t \frac{d}{ds}\Lambda H y_t ds - \int_0^{\tilde{L}} y_t \frac{d}{ds}(s\sqrt{\frac{\beta}{m_0}} H)\Lambda H y_t ds.
\end{aligned}$$

Since Λ is continuous from $L^2(0, \tilde{L})$ to $H^1(0, \tilde{L})$, $s\sqrt{\frac{\beta}{m_0}} H \frac{d}{ds}\Lambda H y_t$ is continuous from $L^2(0, \tilde{L})$ to $L^2(0, \tilde{L})$. There exist, therefore, a constant,say c_2, such that

$$\int_0^T \int_0^{\tilde{L}} \rho y_t M y_t \leq \rho c_2 T E(0). \tag{5.21}$$

Next, for ρ small enough

$$\int_0^T\int_0^{\tilde{L}} s\rho\frac{d}{ds}(\sqrt{\frac{\beta}{m_0}}H)y_t\Lambda Hy_t \leq \rho\tilde{L}c_3\int_0^T\int_0^{\tilde{L}} 2y_t^2 dsdt \leq 4\rho c_3 T\tilde{L}E(0).$$

Here

$$c_3 = \frac{max_{(0,\tilde{L})}(\frac{d}{ds}(\sqrt{\frac{\beta}{m_0}}H))}{\sqrt{m_0\beta}H^2}.$$

Finally,

$$\begin{aligned}\int_0^{\tilde{L}}\sqrt{m_0\beta}H^2(2qysy_s)ds &\leq \int_0^{\tilde{L}}\sqrt{m_0\beta}H^2(2Q\tilde{L}(\int_0^{\tilde{L}}y^2ds)^{\frac{1}{2}}(\int_0^{\tilde{L}}y_sds)^{\frac{1}{2}}\\ &\leq \sqrt{m_0\beta}H^2(2Q\tilde{L}(\frac{1}{\lambda_1}\int_0^{\tilde{L}}(qy^2+y_s^2)ds)^{\frac{1}{2}}(\int_0^{\tilde{L}}y_s^2ds)^{\frac{1}{2}})\\ &\leq 2Q\tilde{L}(\frac{1}{\lambda_1}(2E(t))^{\frac{1}{2}})((2E(t))^{\frac{1}{2}})\\ &= \frac{4Q\tilde{L}}{\sqrt{\lambda_1}}E(t),\end{aligned}$$

Hence

$$\int_0^T y_s^2(\tilde{L},t)dt \leq cE(0)$$

where

$$c = \frac{4\tilde{L}+2T+\rho c_2 T+4\rho c_3 T\tilde{L}+\frac{4Q\tilde{L}T}{\sqrt{\lambda_1}}}{\tilde{L}\sqrt{m_0\beta}|_{s=\tilde{L}}H^2(\tilde{L})}.$$

Corollary 4. *If*

$$(y^0,y^1)\in\mathcal{V}\times\mathcal{H} \quad \textit{then } \partial_n y(\tilde{L},t)\in L^2_{loc}(\mathbf{R}).$$

This is " a hidden regularity " result.

5.3 Definition of the solution to non-homogenous problem

Consider the non-homogenous problem

$$m_0Hu_{tt}+\rho H^{-1}\Lambda Hu_{tt}-m_0Hu_{ss}+m_0Hqu=0 \tag{5.22}$$

$$u(0,t)=0, \quad u(\tilde{L})=f \tag{5.23}$$

$$u(s,0)=u^0, \quad u_t(s,0)=u^1. \tag{5.24}$$

Multiply equation (5.11) by Hu and formally integrate by parts i.e

$$\begin{aligned} 0 &= \int_0^T \int_0^{\tilde{L}} m_0 H^2 u(y_{tt} + \rho m_0^{-1} H^{-1} \Lambda H y_{tt} - y_{ss} + qy) \sqrt{\frac{\beta}{m_0}} dsdt \\ &= \int_0^T \int_0^{\tilde{L}} \sqrt{m_0 \beta} H^2 u(y_{tt} + \rho m_0^{-1} H^{-1} \Lambda H y_{tt} - y_{ss} + qy) dsdt \end{aligned}$$

First consider

$$\begin{aligned} \int_0^T \int_0^{\tilde{L}} \sqrt{m_0\beta} H^2 u(y_{tt} + \rho m_0^{-1} H^{-1} \Lambda H y_{tt}) dsdt &= \int_0^{\tilde{L}} \sqrt{m_0\beta} H^2 u(y_t + \rho m_0^{-1} H^{-1} \Lambda H y_t)|_0^T ds \\ - \int_0^T \int_0^{\tilde{L}} \frac{d}{dt}(\sqrt{m_0\beta} H^2 u)(y_t + \rho m_0^{-1} H^{-1} \Lambda H y_t) dsdt &= \int_0^{\tilde{L}} \sqrt{m_0\beta} H^2 u(y_t + \rho m_0^{-1} H^{-1} \Lambda H y_t)|_0^T ds \\ - \int_0^T \int_0^{\tilde{L}} \sqrt{m_0\beta} H^2 u_t(y_t + \rho m_0^{-1} H^{-1} \Lambda H y_t) dsdt &= \int_0^{\tilde{L}} \sqrt{m_0\beta} H^2 u(y_t + \rho m_0^{-1} H^{-1} \Lambda H y_t)|_0^T ds \\ - \int_0^{\tilde{L}} \sqrt{m_0\beta} H^2 u_t(y + \rho m_0^{-1} H^{-1} \Lambda H y)|_0^T ds &+ \int_0^T \int_0^{\tilde{L}} \sqrt{m_0\beta} H^2 u_{tt}(y + \rho m_0^{-1} H^{-1} \Lambda H y) dsdt \\ &= Y^0 + \int_0^T \int_0^{\tilde{L}} \sqrt{m_0\beta} H^2 u_{tt}(y + \rho m_0^{-1} H^{-1} \Lambda H y) dsdt, \end{aligned} \tag{5.25}$$

where $Y^0 = \int_0^{\tilde{L}} \sqrt{m_0\beta} H^2((uy_t - u_t y) + (\rho m_0^{-1} H^{-1})(u \Lambda H y_t - u_t \Lambda H y))|_0^T ds$.

Next consider

$$\begin{aligned} \int_0^T \int_0^{\tilde{L}} \sqrt{m_0\beta} H^2 u(-y_{ss} + qy) dsdt &= -\int_0^T \sqrt{m_0\beta} H^2 u y_s|_0^{\tilde{L}} + \int_0^T \int_0^{\tilde{L}} \sqrt{m_0\beta} H^2 y_s u_s dsdt \\ + \int_0^T \int_0^{\tilde{L}} \sqrt{m_0\beta} H^2 qyudsdt &= -\int_0^T \sqrt{m_0\beta} H^2 f y_s(\tilde{L}, t) dt + \int_0^T \sqrt{m_0\beta} H^2 u_s y|_0^{\tilde{L}} dt \\ &- \int_0^T \int_0^{\tilde{L}} \sqrt{m_0\beta} H^2 u_{ss} y dsdt)) + \int_0^T \int_0^{\tilde{L}} \sqrt{m_0\beta} H^2 qyudsdt \\ = -\int_0^T \int_0^{\tilde{L}} \sqrt{m_0\beta} H^2 u_{ss} y dsdt - \int_0^T \sqrt{m_0\beta} H^2 f y_s(\tilde{L}, t) dt &+ \int_0^T \int_0^{\tilde{L}} \sqrt{m_0\beta} H^2 qyudsdt. \end{aligned} \tag{5.26}$$

We now put equations (5.25) and (5.26) together, so that

$$0 = Y^0 + \int_0^T \int_0^{\tilde{L}} \sqrt{m_0\beta} H^2 (u_{tt} + \rho m_0^{-1} H^{-1} \Lambda H u_{tt} - u_{ss} + qu) y ds dt$$
$$- \int_0^T \sqrt{m_0\beta}|_{s=\tilde{L}} H^2(\tilde{L}) y_s(\tilde{L}, t) f dt,$$

which reduces to

$$0 = \int_0^{\tilde{L}} \sqrt{m_0\beta} H^2 (u(T)y_t(T) - u_t(T)y(T) + u^1 y^0 - u^0 y^1)$$
$$+\rho m_0^{-1} H^{-1} (u(T)\Lambda H y_t(T) - u_t(T)\Lambda H y(T)$$
$$+u^1 \Lambda H y^0 - u^0 \Lambda H y^1) - \int_0^T \sqrt{m_0\beta}|_{s=\tilde{L}} H^2(\tilde{L}) y_s(\tilde{L}, t) f dt.$$

Therefore

$$\int_0^T \sqrt{m_0\beta}|_{s=\tilde{L}} H^2(\tilde{L}) y_s(\tilde{L}, t) f dt + \int_0^{\tilde{L}} \sqrt{m_0\beta} H^2 (u^0 y^1 - u^1 y^0 + \frac{\rho}{m_0 H} (u^0 \Lambda H y^1 - u^1 \Lambda H y^0)) ds$$
$$= \int_0^{\tilde{L}} \sqrt{m_0\beta} H^2 (u(.,T) y_t(.,T) - u_t(.,T) y(.,T) + \frac{\rho}{m_0 H} (u(.,T) \Lambda H y_t(.,T) - u_t(.,T) \Lambda H y(.,T))) ds.$$

Putting

$$L_T(y^0, y^1) := \int_0^T \sqrt{m_0\beta}|_{s=\tilde{L}} H^2(\tilde{L}) y_s(\tilde{L}, t) f dt + < (-u^1, u^0), (y^0, y^1) >_{\mathcal{V}' \times \mathcal{H}, \mathcal{V} \times \mathcal{H}}, \qquad (5.27)$$

(where

$$< (u, v), (y, z) >= \int_0^{\tilde{L}} \sqrt{m_0\beta} H^2 (uy + vz + \frac{\rho}{m_0 H} (u \Lambda H y + v \Lambda H z)) ds,)$$

identity (5.27) can be rewritten as

$$L_T(y^0, y^1) =< (-u_t(T), u(T)), (y(T), y_t(T)) >_{\mathcal{V}' \times \mathcal{H}, \mathcal{V} \times \mathcal{H}} . \qquad (5.28)$$

Definition 7. We say that (u, u_t) is a solution of (5.22)–(5.24) if $(u, u_t) \in C(\mathbf{R}; \mathcal{H} \times \mathcal{V}')$ and if (5.28) is satisfied for every $T \in \mathbf{R}$ and for every $(y^0, y^1) \in \mathcal{V} \times \mathcal{H}$.

Theorem 6. *Suppose $(u^0, u^1) \in \mathcal{V}' \times \mathcal{H}$ and $f \in L^2(0, T)$, then the problem (5.22)–(5.24) has a unique solution $(u, u_t) \in C((0, t); \mathcal{V}' \times \mathcal{H})$.*

Furthermore, the linear map $(u^0, u^1, f) \mapsto (u, u_t)$ is continuous with respect to these topologies.

Proof:

As a consequence of the hidden regularity estimate, there is a unique continuous linear map

$$\mathbf{L}_1 : H_0^1(0,\tilde{L}) \times L^2(0,\tilde{L}) \mapsto L^2(0,T)$$

such that

$$\mathbf{L}_1(y^0,y^1) = y_s(\tilde{L},t) \quad \forall \quad (y^0,y^1) \in H_0^1(0,\tilde{L}) \times L^2(0,\tilde{L}).$$

It follows that for every given $T \in \mathbf{R}$ the linear form L_T is bounded in $H_0^1(0,\tilde{L}) \times L^2(0,\tilde{L})$. It is also true that the linear map $(y(T), y_t(T)) \mapsto (y^0,y^1)$ is an isomorphism of $H_0^1(0,\tilde{L}) \times L^2(0,\tilde{L})$ onto itself. Hence the linear form

$$(y(T), y_t(T)) \mapsto L_T(y^0,y^1)$$

is also bounded on $H_0^1(0,\tilde{L}) \times L^2(0,\tilde{L})$.

It, therefore, implies that there exists a unique $(u(T), u_t(T)) \in H_0^1(0,\tilde{L}) \times L^2(0,\tilde{L}))$ satisfying (5.28).

Now, we want to show that

$$\text{for } T \in \mathbf{R}, \quad T \mapsto ||u(T), u_t(T)||_{\mathcal{V}' \times \mathcal{H}}$$

is bounded in every bounded interval. This is to say that, for every bounded interval (say, I) and for all $T \in I$ the estimate

$$||(y(T), y_t(T))||_{\mathcal{V} \times \mathcal{H}} \leq c(I)(||f||_{L^2(0,T)} + ||(u^0,u^1)||_{\mathcal{V}' \times \mathcal{H}}). \tag{5.29}$$

Here $c(I)$ is a constant independent of T, f, u^0 and u^1.

Now choose $(y^0,y^1) \in \mathcal{V} \times \mathcal{H}$ arbitrarily and write

$$U = (-u_t, u), \quad U^0 = (-u^1, u^0), \quad Y = (y, y_t), \quad Y^0 = (y^0, y^1).$$

Applying the Hidden regularity theorem (5), we get the following estimate

$$\begin{aligned} | < U(T), Y(T) >_{\mathcal{V}' \times \mathcal{H}, \mathcal{V} \times \mathcal{H}} | &= \int_0^T y_s(t,\tilde{L}) f dt + < U^0, Y^0 >_{\mathcal{V}' \times \mathcal{H}, \mathcal{V} \times \mathcal{H}} \\ &\leq ||y_s(t,\tilde{L})||_I ||f||_I + ||U^0||_{\mathcal{V}' \times \mathcal{H}} ||Y^0||_{\mathcal{V} \times \mathcal{H}} \\ &\leq c(I)(||f||_1 + ||U^0||_{\mathcal{V}' \times \mathcal{H}} ||Y^0||_{\mathcal{V} \times \mathcal{H}}). \end{aligned}$$

Hence (5.29) follows.

It is well known (see Lions and Magenes (35)) that if $(u^0, u^1) \in \mathcal{V} \times \mathcal{H}$ and $f \in C_c^\infty(\mathbf{R})$, such that $f(0) = 0$, then (5.22)–(5.24) has a regular solution $u \in C(\mathbf{R}; \mathcal{V}) \cap C^1(\mathbf{R}; \mathcal{H})$. So in particular $(y, y_t) \in C(\mathbf{R}, \mathcal{V}' \times \mathcal{H})$. From (5.29) the data (y^0, y^1, f) is dense in $\mathcal{V}' \times \mathcal{H} \times L^2(0, T)$. Hence we conclude that $(y, y_t) \in C(\mathbf{R}; \mathcal{V}' \times \mathcal{H})$ in the general case also.

The continuous dependence of the solution on (y^0, y^1, f) follows from (5.29).

5.4 Observability Estimate:

The goal for this section is to establish the inverse inequality

$$\int_0^{\tilde{L}} y_s^2(\tilde{L}, t)dt \geq c' E(0).$$

Set $Q := \sup_{(0,\tilde{L})} q$ and

$$Q^* := \frac{2\tilde{L}Q}{\sqrt{\lambda_1}} + \frac{Q}{\sqrt{\lambda_1}}, \tag{5.30}$$

where λ_1 is the biggest constant such that

$$\int_0^{\tilde{L}} |v_s|^2 + q|v|^2 dx \geq \lambda_1 \int_0^{\tilde{L}} |v|^2 ds, \quad \forall v \in H_0^1(0, \tilde{L}).$$

Theorem 7. *Let*

$$Q^* < 1 \tag{5.31}$$

and T be an interval of length

$$T > \frac{4(\tilde{L} + \rho^* c_0)}{2(1 - Q^*) - \rho^*(c_2 + R)}. \tag{5.32}$$

There exists a ρ^*, where $0 < \rho^* \leq \rho$ and a constant $c' > 0$ such that

$$\int_0^T y_s^2(\tilde{L}, t)dt \geq c' E(0). \tag{5.33}$$

Proof

Rewrite equation (5.19) as

$$\int_0^T \tilde{L}(\sqrt{m_0\beta})|_{s=\tilde{L}} H^2(\tilde{L}) y_s^2(\tilde{L}, t)dt = 2X^0 + \int_0^T \int_0^{\tilde{L}} (\sqrt{m_0\beta} H^2\{y_t^2 + y_s^2 + qy^2$$
$$+\frac{\rho}{m_0 H^2} H y_t \Lambda H y_t\} dsdt + \int_0^T \int_0^{\tilde{L}} \{\rho\{\frac{d}{ds}(s\sqrt{\frac{\beta}{m_0}} H) y_t \Lambda H y_t - \sqrt{\frac{\beta}{m_0}} H y_t \Lambda H y_t +$$
$$y_t M y_t\} + \sqrt{m_0\beta} H^2(-qy^2 + 2qysy_s)\} dsdt$$

The terms $\rho\{\frac{d}{ds}(s\sqrt{\frac{\beta}{m_0}}H)y_t\Lambda Hy_t - \sqrt{\frac{\beta}{m_0}}Hy_t\Lambda Hy_t\}$ simplify to $(\frac{s}{2}b(s) - \frac{m_0'}{m_0}\sqrt{\frac{\beta}{m_0}})(\rho\sqrt{\frac{\beta}{m_0}}Hy_t\Lambda Hy_t)$, so

$$\int_0^T \tilde{L}(\sqrt{m_0\beta})|_{s=\tilde{L}}H^2(\tilde{L})y_s^2(\tilde{L},t)dt = 2X^0 + 2|T|E$$
$$+\int_0^T\int_0^{\tilde{L}}(\frac{s}{2}b(s) - \frac{m_0'}{m_0}\sqrt{\frac{\beta}{m_0}})(\rho\sqrt{\frac{\beta}{m_0}}Hy_t\Lambda Hy_t) + \rho y_t My_t$$
$$+\sqrt{m_0\beta}H^2(-qy^2 + 2qysy_s)dsdt.$$

We first consider the estimate for $\int_0^{\tilde{L}}\sqrt{m_0\beta}H^2(-qy^2+2qysy_s)ds$. We proceed as follows

$$\begin{aligned}
\textstyle\int_0^{\tilde{L}}\sqrt{m_0\beta}H^2(-qy^2+2qysy_s)ds &\geq \textstyle\int_0^{\tilde{L}}\sqrt{m_0\beta}H^2(-Qy^2+2qysy_s)ds \\
&\geq \textstyle\int_0^{\tilde{L}} -Qy^2ds - 2Q\tilde{L}(\int_0^{\tilde{L}}y^2ds)^{\frac{1}{2}}(\int_0^{\tilde{L}}y_s^2ds)^{\frac{1}{2}} \\
&\geq -\tfrac{Q}{\lambda_1}\textstyle\int_0^{\tilde{L}}y_s^2+qy^2ds - 2Q\tilde{L}(\frac{1}{\lambda_1}\int_0^{\tilde{L}}y_s^2+qy^2ds)^{\frac{1}{2}}(\int_0^{\tilde{L}}y_s^2)^{\frac{1}{2}} \\
&\geq -\tfrac{2Q}{\lambda_1}E(t) - 2Q\tilde{L}(\tfrac{1}{\sqrt{\lambda_1}}(2E(t))^{\frac{1}{2}}(2E(t))^{\frac{1}{2}}) \\
&= -2(\tfrac{Q}{\lambda_1} + \tfrac{2Q\tilde{L}}{\sqrt{\lambda_1}})E(0) \\
&= -2Q^*E(0).
\end{aligned} \tag{5.34}$$

From equation (5.21)

$$\int_0^T\int_0^{\tilde{L}}\rho y_tMy_t \geq -\rho c_2TE(0). \tag{5.35}$$

Let $R = max_{(0,\tilde{L})}|(\frac{s}{2}b(s) - \frac{m_0'}{m_0}\sqrt{\frac{\beta}{m_0}})|$. This implies that

$$\int_0^T\int_0^{\tilde{L}}(\frac{s}{2}b(s) - \frac{m_0'}{m_0}\sqrt{\frac{\beta}{m_0}})(\rho\sqrt{\frac{\beta}{m_0}}Hy_t\Lambda Hy_t)dsdt \geq -\rho RTE(0).$$

Finally, from equation (5.20)

$$|X_0| \geq (-2\tilde{L} - 2\rho c_0)E(0). \tag{5.36}$$

Hence

$$\int_0^T y_s^2(\tilde{L},t)dt \geq c'E(0), \tag{5.37}$$

where $c' = \frac{(2T-\rho c_2T-\rho RT-4\rho c_0-4\tilde{L})-2TQ^*}{\tilde{L}(\sqrt{m_0\beta})|_{s=\tilde{L}}H^2(\tilde{L})}$.

There exists $\rho* > 0$ small enough, $2T - \rho^*(c_2T + RT + 4c_0) - 4\tilde{L} - 2TQ^* > 0$, implies that

$$T > \frac{4(\tilde{L} + \rho^*c_0)}{2(1-Q^*) - \rho^*(c_2 + R)}.$$

Remark 15. While (5.31) is restrictive, it is easy to find non trivial examples that satisfy this hypothesis. For instance, if q is constant we have that λ_1 in equation (5.31) is

$$\lambda_1 = (\frac{\pi}{\tilde{L}})^2 + q$$

and the corresponding eigenfunction is

$$\sin(\frac{\pi x}{\tilde{L}}).$$

Then equation (5.31) becomes

$$Q^* = \frac{q(2\tilde{L}\sqrt{(\frac{\pi}{\tilde{L}})^2 + q} + 1)}{(\frac{\pi}{\tilde{L}})^2 + q}.$$

Therefore the condition $Q^* < 1$ implies that

$$4\pi^2\tilde{L}^4q^2 + 4\tilde{L}^6q^3 < \pi^4. \tag{5.38}$$

On solving equation (5.38), we get the estimate

$$-1.73 < \tilde{L}^2 q < 1.47.$$

Hence the estimate (5.32) is admissible.

5.5 The HUM Principle

Here we use the Hilbert Uniqueness method, due to Lions (36), to establish an exact controllability result.

We begin by first looking at (5.39)–(5.41) with terminal state conditions and with control given by $f(t)$ i.e

$$m_0 H u_{tt} + \rho H^{-1} \Lambda H u_{tt} - m_0 H u_{ss} + m_0 H q u = 0, \tag{5.39}$$

with the boundary conditions

$$u(0,t) = 0 \quad u(\tilde{L},t) = f \tag{5.40}$$

and initial conditions

$$u(s,0) = u^0, \quad u_t(s,0) = u^1. \tag{5.41}$$

By theorem (6), given $(u^0, u^1) \in \mathcal{V}' \times \mathcal{H}$ and $f \in L^2(0,T)$, the problem (5.39)–(5.41) has a unique solution

$$u \in C((0,T);\mathcal{V}') \cap C^1((0,T);\mathcal{H}).$$

Definition 8. The problem (5.39)–(5.41) is exactly controllable if for any given $(u^0, u^1), (u_T^0, u_T^1) \in \mathcal{V}' \times \mathcal{H}$ there exists a control $f \in L^2(0,T)$ such that the solution of (5.39)–(5.41) satisfies $u(s,T) = u_T^0, \quad u_t(s,T) = u_T^1$.

Theorem 8. *Assume that (5.31) and (5.32) hold. Then for any pair* $(u^0, u^1), (u_T^0, u_T^1) \in \mathcal{V}' \times \mathcal{H}$ *there exists* $f \in L^2(0,T)$ *such that the solution of (5.39)–(5.41) satisfies*

$$u(s,T) = u_T^0 \text{ and } \quad u_t(s,T) = u_T^1. \tag{5.42}$$

Remark 16. Let us consider the solution of the problem

$$\begin{aligned} m_0 H\hat{u}_{tt} + \rho H^{-1}\Lambda H\hat{u}_{tt} - m_0 H\hat{u}_{ss} + m_0 Hq\hat{u} = 0, \\ \hat{u}(0,t) = \hat{u}(\tilde{L},t) = 0, \\ \hat{u}(s,0) = u_T^0, \quad \hat{u}_t(s,0) = u_T^1, \end{aligned}$$

and assume that there exists a control $f \in L^2(0,T)$ such that the solution of the problem

$$\begin{aligned} m_0 H\acute{u}_{tt} + \rho H^{-1}\Lambda H\acute{u}_{tt} - m_0 H\acute{u}_{ss} + m_0 Hq\acute{u} = 0, \\ \acute{u}(0,t) = 0, \quad \acute{u}(\tilde{L},t) = f, \\ \acute{u}(s,0) = u^0 - \hat{u}(s,0), \quad \acute{u}_t(s,0) = u^1 - \hat{u}_t(s,0) \end{aligned} \tag{5.43}$$

satisfies $\acute{u}(T) = \acute{u}_t(T) = 0$. Then by linearity $u = \hat{u} + \acute{u}$ is a solution to (5.39)–(5.41) and it satisfies (5.42). Therefore, it is sufficient to prove Theorem 8 in the special case where $u_T^0 = u_T^1 = 0$.

The HUM principle involves seeking a control f in the special form $f = \partial_v y$ where y is the solution of the homogeneous problem

$$\begin{aligned} m_0 Hy_{tt} + \rho H^{-1}\Lambda Hy_{tt} - m_0 Hy_{ss} + m_0 Hqy = 0 \\ y(0,t) = 0, \quad y(\tilde{L},t) = 0 \\ y(s,0) = y^0, \quad y_t(s,0) = y^1 \end{aligned}$$

for a suitable choice of $(y^0, y^1) \in H_0^1(0, \tilde{L}) \times L^2(0, \tilde{L})$. From theorem (6), given any $(y^0, y^1) \in H_0^1(0, \tilde{L}) \times L^2(0, \tilde{L})$ problem (5.39)–(5.41) has a unique solution and that $y_s(\tilde{L}, t) \in L^2(0, T)$. Also the linear map $(y^0, y^1) \mapsto y_s(\tilde{L}, t)$ is continuous from $H_0^1(0, \tilde{L}) \times L^2(0, \tilde{L})$ into $L^2(0, T) \times H^{-1}(0, \tilde{L})$.

If (y^0, y^1) is such that $(y(0), y_t(0)) = (u^0, u^1)$ then the control $f := y_s(\tilde{L}, t)$ drives the system (5.39)–(5.41) to rest. Thus Theorem 8 will be proved if we show surjective

$$H_0^1(0, \tilde{L}) \times L^2(0, \tilde{L}) \ni (y^0, y^1) \mapsto (u(0), u_t(0)) \in H^{-1}(0, \tilde{L}) \times L^2(0, \tilde{L}).$$

It is, however, convenient to study the surjectivity of the map

$$\Theta : H_0^1(0, \tilde{L}) \times L^2(0, \tilde{L}) \mapsto H^{-1}(0, \tilde{L}) \times L^2(0, \tilde{L}),$$

defined by $\Theta(y^0, y^1) := (u_t(0), -u(0))$.

Lemma 1. *Assume (5.31 and (5.32) are satisfied. Then Θ is an isomorphism of $H_0^1(0, \tilde{L}) \times L^2(0, \tilde{L})$ onto $H^{-1}(0, \tilde{L}) \times L^2(0, \tilde{L})$.*

Proof

It is clear that Θ is bounded.

We now compute

$$< \Theta(y^0, y^1), (y^0, y^1) >_{\mathcal{V}' \times \mathcal{H}, \mathcal{V} \times \mathcal{H}} .$$

Multiply equation (5.39) by Hy and integrate by parts i.e

$$0 = \int_0^T \int_0^{\tilde{L}} \sqrt{m_0 \beta} H^2 y (u_{tt} + \frac{\rho}{m_0 H} \Lambda H u_{tt} - u_{ss} + qu) ds dt$$
$$= \int_0^{\tilde{L}} \sqrt{m_0 \beta} H^2 (y u_t - y_t u + \frac{\rho}{m_0 H} (u_t \Lambda H y - u \Lambda H y_t))|_0^T$$
$$+ \int_0^T \int_0^{\tilde{L}} \sqrt{m_0 \beta} H^2 (y_{tt} + \frac{\rho}{m_0 H} \Lambda H y_{tt} - y_{ss} + qy) ds dt - \int_0^T \sqrt{m_0 \beta} H^2 |y_s(\tilde{L}, t)|^2 dt$$
$$= \int_0^{\tilde{L}} \sqrt{m_0 \beta} H^2 (y u_t - y_t u + \frac{\rho}{m_0 H} (u_t \Lambda H y - u \Lambda H y_t))|_0^T - \int_0^T \sqrt{m_0 \beta} H^2 |y_s(\tilde{L}, t)|^2 dt$$
$$= \int_0^{\tilde{L}} \sqrt{m_0 \beta} H^2 (-y^0 u_t(0) + y^1 u(0) + \frac{\rho}{m_0 H} (u(0) \Lambda H y^1 - u_t(0) \Lambda H y^0)) ds$$
$$- \int_0^T \sqrt{m_0 \beta} H^2 |y_s(\tilde{L}, t)|^2 dt$$

Now apply the observability estimate to get the estimate

$$< \Theta(y^0, y^1), (y^0, y^1) >_{\mathcal{V}' \times \mathcal{H}, \mathcal{V} \times \mathcal{H}} \quad \geq cE.$$

Thus Θ is an isomorphism of $H_0^1(0, \tilde{L}) \times L^2(0, \tilde{L})$ onto $H^{-1}(0, \tilde{L}) \times L^2(0, \tilde{L})$.

CHAPTER 6. Conclusion

In this thesis, we have studied two cochlear models:

1. The cochlear model without longitudinal elasticity.

2. The cochlear model with longitudinal elasticity.

We saw that, for the first case, we have approximate controllability if a distributed control acts on a portion of the cochlear and exact controllability if the control is applied on the whole basilar membrane. For the second model, exact controllability is achieved if the control (with some restrictions on parameters) is applied at the boundary, the apex of the cochlear.

6.1 Future Research

• While studying controllability of the cochlear model with longitudinal elasticity, we ignored the dynamics of the oval and round windows. It would be interesting to see the results we get if these dynamics are incorporated.

• Also, for the case of the cochlear model with longitudinal elasticity, we only considered the boundary control. It remains to be seen how controllability results would be if we considered distributed control.

• Several other methodologies might be applied to the problem with longitudinal elasticity, for example Carleman estimate and methods based on differential geometry.

BIBLIOGRAPHY

[1] E. D. Sontag. *Mathematical Control Theory*, Deterministic Systems 2nd edition, Texts in Applied Mathematics, Springer-Verlag, New York 1998.

[2] A. Corti. *Recherches sur l′organ de l′ ouï des mammiferes.* Zeitschrift für wissenschafftliche Zoologie, 3(1851), 109–169.

[3] H. V. Helmoltz. *On the Sensation of Tone.* Dover, New York, 1954.

[4] G. V. Békésy. *Experiments in Hearing.* McGraw-Hill inc., New York.

[5] O. F Ranke. *Theory of operation of cochlear: A contribution to the hydrodynamics of the cochlear.* Journal of the Acoustic Society of America, 22(1950),–772-777.

[6] J. J. Zwislocki. *Theory of the acoustical action of the cochlear.* Journal of Acoustical Society of America, 22(1950), 778–784.

[7] N. Y. S. Kiang,T. Watanabe, E. C. Thomas, and L. F. Clark. *Discharge patterns of Single Fibers in the cat's Auditory Nerve.* Mit Press, Cambridge, MA, 1965.

[8] W. S. Rhode. *Measurement of the amplitude and Phase of Vibrations of the Basilar Membrane using Mössbauer Effect.* Ph.D thesis, University of Wisconsin, 1970.

[9] W. S. Rhode. *Observations of the vibration of the Basilar Membrane in squirrel monkeys using the Mössbauer technique.* Journal of the Acoustical Society of America 49(1971), 1218–1231.

[10] P. M. Sellick, R. Patuzzi, and B. M. Johnstone. *Measurements of Basilar Membrane motion in the guinea pig using the Mössbauer technique.* Journal of the Acoustical Society of America 72(1982), 131–141.

[11] H. Davies. *An active process in cochlear mechanics.* Hearing Research 9(1983), 79-90.

[12] W. E. Brownell, B. Kachar, R. Altschuler. *Electrokinetic changes of cochlear outer hair cells.* Nature 322(1986), 365–368.

[13] W. E. Brownell, W. E Shehata. *The effect of cytoplasmic turgor pressure on the static and dynamic mechanical properties of outer hair cells.* In P. Dallos, C. D. Geisler, J. W. Mathews, M. A. Ruggero and C. R. Steele, editors, *The mechanics and Biophysics of Hearing.* Springer-Verlag, Berlin, 1990. Spinger-Verlag, Berlin, 1990.

[14] B. N Evans , R. Hallworth, and P. Dallos. *The nonlinearity of outer hair cell motility:* Implications for cochlear physiology and pathology. In P. Dallos, C. D. Geisler, J. W. Mathews, M. A. Ruggero and C. R. Steele, editors. *The mechanics and Biophysics of Hearing.* Springer-Verlag, Berlin, 1990.

[15] M. A. Ruggero and N. C. Rich. *System injection of furosemide alters the mechanical response to sound of the basilar membrane.* In P. Dallos, C. D. Geisler, J. W. Mathews, M. A. Ruggero and C. R. Steele, editors, *The mechanics and Biophysics of Hearing.* Springer-Verlag, Berlin, 1990.

[16] J. Santos-Sacchi. *Fast outer cell motility:* How fast is fast? In P. Dallos, C. D. Geisler, J. W. Mathews, M. A. Ruggero and C. R. Steele, editors, *The mechanics and Biophysics of Hearing.* Springer-Verlag, Berlin, 1990.

[17] P. Dallos, D. Oliver, D. Z He, N. Klocker, J. Ludwig, U. Shulte, S. Waldegger, J. P. Rusppersberg, B. Fakler. *Intracellular anions as the voltage sensor of prestin, the outer hair cell motor protein* Science, 292(2001), 2340–2343.

[18] M. A. Ruggero, S. S. Narayan, A. N. Temchin, A. Recio. *Frequency tuning of basilar membrane and auditory nerve fibers in the same cochleae.* Science vol.282, 5395(1998), 1882–1884.

[19] M. C. Liberman, J. Gao, D. Z. He, X. Wu, S. Jia, J. Zuo. *Prestin is required for electromotility of the outer hair cell and for the cochlear amplifier.* Nature 419(2002), 300–304.

[20] M. H. Holmes. *A mathematical model of the dynamics of the inner ear.* J. Fluid mech., 116(1982), 59–75.

[21] M. H. Holmes and L. A. Rubenfeld, eds.. *Mathematical modeling of the hearing process, Lecture notes in Biomath.*, 43, Springer, Berlin, New york, 1980.

[22] C. S. Peskin. *Partial differential equations in biology*, courant inst. Lecture notes, courant inst.of mathematical sciences, New york Univ., New york, 1975-76.

[23] C. R. Steele. *Behavior of basilar membrane with pure tone excitation*, J. Acoust. Soc. Amer., 55(1974) 148–162.

[24] C. R. Steele and L. Tabor. *Comparison of WKB and finite difference calculations for two dimensional cochlear model,* J. Acoust. Soc. of Amer., 65(1979) 1001-1006.

[25] J.L. Randall, S. C. Peskin and D. L. Peter. *Solution of two dimensional Cochlear Model with fluid viscosity*, Siam J. Appl. Math. vol. 48, no. 1, 1998, pp 191–213.

[26] S. W. Hansen. *Exact cxontrollability of an elastic membrane coupled with a potential fluid*, Int. J. Appl. Math. Comput. Sci., Vol. 11, No. 6,(2001), 1231-1248.

[27] G. Avalos. *The exponential stability of coupled hyperbolic/ parabolic system arising in structural acoustics.* Abstr. Appl. Anal. Vol. 1, No. 2, (1996), pp. 203-219.

[28] H. T. Banks, R. J. Silcox and R. C. Smith. *The modeling and control of acoustic / structure interaction problems in pizoceramic actuators: 2-d numerical examples.* ASME J. Vibr. Acoust., Vol. 2, (1993), pp. 343–390.

[29] J. -L. and E. Zuazua. *Approximate controllability of hydro-elastic coupled system.* ESAIM: Contr. Opti. Calc. Var., Vol. 1, (1995) pp. 1–15.

[30] S. Micu and E. Zuazua. *Boundary controllability of a linear hybrid system arising in the control of noise.* SIAM J. Contr. Optim., Vol. 35, No. 5, (1997), pp. 1614-1637.

[31] J. Xin and Y. Kim. *A two-dimensional nonlinear nonlocal feed-forward cochlear model and time domain computation of multitone interactions* . SIAM J. Multiscale Modeling and Simulation, Vol. 4, No. 2, pp 664-690, 2005.

[32] S. T. Neely and D. O. Kim. *An active cochlear model showing sharp tuning and high sensitivity.* Hearing Research, 9(1983) 123-130.

[33] I. Lasiecka and G. Avalos. *Exact controllability of structural acoustic interactions.* Journal de Mathematiques Pure et Applique, **82** (2003), 1047-1073, .

[34] S. W. Hansen and A. Lyashenko. *Exact controllability of a beam in an incompressible inviscid fluid.* Disc. Cont. Dyn. Syst., **3** (1997) no. 1, 59–78.

[35] J.-L. Lions and E. Magenes. *Problèmes aux limites non homogenes et applications I–III*, Dunod, Paris, 1968-1970.

[36] J.-L. Lions. *Contrôlabilité exacte des systèmes distribueés,* C. R. Acad. Sci. Paris Sér I Math. 302(1986), 471–475.

[37] A. Pazy. *Semigroup of linear operators and applications to partial differential equations,* Springer-Verlag, New York, 1983.

[38] R. F. Curtain and H. Zwart. *An introduction to infinite-dimensional linear systems theory,* Springer-Verlag, New York, 1995

[39] D. Manoussaki and R. S. Chadwick. *Effects of geometry on fluid loading in a coiled cochlear.* SIAM J. APPL. MATH. VOL. 61, NO. 2, (2002), PP 369-386.

[40] J. Nečas. *Les Méthodes directes en théorie des équations elliptiques.* Paris: Masson, 1967.

[41] S. W. Hansen and E. Zuazua. *Exact controllability and stabilization of a vibrating string with an interior point mass,* SIAM J. Control and optimization, VOL. 33, NO. 5, PP 1357–1391, 1995.

[42] V. Komornik. *Exact controllability and stabilization: The multiplier method*, John Wiley and Sons, New York,1994.

[43] J. Xin. *Dispersive instability and its minimization in time-domain computation of steady- state responses of cochlera models.* J. Acoust. Soc. Am. 115(5), pt. 1, May 2004.

[44] R. G. Kessel and R. H. Kardon. *Tissues and Organs: a text-atlas of scanning electron Microscopy*

[45] R. D. Luce *Sound and Hearing. A conceptual Introduction.* Lawrence Erlbaum Ass., Publishers Hillsdale, New Jersey.

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