**Weak homomorphisms of coalgebras**

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A dissertation submitted to the graduate faculty

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

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2007

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## ABSTRACT

Although coalgebras have primarily been used to model various structures in theoretical computer science, it has been observed that they may also model mathematical structures. For example, there is a natural way to turn graphs into coalgebras obtained from the finite powerset functor. There is also a natural way to turn topological spaces into coalgebras for the filter functor. We observe that when coalgebras are used to model these mathematical structures, standard coalgebraic notion of a homomorphism is too strict. In this thesis, we propose a relaxation of the condition for the definition of a homomorphism and we show that our weak version induces the proper level homomorphisms between mathematical structures.

Based on this appropriate relaxation of the concept of coalgebra homomorphism, we demonstrate the finite completeness and non-cocompleteness of the category of locally finite graphs (including loops) and graph homomorphisms by using existing results from the theory of coalgebras. We also prove the equivalence between the usual category of topological spaces and the category of coalgebras obtained from topological spaces. Because of our relaxation, we gains and loses some aspects of coalgebras properties. We illustrate it by giving an example of complete category of all coalgebras for the powerset functor having a simple construction with our relaxation. We give another example of noncocomplete category of all coalgebras for the powerset functor with our relaxation.

# CHAPTER 1. Introduction

In universal algebra, it sometimes happens that we still get an interesting map even though we relax the condition of homomorphism. Here is an example. Let $I^o$ denote the open unit interval $(0, 1)$. According to [13], an algebra $(B, I^o)$ of type $I^o \to \{2\}$ is a *barycentric algebra* if and only if it satisfies the indentities

1. $xx\underline{p} = x$ (idempotence)
2. $xy\underline{p} = yx\underline{p'}$ (skew-commutativity), and
3. $xy\underline{p}z\underline{q} = xyz\underline{(q/(p'q'))}\ \underline{(p'q')'}$ (skew-associativity),

for $p, q$ in $I^0$, where $\underline{p}$ is a binary operation and $p' = 1 - p$. For example, convex sets $(C, I^0)$ as subalgebras of the reduct $(E, I^0)$ of an affine space $(E, \mathbb{R})$, where $xy\underline{r} = x(1 - r) + yr$, are barycentric algebras. If $f : (C, I^0) \to (\mathbb{R}, I^0)$ is a homomorphism between two barycentric algebras, then

$$(x(1 - r) + yr)f = x^f(1 - r) + y^f r.$$

If we replace the equality by the inequality $\leq$, then we obtain a convex function. Similarly, with the inequality $\geq$, we obtain a concave function.

Join semilattices $(H, I^o) = (H, \vee)$ with $xy\underline{p} = x \vee y$ are other examples of barycentric algebras. If $f : (H_1, \vee) \to (H_2, \vee)$ is a homomorphism between two semilattices, then

$$(x \vee y)f = x^f \vee y^f.$$

If we replace the equality by the inequality $\geq$, then we obtain an order-preserving map, which is still an interesting map.

Now let's turn to coalgebras. Coalgebras have two aspects: as the dual of algebras; and as a common framework for many structures in theoretical computer science, including automata, transition systems, and object oriented systems. As the dual of algebras, it would be natural to expect the relaxation of the homomorphism concept to give nice behavior as in the case of barycentric algebras.

Let us be more specific. Although coalgebras have primarily been used to model various structures in theoretical computer science, it has been observed that they may also model mathematical structures. The main idea is the following: A certain packet of information is associated with each state or element. The structure map of the coalgebra then assigns the packet of information to each state or element of the system being considered. For example, in a locally finite graph $G$, the neighborhood of $x$ is associated with each vertex $x$ of $G$. By using the finite powerset functor, an undirected graph can be turned into a graphic coalgebra where the structure map describes the neighborhood for each vertex. In a topological space $X$, the set of all open sets containing $x$ is associated with each point $x$ of $X$. It is known that a topological space can be modeled as a topological coalgebra using the filter functor [7].

This thesis adopts the coalgebraic point of view for the study of graph theory and topological spaces. Chapter 2 reviews basic notions of category theory and the standard background for coalgebra, including the naive homomorphism concept. The subsequent Chapter 3 introduces coalgebraic view of mathematical structures. In Section 3.1, the powerset functor and the filter functor are introduced. In Section 3.2, we express locally finite graphs allowing loops in coalgebraic language. Also, we observe that the standard coalgebraic notion of a homomorphism is too strict: coalgebra homomorphisms turn out to be full rather than general graph homomorphisms. Because of this strictness, we do not have a simple construction for graph products. In Section 3.3, we express topological spaces as coalgebras for the filter functor. Recalling that naive coalgebra homomorphisms are too strict in this context, since they only yield open continuous maps [7], we observe that there can be no equivalence between the category **Top** of continuous maps and the category **Tp** of naive coalgebra homomorphisms between topological spaces.

Chapter 4 suggests two kinds of relaxations of the condition for the definition of homomorphism. One is called a lower morphism which is a map preserving information. The other is called an upper morphism reflecting the information. The class of all lower (or upper) homomorphisms forms a category in which each coproduct exists. To formalize the key properties of this category, we introduce the concept of a weakly closed class of maps which satisfy what the class of all lower (or upper) homomorphisms do.

Section 4.2 introduces weak coquasivarieties. A weak coquasivarity is an interesting subclass of the category of a weakly closed class since it inherits categorical completeness under an assumption about the existence of surjective-injective factorizations. The category $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$ of all coalgebras for the powerset functor $\mathcal{P}$, with the lower morphisms, forms the topic of Section 4.4. It is known (see [6]) that the category $\mathbf{Set}_{\mathcal{P}}$ of all coalgebras for the powerset functor, with standard homomorphisms, does not have a terminal coalgebra, and so is not complete. The category $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$ is shown to be bicomplete in which every limit and colimit is constructed exactly as in the underlying category of sets. Moreover, the category $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$ has the surjective-injective factorization property and its weak coquasivarities are also complete. Section 4.5 gives an example of noncocomplete category $\underleftarrow{\mathbf{Set}}_{\mathcal{P}}$ of all coalgebras for the powerset functor, with upper morphisms.

With the upper morphisms introduced in Section 4.1, we study the category $\underrightarrow{\mathbf{Set}}_{N}$ of all coalgebras for the finite powerset functor in Section 5.1. The category is shown to be finitely complete (Theorem 5.1.5). Moreover, the category $\underrightarrow{\mathbf{Set}}_{N}$ does have the factorization property required to guarantee that its finite completeness is inherited by its weak coquasivarieties (Proposition 5.1.6). It is shown that the category $\underrightarrow{\mathbf{Set}}_{N}$ is non-complete (Theorem 5.1.8).

Section 5.2 shows that lower morphism between graphic coalgebras has a proper level of power, corresponding an edge-preserving map (Theorem 5.2.2). As an example of the merit of the coalgebraic point of view, we demonstrate the finite completeness (Corollary 5.2.6) of the category $\underrightarrow{\mathbf{Gph}}$ of locally finite graphs including loops by showing that the class $\mathcal{G}$ of graphic coalgebras forms a weak coquasivariety (Proposition 5.2.5). Subsection 5.2.3 shows non-cocompleteness of the category $\underrightarrow{\mathbf{Gph}}$.

The category $\underleftarrow{\mathbf{Set}}_{\mathcal{F}}$ of all coalgebras for the filter functor, with the upper morphisms introduced in Section 4.1, forms the topic of Section 6.1. The category is shown to be complete (Theorem 6.1.5), in contrast to the corresponding category $\mathbf{Set}_{\mathcal{F}}$ of standard homomorphisms (Proposition 3.1.6). Moreover, the category $\underleftarrow{\mathbf{Set}}_{\mathcal{F}}$ of upper morphisms does have the factorization property required to guarantee that its completeness is inherited by its weak coquasivarieties (Proposition 6.1.7).

Section 6.2 shows that upper morphisms are the correct coalgebra homomorphisms for topological coalgebras, coinciding with continuous maps (Theorem 6.2.1). Thus the category $\underleftarrow{\mathbf{Tp}}$ of upper morphisms between topological coalgebras turns out to be bicomplete, since it is equivalent to the usual bicomplete category **Top** of continuous maps between topological spaces (Proposition 6.2.3). Despite the completeness, it transpires in Section 6.3 that the class $\mathcal{T}$ of topological coalgebras does not have the closure properties over $\underleftarrow{\mathbf{Set}}_{\mathcal{F}}$ required for weak coquasivarieties.

For notations used in this thesis, readers are referred to [14]. In particular, mappings are generally placed on the right of their arguments, either in line $xf$ or as a superfix $x^f$. These conventions help to minimize the number of brackets and follow the arrows in diagrams since text is read from left to right.

# CHAPTER 2. Coagebra fundamentals

In this chapter we will introduce the concept of a coalgebra and their basic properties.

## 2.1 Algebras and the concept of a coalgebra

**Definition 2.1.1.** A *type* is a function $\tau : \Omega \to \mathbb{N}$. The domain $\Omega$ of the type $\tau$ is called its *operator domain*, and the elements of $\Omega$ are called *operators*.

**Definition 2.1.2.** Given a type $\tau : \Omega \to \mathbb{N}$, a $\tau$-*algebra* or an *algebra* $(A, \Omega)$ *of type* $\tau$ is defined to be a set $A$ equipped with an operation $\omega : A^{\omega\tau} \to A$ corresponding to each operator $\omega$ of the domain $\Omega$ of $\tau$.

For a given algebra $(A, \Omega)$ of type $\tau$, the operations of $\Omega$ may be combined into a single map

$$\sum_{\omega\in\Omega} \omega : \sum_{\omega\in\Omega} A^{\omega\tau} \to A\,,$$

using the disjoint sum operator. If we write $AF$ instead of $\sum_{\omega\in\Omega} A^{\omega\tau}$, then the algebra $(A, \Omega)$ can be described by a pair $(A, \alpha)$, where $\alpha : AF \to A$ is the map $\sum_{\omega\in\Omega} \omega$. A coalgebra is the dual notion of an algebra $(A, \alpha)$ of type $\tau$. That is, a coalgebra is a pair $(A, \alpha)$, where $A$ is a set and $\alpha : A \to AF$ is a map. This rather blurred notion of a coalgebra can be formalized by category theory.

## 2.2 Categories

This section provides basic notions of category theory needed in the theory of coalgebras. For more details, readers are referred to [10].

**Definition 2.2.1.** A *category* $\mathbf{C}$ is a pair $(\text{Ob}(\mathbf{C}), \text{Mor}(\mathbf{C}))$ where $\text{Ob}(\mathbf{C})$ is a class of *objects* and $\text{Mor}(\mathbf{C}))$ is a class of *morphisms*. There are two functions assigning objects to a morphism: the *domain* $d_0 : \text{Mor}(\mathbf{C}) \to \text{Ob}(\mathbf{C})$ and the *codomain* $d_1 : \text{Mor}(\mathbf{C}) \to \text{Ob}(\mathbf{C})$. The class of all morphisms with domain $x$ and codomain $y$ is denoted by $\mathbf{C}(x, y)$. For each object $a \in \text{Ob}(\mathbf{C})$, there is an *identity morphism* $id_a \in \mathbf{C}(a, a)$. Finally, for any two morphisms $f$ and $g$ with $fd_1 = gd_0$, there is a *composition* $fg \in \mathbf{C}(fd_0, gd_1)$. It satisfies the following two laws:

1. *Associate law* : $\forall x, y, z, t \in \text{Ob}(\mathbf{C})$, $\forall f \in \mathbf{C}(x, y)$, $\forall g \in \mathbf{C}(y, z)$, $\forall h \in \mathbf{C}(z, t)$, $(fg)h = f(gh)$

2. *Identity law* : $\forall x, y \in \text{Ob}(\mathbf{C})$, $\forall f \in \mathbf{C}(x, y)$, $id_x f = f id_y = f$.

The following three examples of category will be used in this thesis.

- **Set**: The category of sets, where $\text{Ob}(\mathbf{Set})$ is the class of all sets, and $\text{Ob}(\mathbf{Set})$ is the class of all mappings between sets.

- **Graph**: The category of graphs, where $\text{Ob}(\mathbf{Graph})$ is the class of all undirected locally finite graphs including loops, and $\text{Mor}(\mathbf{Graph})$ is the class of all edge-preserving maps.

- **Top**: The category of topological spaces, where $\text{Ob}(\mathbf{Top})$ is the class of all topological spaces, and $\text{Mor}(\mathbf{Top})$ is the class of all continuous function.

A morphism $f$ is called an *isomorphism* if it has left and right inverses in $\text{Mor}(\mathbf{C})$.

**Definition 2.2.2.** A category $\mathbf{B}$ is said to be a *subcategory* of the category $\mathbf{C}$ if

1. $\text{Ob}(\mathbf{B}) \subseteq \text{Ob}(\mathbf{C})$ and $\text{Mor}(\mathbf{B}) \subseteq \text{Mor}(\mathbf{C})$;

2. The domain, codomain, and compositions of $\mathbf{B}$ are restrictions of $\mathbf{C}$;

3. Every $\mathbf{B}$-identity is a $\mathbf{C}$-identity.

A subcategory $\mathbf{B}$ of a category $\mathbf{C}$ is said to be a *full subcategory* if for all $x, y \in \text{Ob}(\mathbf{B})$, $\mathbf{B}(x, y) = \mathbf{C}(x, y)$.

**Definition 2.2.3.** Let $\mathbf{D}$ and $\mathbf{C}$ be categories. A *functor* $F : \mathbf{D} \to \mathbf{C}$ consists of two functions, an *object part* $F : \mathrm{Ob}(\mathbf{D}) \to \mathrm{Ob}(\mathbf{C})$ and a *morphism part* $F : \mathrm{Mor}(\mathbf{D}) \to \mathrm{Mor}(\mathbf{C})$, with the following properties:

1. If $f \in \mathbf{D}(x, y)$, then $f^F \in \mathbf{C}(xF, yF)$;
2. $\forall x \in \mathrm{Ob}(\mathbf{D})$, $id_x^F = id_{xF}$;
3. For any composable pair $f, g \in \mathrm{Mor}(\mathbf{D})$, $(fg)^F = f^F g^F$.

A functor $F$ from $\mathbf{C}$ to itself is called an *endofunctor*. Let $\mathbf{C}$ be a category and let $D$ be a *diagram* in $\mathbf{C}$, that is $D$ is a collection $(D_i)_{i\in I}$ of objects and a collection $(f_k)_{k\in K}$ of morphisms between the objects of $(D_i)$.

**Definition 2.2.4.** Given a diagram $D$, a *cone* over $D$ will be a single object $L$ together with morphisms $\pi_i : L \to D_i$ for each $i \in I$, so that for every morphism $f_k : D_i \to D_j$, we have $\pi_i f_k = \pi_j$. A cone $(L, (\pi_i))$ is called the *limit* of $D$ if for every other cone $(L', (\pi'_i))$ over $D$, there is a unique morphism $\tau : L' \to L$ so that $\pi'_i = \tau\pi_i$ for every $i \in I$.

Colimits are defined dually to limits by reversing arrows. To be precise, a *cocone* over $D$ is a single object $S$ together with morphisms $\varepsilon_i : D_i \to S$ for each $i \in I$, so that for every morphism $f_k : D_j \to D_i$, we have $f_k\varepsilon_i = \varepsilon_j$. A cocone $(S, (\varepsilon_i))$ is called the *colimit* of $D$ if for every other cocone $(S', (\varepsilon'_i))$ over $D$, there is a unique morphism $\tau : S \to S'$ so that $\varepsilon'_i = \varepsilon_i\tau$ for every $i \in I$. Some examples of limits and colimits are presented below:

- Let $D$ be a diagram with a class of objects and no morphisms between them. Then, a limit of $D$, if it exists, is called a *product* of $D$. Dually, a colimit of $D$, if it exists, is called a *coproduct* of $D$.
- Let $D$ be a diagram with object set $x, y, z$ and two morphisms $f \in \mathbf{C}(x, z)$ and $g \in \mathbf{C}(y, z)$. Then, a limit of $D$, if it exists, is called a *pullback* of $f$ and $g$. The dual notion is called a *pushout*.

- An object 1 is called a *terminal object* if for every object $X$, there is exactly one morphism from $X$ to 1. Note that a terminal object is the limit of the empty set. The dual notion is called an *initial object*.

**Definition 2.2.5.** A category is called *complete* if every limit exists. Dually, a category is *cocomplete* if every colimit exists. If a category is complete and cocomplete, then the category is said to be *bicomplete*.

**Theorem 2.2.6.** [10] A given category $\mathbf{C}$ is complete if and only if $\mathbf{C}$ has products and pullbacks. Dually, $\mathbf{C}$ is cocomplete if and only if $\mathbf{C}$ has coproducts and pushouts.

**Definition 2.2.7.** If a given diagram $D$ is finite, i.e. $D$ is a collection of finite number of objects and finite number of morphisms, then the limit of $D$ is called a *finite limit*. A category is called *finitely complete* if every finite limit exists.

**Theorem 2.2.8.** [10] A given category $\mathbf{C}$ is finitely complete if and only if $\mathbf{C}$ has finite products and pullbacks.

## 2.3 Coalgebras and their properties

Let $F : \mathbf{Set} \to \mathbf{Set}$ be an endofunctor.

**Definition 2.3.1.** An $F$*-coalgebra* is a pair $(X, \alpha)$ consisting of a set $X$ and a map $\alpha : X \to XF$. $X$ is called the *base set* (or *state set*) and $\alpha$ is the *structure map* on $X$.

$$\begin{array}{c} X \\ \alpha \big\downarrow \\ XF \end{array}$$

**Definition 2.3.2.** Let $(X, \alpha)$ and $(Y, \beta)$ be $F$-coalgebras. An $F$*-homomorphism* from $(X, \alpha)$ to $(Y, \beta)$ is a map $f : X \to Y$ for which the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \alpha \big\downarrow & & \big\downarrow \beta \\ XF & \xrightarrow[f^F]{} & YF \end{array}$$

It is known that the class of all $F$-homomorphisms forms a category which we shall denote by $\mathbf{Set}_F$.

**Definition 2.3.3.** An $F$-coalgebra $(S, \alpha_S)$ is called a *subcoalgebra* (or *substructure*) of $(X, \alpha_X)$ if $S \subseteq X$ and the canonical inclusion map $\iota : S \hookrightarrow X$ is an $F$-homomorphism. We write

$$(S, \alpha_S) \leq (X, \alpha_X)$$

if $(S, \alpha_S)$ is a subcoalgebra of $(X, \alpha_X)$.

The functor $U : \mathbf{Set}_F \to \mathbf{Set}$, defined by $(X, \alpha)U = X$ and $f^U = f$ for $(X, \alpha) \in \mathrm{Ob}(\mathbf{Set}_F)$ and $f \in \mathrm{Mor}(\mathbf{Set}_F)$, is called the *underlying set functor*. We say that the *limit of a diagram $D$ is preserved by the underlying set functor* if $((L, \alpha)U, (\pi_i^U))$ is a limit of $DU$ in $\mathbf{Set}$, where $D$ is a diagram in $\mathbf{Set}_F$ and $((L, \alpha), (\pi_i))$ is a limit of $D$ in $\mathbf{Set}_F$.

**Theorem 2.3.4.** [1] Every colimit exists in $\mathbf{Set}_F$ and is preserved by the underlying set functor.

An $F$-coalgebra is called a *terminal coalgebra* if it is a terminal object in $\mathbf{Set}_F$.

**Theorem 2.3.5.** [6] If $(P, \pi)$ is a terminal coalgebra, then the structure map $\pi$ is an isomorphism in $\mathbf{Set}_F$.

A coalgebra $(Y, \alpha_Y)$ is called an *$F$-homomorphic image of a coalgebra* $(X, \alpha_X)$ if there exists a surjective $F$-homomorphism $f : X \to Y$.

**Definition 2.3.6.** Let $\mathcal{K}$ be a class of $F$-coalgebras. We define the following classes:

1. $\mathsf{H}(\mathcal{K})$ : the class of all $F$-homomorphic images of objects from $\mathcal{K}$,
2. $\mathsf{S}(\mathcal{K})$ : the class of all $F$-coalgebras which are isomorphic to subcoalgebras of objects from $\mathcal{K}$,
3. $\Sigma(\mathcal{K})$ : the class of all $F$-coalgebras which are isomorphic to coproducts of objects from $\mathcal{K}$.

A class $\mathcal{K}$ is called *closed* under $\mathsf{H}$, $\mathsf{S}$, or $\Sigma$, provided that $\mathsf{H}(\mathcal{K}) \subseteq \mathcal{K}$, $\mathsf{S}(\mathcal{K}) \subseteq \mathcal{K}$, or $\Sigma(\mathcal{K}) \subseteq \mathcal{K}$.

**Definition 2.3.7.** A *co-variety* is a class $\mathcal{K}$ of coalgebras which is closed under $\mathsf{H}$, $\mathsf{S}$, and $\Sigma$. A *co-quasivariety* is a class closed under $\mathsf{H}$ and $\Sigma$.

**Proposition 2.3.8.** [8] If $\mathbf{Set}_F$ is complete, then so is every co-quasivariety of $\mathbf{Set}_F$.

**Definition 2.3.9.** $F$ is called *bounded* if there is a cardinality $\kappa$ so that for each $F$-coalgebra $(X, \alpha)$ and any $x \in X$, there exists a subcoalgebra $(U_x, \beta) \leq (X, \alpha)$ of cardinality at most $\kappa$ with $x \in U_x$.

**Theorem 2.3.10.** [6], [8] If the functor $F$ is bounded, then $\mathbf{Set}_F$ is complete.

# CHAPTER 3. Coalgebraic view of mathematical sturctures

This chapter introduces the coalgebraic point of view for the study of graph theory and topological spaces. In each case, it will be shown that naive coalgebra homomorphisms are too strict.

## 3.1 The powerset functor and the filter functor

### 3.1.1 The powerset functor

In this thesis, we denote the covariant powerset functor on the category **Set** by $\mathcal{P}$. It is defined as follows:

1. For a given set $X$, the set $X\mathcal{P}$ is the set of all subsets of $X$.

2. For a given map $f \in \mathbf{Set}(X, Y)$, the map $f^{\mathcal{P}} \in \mathbf{Set}(X\mathcal{P}, Y\mathcal{P})$ is given by $Uf^{\mathcal{P}} = Uf$ for $U \in X\mathcal{P}$.

The finite powerset functor, denoted by $N$, is defined in the same manner except that $XN$ is the set of all finite subsets of $X$. The finite powerset functor is very interesting because $N$ is bounded and an $N$-coalgebra is just an image-finite transition system, where from every state there are only finitely many possible transitions into the next state [8]. Furthermore, for a given locally finite graph, each vertex determines a finite set whose elements are adjacent to the vertex. So we can turn a given locally finite graph into an $N$-coalgebra.

**Proposition 3.1.1.** The category $\mathbf{Set}_N$ is bicomplete.

### 3.1.2 The filter functor

**Definition 3.1.2.** [7] Let $X$ be a set. A collection $H$ of subsets of $X$ is called *downward directed* if for $U$, $V \in H$, there always exists $W \in H$ with $W \subseteq U \cap V$. A nonempty downward directed collection $H$ is called a *filter* on $X$ if $V \supseteq U \in H$ always implies $V \in H$.

Given a nonempty downward directed set $H \subseteq 2^X$,

$$\uparrow H = \{U \subseteq X \mid \exists U' \in H \text{ such that } U' \subseteq U\}$$

is the filter generated by $H$. We denote the set of all filters on $X$ by $X\mathcal{F}$. The assignment $\mathcal{F}$ may be made into a functor $\mathcal{F} : \mathbf{Set} \to \mathbf{Set}$, called the *filter functor*, by defining it on a map $f : X \to Y$ as $Hf^{\mathcal{F}} := \ \uparrow (Hf)$, where $H$ stands for an arbitrary filter on $X$ [6], [7].

**Proposition 3.1.3.** [7] A map $f : (X, \alpha) \to (Y, \beta)$ is an $\mathcal{F}$-homomorphism if and only if for all $x \in X$ and all $V \subseteq Y$,

$$V \in xf\beta \Leftrightarrow f^{-1}(V) \in x\alpha\,.$$

**Proposition 3.1.4.** [7] A subset $S \subseteq X$ is an $\mathcal{F}$-subcoalgebra of $(X, \alpha)$ if and only if $S \in s\alpha$ for each $s \in S$.

**Lemma 3.1.5.** For a given set $X$, there is no bijection $\alpha : X \to X\mathcal{F}$.

*Proof.* When $X$ is empty, this is easy to prove. Assume that $X$ is not empty. Then for distinct $A$, $B \in X\mathcal{P}$,

$$\uparrow \{A\} \neq \uparrow \{B\}.$$

Therefore, there are at least $2^{|X|}$ filters. By Cantor's Theorem, there is no bijection $\alpha : X \to X\mathcal{F}$. □

By Lemma 3.1.5 and Theorem 2.3.5, we obtain the following.

**Proposition 3.1.6.** The category $\mathbf{Set}_{\mathcal{F}}$ is not complete. In particular, there is no final $\mathcal{F}$-coalgebra.

## 3.2 Graphic coalgebras with homomorphisms

In this section, we will see how to turn undirected locally finite graphs into coalgebras obtained from the finite powerset functor. We also show that the standard coalgebraic notion of a homomorphism is too strict.

### 3.2.1 Graphic coalgebras

**Definition 3.2.1.** A $N$-coalgebra $(X, \alpha)$ is called *graphic* if

$$\forall x, y \in X,\ \ x \in y\alpha \text{ iff } y \in x\alpha.$$

We denote the class of all graphic $N$-coalgebras by $\mathcal{G}$. We can easily see that a graphic $N$-coalgebra $(X, \alpha)$ induces an undirected locally finite graph $G_{(X,\alpha)}$ including loops such that $V(G_{(X,\alpha)}) = X$ and for given $x \in X$, the set of neighbors of $x$ is $x\alpha$. Also, an undirected locally finite graph can be turned into the graphic $N$-coalgebra where the structure map is defined by the set of neighborhood for each point. In this sense, we may call a graphic $N$-coalgebra a graph.

**Lemma 3.2.2.** An $N$-homomorphism preserves loops.

*Proof.* Let $f : (X, \alpha) \to (Y, \beta)$ be an $N$-homomorphism on $\mathcal{G}$. If $x \in X$ has a loop, then $x \in x\alpha$. Since $x\alpha f^N = xf\beta$, $xf \in xf\beta$. Therefore $xf$ has a loop. □

From the above Lemma 3.2.2, the preimage of a $N$-coalgebra with no loops in $\mathcal{G}$ under $N$-homomorphisms has no loops. Let $\varphi : X \to Y$ be a map. Then $\varphi = \widetilde{\varphi}\iota$, where $\widetilde{\varphi} : X \to X\varphi$ is the corestriction of $\varphi$ to its image defined by $x\widetilde{\varphi} = x\varphi$ for $x \in X$, and $\iota : X\varphi \hookrightarrow Y$ is the natural inclusion. According to [6], every $\mathcal{P}$-homomorphism $\varphi : (X, \alpha) \to (Y, \beta)$ in $\mathbf{Set}_{\mathcal{P}}$ has a unique coalgebra structure $\gamma$ on $X\varphi$ so that both $\widetilde{\varphi}$ and $\iota$ are $\mathcal{P}$-homomorphisms. Indeed, for a given $x\varphi \in X\varphi$, $x\varphi\gamma$ is defined by $x\varphi\beta$. The coalgebra $(X\varphi, \gamma)$ is called the *image of* $\varphi$.

**Proposition 3.2.3.** Let $f$ be an $N$-homomorphism from $(X, \alpha)$ to $(Y, \beta)$ on $\mathcal{G}$. Then $f$ is a full graph homomorphism for induced graphs, i.e. $f$ preserves edges and every edge in the image is induced by some edge in the preimage.

*Proof.* If $x' \in x\alpha$, then $x'f \in xf\beta$. So $f$ preserves edges. Let $y' \in xf\beta$ where $x \in X$. Since $x\alpha f^N = xf\beta$, there is a vertex $x' \in x\alpha$ such that $x'f^N = y'$. Hence $f$ is a full graph homomorphism. □

Although $N$-coalgebras give a natural way to express graphs, the concept of $N$-homomorphism is too strict since we usually define graph homomorphisms as edge-preserving maps. Because of this strictness, we obtain the following property which does not hold within usual edge-preserving maps.

**Proposition 3.2.4.** Let $(X, \alpha) \in \mathcal{G}$ be connected and $(Y, \beta) \in \mathcal{G}$. Let $f : X \to Y$ be an $N$-homomorphism. Then $(Xf, \beta|_{Xf})$ is a connected component of $(Y, \beta)$, i.e. $f$ preserves connected components. In particular, if $(Y, \beta)$ is connected, then $f$ is surjective.

We denote the graphic $N$-coalgebra whose graph structure is

$$0 \text{ --- } 1$$

by $2 = (\{0, 1\}, \alpha_2)$. By combining Lemma 3.2.2, Lemma 3.2.3, and Proposition 3.2.4, we obtain the following result.

**Proposition 3.2.5.** Let $(X, \alpha) \in \mathcal{G}$ and let $f : (X, \alpha) \to 2$ be an $N$-homomorphism. Then $(X, \alpha)$ is a nontrivial bipartite graph.

*Proof.* By Proposition 3.2.4, $f$ is surjective. By Lemmas 3.2.2 and 3.2.3, $(X, \alpha)$ is nontrivial and has no loop. Now, it is enough to show that $(X, \alpha)$ has no odd cycles. Suppose that it has an odd cycle with distinct vertices $(v_1, \ldots, v_{2n+1})$. Without loss of generality, we may assume that $v_1 f = 1$. Since $\{v_2, v_{2n+1}\} \subseteq v_1\alpha$, $\{v_2, v_{2n+1}\}f^N \subseteq v_1\alpha f^N = v_1 f \alpha_2 = \{0\}$. So, $v_2 f = v_{2n+1} f = 0$. By continuing this process, we have $v_{n+1} f = v_{n+2} f$. However, since $v_{n+1} \in v_{n+2}\alpha$, $v_{n+1} f \neq v_{n+2} f$ which is a contradiction. Therefore, $(X, \alpha)$ is a bipartite graph. □

### 3.2.2 Completeness

We denote the full subcategory of $\mathbf{Set}_N$ with the object class $\mathcal{G}$ by **Gph**. In this subsection, we show the completeness of **Gph**. Note that $\mathbf{Set}_N$ is complete by Theorem 2.3.10 since $N$

is bounded by $\omega$. In order to show that **Gph** is complete, it is enough to prove that $\mathcal{G}$ is a co-quasivariety by Proposition 2.3.8, without mentioning the existence of limits. This is one of the advantages of the coalgebraic point of view.

**Proposition 3.2.6.** $\mathcal{G}$ forms a co-variety.

*Proof.* (i) $\mathcal{G}$ is closed under $\mathsf{H}$;

Let $f : (X, \alpha) \to (Y, \beta)$ be a surjective $N$-homomorphism where $(X, \alpha) \in \mathcal{G}$. We need to show that $(Y, \beta)$ is graphic. For given $a'$, $b' \in Y$, suppose that $a' \in b'\beta$. Since $f$ is surjective, $\exists b \in X$ such that $bf = b'$. Since $f$ is an $N$-homomorphism, $b\alpha f^N = b'\beta$. Since $a' \in b'\beta = b\alpha f^N$, $\exists a \in b\alpha$ such that $af = a'$. Since $af = a'$, $a\alpha f^N = a'\beta$. Since $(X, \alpha) \in \mathcal{G}$ and $a \in b\alpha$, $b \in a\alpha$. Therefore $bf = b' \in a'\beta$. Hence $(Y, \beta) \in \mathcal{G}$.

(ii) $\mathcal{G}$ is closed under $\mathsf{S}$;

Let $(A, \alpha)$ be a subcoalgebra of $(B, \beta)$ with the natural inclusion map $\iota : A \to B$ and assume that $(B, \beta) \in \mathcal{G}$. For each $a \in A$,

$$a\alpha = a\alpha\iota^N = a\iota\beta = a\beta. \tag{3.1}$$

Now, for given $a$, $b \in A$, assume that $a \in b\alpha$. By the above (3.1), $a \in b\beta$. Since $(B, \beta) \in \mathcal{G}$, $b \in a\beta$. Again, by (3.1), $b \in a\alpha$. Therefore $(A, \alpha) \in \mathcal{G}$.

(iii) $\mathcal{G}$ is closed under $\Sigma$;

Let $(X, \alpha)$ be the sum of $\{(X_i, \alpha_i)\}_{i \in I}$, where for each $i \in I$, $(X_i, \alpha_i) \in \mathcal{G}$. Note that $X$ is the disjoint union of $\{X_i\}_{i \in I}$ with the insertion maps $\iota_i$'s. For given $x$, $y \in X$, if $x \in y\alpha$, then $\exists i \in I$ such that $y \in X_i$. Since $y\alpha = y\alpha_i$, $x \in X_i$. Therefore $y \in x\alpha$. Hence $(X, \alpha) \in \mathcal{G}$.

□

**Corollary 3.2.7.** **Gph** is complete.

Since $\mathbf{Set}_N$ is complete, every product exists. However, it seems there is no simple construction for the products of $N$-coalgebras. For example, let $C_2$ be the graphic $N$-coalgebra whose corresponding graph is the complete graph on $\{0, 1\}$. Then, the product $C_2 \times C_2$ in $\mathbf{Set}_N$ is infinite [8]. What if we restrict our concern to the subcategory **Gph**? Since **Gph** is complete, every product exists. However, for given graphic $N$-coalgebras $(X, \alpha)$ and $(Y, \beta)$, the product of them in **Gph** could be different from the product in $\mathbf{Set}_N$. We show that the product $C_2 \times C_2$ in **Gph** is infinite and there is still no simple construction for the products.

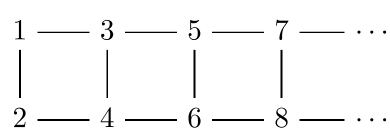
**Lemma 3.2.8.** Let $(P, \gamma)$ denote the product $C_2 \times C_2$ in **Gph** with homomorphisms $\eta_i : (P, \gamma) \to C_2$, $i = 1, 2$. Let $(G, \beta)$ be a graphic coalgebra. Let $f, g : (G, \beta) \to C_2$ be two $N$-homomorphisms. Let $(f, g) : (G, \beta) \to (P, \gamma)$ denote the unique $N$-homomorphism so that $(f, g)\eta_1 = f$ and $(f, g)\eta_2 = g$. Then, for given $a, b \in G$, if $af \neq bf$ or $ag \neq bg$, then $a(f, g) \neq b(f, g)$.

*Proof.* Suppose $a(f, g) = b(f, g)$. Then, $af = a(f, g)\eta_1 = b(f, g)\eta_1 = bf$. Similarly, $ag = bg$. This is a contradiction to our assumption. □

By Lemma 3.2.8, $af = bf$ and $ag = bg$ is necessary for $a(f, g) = b(f, g)$.

**Theorem 3.2.9.** The product $C_2 \times C_2$ in **Gph** is infinite.

*Proof.* Let $(G, \beta) \in \mathsf{G}$ be the graphic coalgebra whose graph structure is the infinite ladder graph displayed as follows:



Define $f : G \to \{0, 1\}$ by

$$nf = \begin{cases} 0, & \text{if } n \equiv 1 \text{ or } 2 \pmod 4; \\ 1, & \text{otherwise.} \end{cases}$$

Define $g : G \to \{0, 1\}$ by

$$ng = \begin{cases} 0, & \text{if } n \equiv 0, 1, 3 \text{ or } 6 \pmod 8; \\ 1, & \text{otherwise.} \end{cases}$$

Then it can readily be seen that $f$ and $g$ are $N$-homomorphisms from $(G, \beta)$ to $C_2$. So there is a unique $N$-homomorphism $(f, g) : (G, \beta) \to (P, \gamma)$ such that $(f, g)\eta_1 = f$ and $(f, g)\eta_2 = g$.

We will show that $(f,g)$ is injective. By Lemma 3.2.8, it is possible *a priori* that $2(f,g) = 5(f,g)$. Since $3f \neq 6f$, $6f \neq 7f$, and $3g \neq 7g$, $|5\beta(f,g)^N| = |\{3,6,7\}(f,g)^N| = 3$. Similarly, $|2\beta(f,g)^N| = 2$. Since $5\beta(f,g)^N = 5(f,g)\gamma$ and $2\beta(f,g) = 2(f,g)\gamma$, $2(f,g) \neq 5(f,g)$. Similarly, $1(f,g) \neq 6(f,g)$. Now assume that $n(f,g) = m(f,g)$ for some $n$ and $m$ with $1 < n < m$. We assume that $n$ is odd. By the definition of $f$ and $g$,

$$|n(f,g)\gamma| = |n\beta(f,g)^N| = |\{n-2, n+1, n+2\}(f,g)^N| = 3.$$

Also, $|m\beta(f,g)^N| = 3$. Since $(n-2)f = (m-2)f$ and $(n-2)g = (m-2)g$, we must have $(n-2)(f,g) = (m-2)(f,g)$. Therefore $1(f,g) = m'(f,g)$ for some $m' > 1$. However,

$$|1\beta(f,g)^N| = |1(f,g)\gamma| = 2 \neq 3 = |m'(f,g)\gamma| = |m'\beta(f,g)^N|.$$

This is a contradiction to $1(f,g) = m'(f,g)$. The case where $n$ is even can be treated similarly. Therefore $(f,g)$ is injective and $C_2 \times C_2$ in **Gph** is infinite. □

### 3.3 Topological coalgebras with homomorphisms

There is a natural way to turn topological spaces into coalgebras for the filter functor. However, naive coalgebra homomorphisms correspond to open continuous maps. Let $(X,\tau)$ be a topological space. For a given $x \in X$, let $U_\tau(x)$ denote the filter generated by the set of all open sets containing $x$.

**Definition 3.3.1.** [7] An $\mathcal{F}$-coalgebra $(X,\alpha)$ is called *topological* if there exists a topology $\tau$ on $X$ so that for all $x \in X$, $x\alpha = U_\tau(x)$.

A given topological space $(X,\tau)$ yields a topological $\mathcal{F}$-coalgebra $(X,U_\tau)$. We denote the class of all topological $\mathcal{F}$-coalgebras by $\mathcal{T}$, and write **Tp** for the full subcategory of $\mathbf{Set}_{\mathcal{F}}$ with object class $\mathcal{T}$. Let **Top** denote the (usual) category of topological spaces, where the object class is the class of all topological spaces, and the morphism class is the class of all continuous maps.

By Proposition 3.1.3 and 3.1.4, we obtain the following proposition.

**Proposition 3.3.2.** [7] Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be topological spaces. Then a map $f : (X, U_{\tau_X}) \to (Y, U_{\tau_Y})$ is an $\mathcal{F}$-homomorphism if and only if $f$ is continuous and open. A subset $S$ of $X$ is a subcoalgebra if and only if it is open.

**Proposition 3.3.3.** [7] An $\mathcal{F}$-coalgebra $(X, \alpha)$ is topological if and only if for every $x \in X$ and $U \subseteq X$ we have

$$U \in x\alpha \Rightarrow \exists S \leq X \text{ such that } x \in S \subseteq U\,.$$

**Proposition 3.3.4.** [7] $\mathcal{T}$ forms a co-variety over $\mathbf{Set}_{\mathcal{F}}$.

Although $\mathcal{F}$-coalgebras give a natural way to express topological spaces, the concept of $\mathcal{F}$-homomorphism is too strict, since we usually define homomorphisms of topological spaces as continuous maps. Indeed, we have the following non-equivalence between **Tp** and **Top**.

**Proposition 3.3.5.** There is no equivalence $G : \mathbf{Tp} \to \mathbf{Top}$.

*Proof.* Let $(X = \{*\}, \tau_X)$ be a topological space with $\tau_X = \{\varnothing, X\}$, and let $(Y, \tau_Y)$ be a topological space with $|Y| \geq 2$ and $\tau_Y = \{\varnothing, Y\}$. Proposition 3.3.2 gives $\mathbf{Tp}((X, U_{\tau_X}), (Y, U_{\tau_Y})) = \varnothing$. Now suppose that there is an equivalence $G : \mathbf{Tp} \to \mathbf{Top}$. Then $G$ is full, faithful, and dense. If $|\mathbf{Top}((A, \tau_A), (B, \tau_B))| = 0$ for some topological spaces $(A, \tau_A)$ and $(B, \tau_B)$, then $A \neq \varnothing$ and $B = \varnothing$. Therefore $(Y, U_{\tau_Y})G = (\varnothing, \{\varnothing\})$ for any topological space $(Y, \tau_Y)$ with $|Y| \geq 2$ and $\tau_Y = \{\varnothing, Y\}$. Now let $(Y_1 = \{1, 2\}, \tau_{Y_1})$ be a topological space with $\tau_{Y_1} = \{\varnothing, Y_1\}$, and let $(Y_2 = \{a, b, c, d\}, \tau_{Y_2})$ be a topological space with $\tau_{Y_2} = \{\varnothing, Y_2\}$. Then $|\mathbf{Tp}((Y_1, U_{\tau_{Y_1}}), (Y_2, U_{\tau_{Y_2}}))| = 0$ but $|\mathbf{Top}((Y_1, U_{\tau_{Y_1}})G, (Y_2, U_{\tau_{Y_2}})G)| = 1$, which is a contradiction. Therefore there is no equivalence $G : \mathbf{Tp} \to \mathbf{Top}$. □

# CHAPTER 4. Weak homomorphisms

## 4.1 Weak homomorphisms of coalgebras

Let $T : \mathbf{Set} \to \mathbf{Set}$ be an endofunctor on the category of sets, such that for a given set $X$, the image $XT$ is a set of sets. Suppose further that for a given map $f : X \to Y$ and for given $A, B \in XT$, the containment $A \subseteq B$ implies $Af^T \subseteq Bf^T$. Then the endofunctor $T$ is described as *monotonic*. For example, $T$ could be the covariant powerset functor or the filter functor. Note that monotonic endofunctors are examples of "functors with order" as described in [12].

**Definition 4.1.1.** Let $(X, \alpha)$ and $(Y, \beta)$ be $T$-coalgebras for a monotonic endofunctor $T$. Then a *lower $T$-morphism* from $(X, \alpha)$ to $(Y, \beta)$ is a map $f : X \to Y$ such that for each $x \in X$, we have the inclusion $x\alpha f^T \subseteq xf\beta$. Similarly, an *upper $T$-morphism* from $(X, \alpha)$ to $(Y, \beta)$ is a map $f : X \to Y$ such that for each $x \in X$, the inclusion $x\alpha f^T \supseteq xf\beta$ holds.

In transition systems, if one wants maps only preserve transitions, then we need a lower $T$-morphism concept. Indeed, the idea of lower $T$-morphisms between transition systems was introduced in [5]. If we recall that a structure map of a coalgebra assigns the packet of information to each state of the system being considered, then we may regard a lower $T$-morphism as a map preserving the information. Likewise, an upper $T$-morphism could be considered as a map reflecting the information. This way of interpretation makes us to expect that edge-preserving maps might be described as lower morphisms and continuous maps between topological spaces might be described as upper morphisms. Indeed, it is shown that lower morphisms between graphic coalgebras agree with the edge-preserving maps of graphs in Chapter 5. Likewise, upper morphisms between topological coalgebras agree with the correct homomorphisms of topological spaces, namely continuous maps, see Chapter 6.

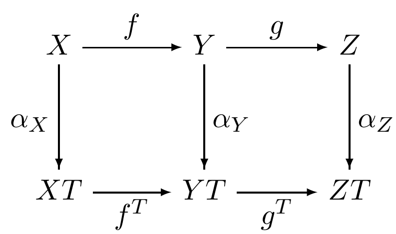
Since $T$ is a functor, the identity map is always a lower and upper $T$-morphism.

**Lemma 4.1.2.** The composition of two lower (resp. upper) $T$-morphisms is again a lower (resp. upper) $T$-morphism.

*Proof.* Let $(X, \alpha_X)$, $(Y, \alpha_Y)$, and $(Z, \alpha_Z)$ be $T$-coalgebras. Let $f : X \to Y$ and $g : Y \to Z$ be lower $T$-morphisms. For given $x \in X$, since $f$ is a lower $T$-morphism, $x\alpha_X f^T \subseteq xf\alpha_Y$. Since $g$ is a lower $T$-morphism, $xf\alpha_Y g^T \subseteq xfg\alpha_Z$. So

$$x\alpha_X(fg)^T = x\alpha_X f^T g^T \subseteq xf\alpha_Y g^T \subseteq xfg\alpha_Z.$$

Therefore $fg$ is a lower $T$-morphism.

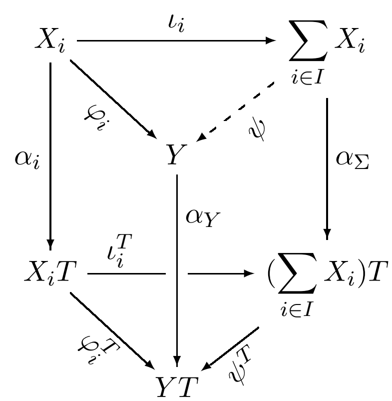



For the case of upper morphisms, we may just change the direction of the inclusions in the above proof. □

From the previous observation and the above Lemma 4.1.2, the class of all $T$-coalgebras forms a category denoted by $\underrightarrow{\mathbf{Set}}_T$ (resp. $\underleftarrow{\mathbf{Set}}_T$) with lower (resp. upper) morphisms.

**Proposition 4.1.3.** For every family $(X_i, \alpha_i)_{i \in I}$ of $T$-coalgebras, there exists a sum $\sum_{i \in I}(X_i, \alpha_i)$ in $\underrightarrow{\mathbf{Set}}_T$ (resp. $\underleftarrow{\mathbf{Set}}_T$). The sum is preserved by the underlying set functor and its structure map $\alpha_\Sigma$ is given by $x_i\alpha_\Sigma = x_i\alpha_i\iota_i^T$ for $x_i \in X_i$, where $\iota_i : X_i \to \sum X_i$ is the insertion map.

*Proof.* Let $(Y, \alpha_Y)$ be a $T$-coalgebra and let $\varphi_i : X_i \to Y$ be a lower $T$-morphism. Then there is a unique map $\psi : \sum X_i \to Y$ in **Set** with $\iota_i \psi = \varphi_i$.



Since $\varphi_i^T = \iota_i^T \psi^T$,

$$\alpha_i \varphi_i^T = \alpha_i \iota_i^T \psi^T = \iota_i \alpha_\Sigma \psi^T.$$

Since $\varphi_i$ is a lower morphism, for given $x \in X_i$,

$$x\alpha_i \varphi_i^T \subseteq x\varphi_i \alpha_Y = x\iota_i \psi \alpha_Y.$$

So, for given $i \in I$ and $x \in X_i$, $x\iota_i \alpha_\Sigma \psi^T \subseteq x\iota_i \psi \alpha_Y$. Therefore $\psi$ is a lower $T$-morphism. For the case of upper morphisms, we may just change the direction of the inclusions in the above proof. □

Note that each insertion map $\iota_i$ is a $T$-homomorphism. To formalize the key properties of the categories $\underrightarrow{\mathbf{Set}}_T$ and $\underleftarrow{\mathbf{Set}}_T$ (and other, similar categories), we introduce the following concept.

**Definition 4.1.4.** Let $F$ be an endofunctor on the category of sets. A class $\mathcal{W}$ of maps between the underlying sets of $F$-coalgebras is said to be *weakly closed* if the following conditions are satisfied:

(a) $\mathcal{W}$ contains the class of all $F$-homomorphisms;

(b) $\mathcal{W}$ forms a category in which each coproduct exists;

(c) Each coproduct in $\mathcal{W}$ is preserved by the underlying set functor.

If $\mathcal{W}$ is weakly closed, then a member of $\mathcal{W}$ is called a *weak $F$-homomorphism*. The corresponding category is called a *weak category of $F$-coalgebras*, and denoted by $\mathbf{W}$.

Assume that we have a weak category $\mathbf{W}$. Then *subcoalgebras*, *covarieties* and *coquasivarieties* under weak $F$-homomorphisms are defined as in Section 2.3 simply replacing $F$-homomorphisms with weak $F$-homomorphisms. We write

$$(S, \alpha_S) \leq_w (X, \alpha_X)$$

if $(S, \alpha_S)$ is a subcoalgebra of $(X, \alpha_X)$ over $\mathbf{W}$.

**Theorem 4.1.5.** For a monotonic endofunctor $T$ on the category of sets, the class of all lower (resp. upper) $T$-morphisms is weakly closed.

## 4.2 Weak coquasivarieties

Let $\mathbf{W}$ be a weak category of $F$-coalgebras and let $\mathcal{K}$ be a subclass of $F$-coalgebras. We denote the full subcategory of $\mathbf{W}$ with the object class $\mathcal{K}$ by $\mathbf{K}$. In particualr, we denote the full subcategory of $\underrightarrow{\mathbf{Set}}_T$ (resp. $\underleftarrow{\mathbf{Set}}_T$) with the object class $\mathcal{K}$ by $\underrightarrow{\mathbf{K}}$ (resp. $\underleftarrow{\mathbf{K}}$).

**Definition 4.2.1.** Let $\mathbf{W}$ be a weak category of $F$-coalgebras and let $\mathcal{K}$ be a subclass of $F$-coalgebras. A *weak coquasivariety of* $\mathbf{K}$ is a subclass $\mathcal{L}$ of $\mathcal{K}$ closed under $\Sigma$ over $\mathbf{K}$, and such that for a given surjective morphism $f : (X, \alpha_X) \twoheadrightarrow (Y, \alpha_Y)$ over $\mathbf{K}$ with $(X, \alpha_X) \in \mathcal{L}$, there is a structure map $\alpha$ on $Y$ with the following properties:

(a) $(Y, \alpha) \in \mathcal{L}$ ;

(b) $(Y, \alpha) \leq_w (Y, \alpha_Y)$ ;

(c) $f : (X, \alpha_X) \to (Y, \alpha)$ is a morphism over $\mathbf{K}$.

Note that a coquasivariety is a weak coquasivariety. In Section 5.2.2, it is shown that $\mathcal{G}$ is not closed under lower $\mathcal{P}$-morphic images but $\mathcal{G}$ forms a weak coquasivariety in the category $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$.

Let $\varphi : X \to Y$ be a map. We say that $\varphi$ *$SI$-factors through* $Z$ if there is a surjective map $f : X \twoheadrightarrow Z$ and an injective map $g : Z \rightarrowtail Y$ such that $\varphi = fg$. If $\varphi$ $SI$-factors through $Z$, then $\varphi$ is also said to be *$SI$-factorizable by* $Z$. One natural way to $SI$-factorize $\varphi$ is to take $\varphi = \widetilde{\varphi}\iota$, where $\widetilde{\varphi}$ is the corestriction of $\varphi$ to its image, and $\iota$ is the natural inclusion.

**Definition 4.2.2.** Let $\mathbf{W}$ be a weak category of $F$-coalgebras and let $\mathcal{K}$ be a subclass of $F$-coalgebras. Let $\varphi : (X, \alpha_X) \to (Y, \alpha_Y)$ be a weak $F$-homomorphism over $\mathbf{K}$. Then $\varphi$ is said to be *weakly* $SI$*-factorizable over* $\mathbf{K}$ if for any set $Z$, and for a given $SI$-factorization $f : X \twoheadrightarrow Z$ and $g : Z \rightarrowtail Y$ with $fg = \varphi$, there is a structure map $\alpha$ on $Z$ such that $(Z, \alpha) \in \mathcal{K}$ and both $f : (X, \alpha_X) \twoheadrightarrow (Z, \alpha)$ and $g : (Z, \alpha) \rightarrowtail (Y, \alpha_Y)$ are weak $F$-homomorphisms over $\mathbf{K}$. The full subcategory $\mathbf{K}$ is called *weakly* $SI$*-factorizable* if every weak $F$-homomorphism over $\mathbf{K}$ is weakly $SI$-factorizable over $\mathbf{K}$.

Suppose that $\mathbf{W}$ is a weakly $SI$-factorizable weak category. Then $\mathbf{W}$ is a weak coquasivariety of itself.

**Lemma 4.2.3.** Let $\mathbf{W}$ be a weak category of $F$-coalgebras and let $\mathcal{K}$ be a subclass of $F$-coalgebras such that $\mathbf{K}$ is weakly $SI$-factorizable. Let $\mathcal{L}$ be a weak coquasivariety of $\mathbf{K}$. Let $f : (X, \alpha_X) \to (Y, \alpha_Y)$ be a weak homomorphism over $\mathbf{K}$ with $(X, \alpha_X) \in \mathcal{L}$. Then there is a structure map $\alpha$ on $Xf$ such that $\widetilde{f} : (X, \alpha_X) \to (Xf, \alpha)$ is a weak homomorphism over $\mathbf{K}$, $(Xf, \alpha) \in \mathcal{L}$, and $(Xf, \alpha) \leq_w (Y, \alpha_Y)$.

*Proof.* Since $\mathbf{K}$ is weakly $SI$-factorizable, there is a structure map $\alpha'$ on $Xf$ such that $(Xf, \alpha') \in \mathcal{K}$, $\widetilde{f} : (X, \alpha_X) \twoheadrightarrow (Xf, \alpha')$ is a weak homomorphism over $\mathbf{K}$, and $(Xf, \alpha') \leq_w (Y, \alpha_Y)$. Since $\mathcal{L}$ is a weak coquasivariety of $\mathbf{K}$, there is a structure map $\alpha$ on $Xf$ such that $\widetilde{f} : (X, \alpha_X) \to (Xf, \alpha)$ is a weak homomorphism over $\mathbf{K}$, $(Xf, \alpha) \in \mathcal{L}$, and $(Xf, \alpha) \leq_w (Xf, \alpha') \leq_w (Y, \alpha_Y)$. □

**Proposition 4.2.4.** Let $\mathbf{W}$ be a weak category of $F$-coalgebras and let $\mathcal{K}$ be a subclass of $F$-coalgebras such that $\mathbf{K}$ is weakly $SI$-factorizable. Let $\mathcal{L}$ be a weak coquasivariety of $\mathbf{K}$. Then the union of a family of subcoalgebras in $\mathcal{L}$ of $(X, \alpha_X) \in \mathcal{K}$ is a subcoalgebra in $\mathcal{L}$ of $(X, \alpha_X)$.

*Proof.* Since $\mathcal{L}$ is a weak coquasivariety, for a given family $(S_i, \alpha_i)_{i \in I}$ in $\mathcal{L}$ of subcoalgebras of $(X, \alpha_X) \in \mathcal{K}$, there exists a sum $\sum_{i \in I}(S_i, \alpha_i)$ in $\mathcal{L}$, which is preserved by the underlying set functor with the insertion map $e_i : S_i \to \sum_{i \in I} S_i$. Since for each $i \in I$, the inclusion map $\iota_i : S_i \to X$ is a weak $F$-homomorphism, there exists a unique weak $F$-homomorphism

$\psi : \sum_{i\in I} S_i \to X$ such that $e_i\psi = \iota_i$. Since **K** is weakly SI-factorizable, there is a structure map $\alpha$ on $(\sum_{i\in I} S_i)\psi$ such that $((\sum_{i\in I} S_i)\psi, \alpha) \in \mathcal{L}$ and $((\sum_{i\in I} S_i)\psi, \alpha) \leq_w (X, \alpha_X)$ by Lemma 4.2.3. Since $(\sum_{i\in I} S_i)\psi = \bigcup S_i$, $(\bigcup S_i, \alpha) \in \mathcal{L}$ and $(\bigcup S_i, \alpha) \leq_w (X, \alpha_X)$. □

From the proof of Proposition 4.2.4, we obtain the following.

**Corollary 4.2.5.** Let **W** be a weak category of $F$-coalgebras and let $\mathcal{K}$ be a subclass of $F$-coalgebras such that **K** is weakly $SI$-factorizable. Let $(S_i, \alpha_i)_{i\in I}$ be a family of subcoalgebras in $\mathcal{L}$ of $(X, \alpha_X) \in \mathcal{K}$. Then $(S_i, \alpha_i) \leq_w (\bigcup S_i, \alpha)$ where $\alpha$ is the structure map on $\bigcup S_i$ as in the proof of Proposition 4.2.4.

**Theorem 4.2.6.** Let **W** be a weak category of $F$-coalgebras and let $\mathcal{K}$ be a subclass of $F$-coalgebras such that **K** is weakly $SI$-factorizable. If **K** is complete, then so is every weak coquasivariety of **K**. Similarly, if **K** is finitely complete, then so is every weak coquasivariety of **K**.

*Proof.* Let $\mathcal{L}$ be a weak co-quasivariety of **K**. Let $I$ be a small category and let $D : I \to \mathbf{L}$ be a functor. Then since **K** is complete, we have the limit $((L, \alpha), (\eta_i)_{i\in I})$ in **K**. Let $((L', \alpha'), (\eta'_i)_{i\in I})$ be a cone of $D$ in $\mathcal{L}$. Then there is a unique weak homomorphism $\tau : (L', \alpha') \to (L, \alpha)$ such that there is a structure map $\beta'$ on $L'\tau$ such that $\widetilde{\tau} : (L', \alpha') \to (L'\tau, \beta')$ is a weak homomorphism, $(L'\tau, \beta') \in \mathcal{L}$, and $(L'\tau, \beta') \leq_w (L, \alpha)$ by Lemma 4.2.3. By Proposition 4.2.4, we have $(S, \beta) \in \mathcal{L}$, the union of all subcoalgebras in $\mathcal{L}$ of $(L, \alpha)$. So, the inclusion map $\iota : (L'\tau, \beta') \hookrightarrow (S, \beta)$ is a weak homomorphism. Therefore $\widetilde{\tau}\iota : (L', \alpha') \to (S, \beta)$ is the unique weak homomorphism such that $(S, \beta)$ is the limit in **L**. □

## 4.3 Weak congruences

**Definition 4.3.1.** A *weak congruence* on an $F$-coalgebra $(X, \alpha_X)$ is defined as the kernel of a weak $F$-homomorphism $\varphi : (X, \alpha_X) \to (Y, \alpha_Y)$ for some $F$-coalgebra $(Y, \alpha_Y)$.

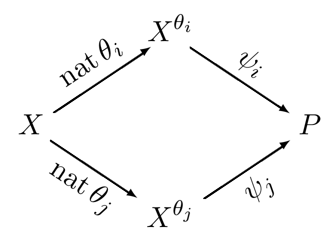
**Lemma 4.3.2.** Assume that a weak category **W** is weakly $SI$-factorizable. For an equivalence relation $\theta$ on an $F$-coalgebra $(X, \alpha_X)$, $\theta$ is a weak congruence if and only if there is a structure map $\alpha_\theta$ on $X^\theta$ for which the natural projection $(\operatorname{nat}\theta) : X \to X^\theta$ is a weak $F$-homomorphism.

*Proof.* The 'if' part is clear. Suppose $\theta$ is a weak congruence. Then there is a weak $F$-homomorphism $\varphi : (X, \alpha_X) \to (Y, \alpha_Y)$ for some $F$-coalgebra $(Y, \alpha_Y)$ with $\ker \varphi = \theta$. Since $\mathbf{W}$ is weakly $SI$-factorizable, there is a structure map on $X\varphi$ such that $\widetilde{\varphi} : X \twoheadrightarrow X\varphi$ is a weak $F$-homomorphism. Note that $\ker \widetilde{\varphi} = \theta$. Since $\widetilde{\varphi}$ is surjective, $\widetilde{\varphi}$ can be $SI$-factorized as $(\operatorname{nat}\theta)\overline{\widetilde{\varphi}}$ where $\overline{\widetilde{\varphi}} : X^\theta \to X\varphi$ is the bijection defined by $x^\theta\overline{\widetilde{\varphi}} = a\widetilde{\varphi}$ for $x^\theta \in X^\theta$. Since $\mathbf{W}$ is weakly $SI$-factorizable, there is a structure map $\alpha_\theta$ on $X^\theta$ for which the natural projection $(\operatorname{nat}\theta : X \to X^\theta)$ is a weak $F$-homomorphism. $\square$

The set of all weak congruences on a $F$-coalgebra $(X, \alpha_X)$ is ordered by set inclusion.

**Proposition 4.3.3.** Assume that a weak category $\mathbf{W}$ is weakly $SI$-factorizable and cocomplete, with every colimit is preserved by the underlying set functor. Let $(\theta_i)_{i\in I}$ be a nonempty family of weak congruences on $(X, \alpha_X)$. Then the supremum of the $(\theta_i)_{i\in I}$ exists, and is given as the transitive closure of their union.

*Proof.* Let $\Phi$ denote the transitive closure of $\bigcup_{i\in I}\theta_i$. Then $\Phi$ is the smallest equivalence relation containing all the $\theta_i$. For each $i \in I$, since $\theta_i$ is a weak congruence, there is a structure map on $X^{\theta_i}$ such that $(\operatorname{nat}\theta_i) : X \to X^{\theta_i}$ is a weak $F$-homomorphism by Lemma 4.3.2. Since $\mathbf{W}$ is cocomplete, we can form the pushout $((P, \alpha), \psi_i)$ of all $\operatorname{nat}\theta_i$.




Since $(\operatorname{nat}\theta_i)\psi_i$ is a weak $F$-homomorphism, it suffices to show that

$$\ker((\operatorname{nat}\theta_i)\psi_i) = \Phi.$$

Since the pushout is preserved by the underlying set functor, $P = (\Sigma_{i\in I}X^{\theta_i})^\theta$ where $\theta$ is the smallest equivalence relation including $(x^{\theta_i}, x^{\theta_j})$ for any $i, j \in I$, and $\psi_i$ is the natural projection. For any $j \in I$, if $(x, y) \in \theta_j$, then $(x^{\theta_i})^\theta = (x^{\theta_j})^\theta = (y^{\theta_j})^\theta = (y^{\theta_i})^\theta$. So $\bigcup_{i\in I}\theta_i \subseteq \ker((\operatorname{nat}\theta_i)\psi_i)$ and hence $\Phi \subseteq \ker((\operatorname{nat}\theta_i)\psi_i)$. Now let $(x, y) \in \ker((\operatorname{nat}\theta_i)\psi_i)$. Then there exists a finite sequence $\{x_1, \ldots, x_n\}$ on $X$ such that $(x, x_1) \in \theta_i$, $(x_k, x_{k+1}) \in \theta_{i_k}$ for some $\theta_{i_k} \in (\theta_i)_{i\in I}$ where $k = 1, \ldots, n-1$, and $(x_n, y) \in \theta_i$. Therefore $(x, y) \in \Phi$.

□

## 4.4 Lower $\mathcal{P}$-morphisms

In this section, it will be shown that the category of lower $\mathcal{P}$-morphisms is bicomplete.

**Lemma 4.4.1.** If a colimit exists in $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$, then it is preserved by the underlying set functor. Similarly, if a limit exists in $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$, then it is preserved by the underlying set functor.

*Proof.* Let $I$ be a small category and let $D : I \to \underrightarrow{\mathbf{Set}}_{\mathcal{P}}$ be a functor. Let $U : \underrightarrow{\mathbf{Set}}_{\mathcal{P}} \to \mathbf{Set}$ denote the underlying set functor and for each $i \in I$, let $(D_i, \alpha_i)$ denote $iD$. Assume that a colimit $((S, \beta), (\varepsilon_i)_{i\in I})$ of $D$ exists in $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$. Let $(S', (\varepsilon'_i)_{i\in I})$ be a cocone of $DU$. We define a structure map $\beta'$ on $S'$ by $x\beta' = S'$ for each $x \in S'$. Then, for any $i \in I$, since for any $d \in D_i$, $d\alpha_i {\varepsilon'_i}^{\mathcal{P}} \subseteq S' = d\varepsilon'_i\beta'$, $\varepsilon'_i$ is a lower $\mathcal{P}$-morphism. So, $((S', \beta'), (\varepsilon'_i)_{i\in I})$ is a cocone of $D$ and there exists a unique lower $\mathcal{P}$-morphism $\tau : S \to S'$ such that $\varepsilon'_i = \varepsilon_i\tau$. Therefore $(S, (\varepsilon_i)_{i\in I})$ is the colimit of $DU$ in $\mathbf{Set}$.

Now assume that a limit $((L, \alpha), (\eta_i)_{i\in I})$ of $D$ exists in $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$. Let $(L', (\eta'_i)_{i\in I})$ be a cone of $DU$. We define a structure map $\alpha'$ on $L'$ by $x\alpha' = \varnothing$ for each $x \in L'$. Then, for any $i \in I$, since for any $x \in L'$, $x\alpha' {\eta'_i}^{\mathcal{P}} = \varnothing \subseteq x\eta'_i\alpha_i$, $\eta'_i$ is a lower $\mathcal{P}$-morphism. So, $((L', \alpha'), (\eta'_i)_{i\in I})$ is a cone of $D$ and there exists a unique lower $\mathcal{P}$-morphism $\tau : L' \to L$ such that $\eta'_i = \tau\eta_i$. Therefore $(L, (\eta_i)_{i\in I})$ is the limit of $DU$ in $\mathbf{Set}$. □

### 4.4.1 Completeness

By Theorem 2.3.5, the terminal coalgebra does not exist in $\mathbf{Set}_{\mathcal{P}}$. However, the terminal coalgebra exists in $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$. In this subsection, we show that $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$ is complete, and that the limits are preserved by the underlying set functor.

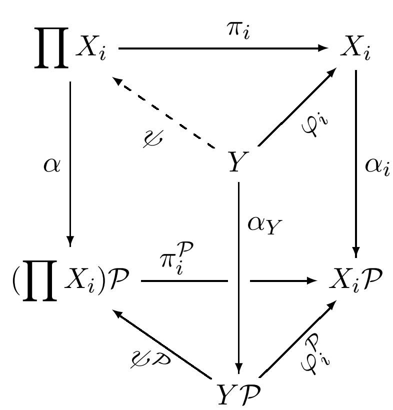
For a given family $(X_i)_{i\in I}$ of sets, let $\prod_{i\in I} X_i$ denote the set

$$\{f : I \to \textstyle\bigcup_{i\in I} X_i \mid \forall i \in I, if \in X_i\}.$$

A choice function $f \in \prod_{i\in I} X_i$ is written by $\prod_{i\in I} x_i$ if for each $i \in I$, $if = x_i$. It is well-known that $(\prod_{i\in I} X_i, (\pi_i)_{i\in I})$ is the product of $(X_i)_{i\in I}$ in **Set**, where each *projection map* $\pi_i : \prod X_i \to X_i$ is given by $f\pi_i = if$.

**Proposition 4.4.2.** For every family $(X_i, \alpha_i)_{i\in I}$ of $\mathcal{P}$-coalgebras, there exists a product of $(X_i, \alpha_i)_{i\in I}$ in $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$, which is preserved by the underlying set functor. If $I = \varnothing$, then the product is the terminal coalgebra $(\{*\}, \alpha)$ with $*\alpha = \{*\}$. If $I \neq \varnothing$, then its structure map $\alpha$ is given by $(\prod_{i\in I} x_i)\alpha = \prod_{i\in I}(x_i\alpha_i)$, i.e. each projection map $\pi_i$ is a $\mathcal{P}$-homomorphism.

*Proof.* It is easy to check that $(\{*\}, \alpha)$ is the terminal coalgebra. Suppose $I \neq \varnothing$. Let $(Y, \alpha_Y)$ be a $\mathcal{P}$-coalgebra. For each $i \in I$, let $\varphi_i : Y \to X_i$ be a lower $\mathcal{P}$-morphism. Then there is a unique map $\psi : Y \to \prod X_i$ in **Set** with $\psi\pi_i = \varphi_i$.



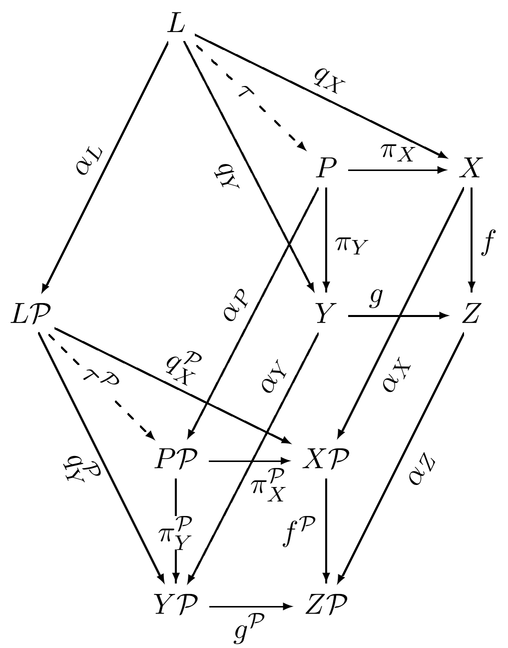

For given $y \in Y$, since $\psi\pi_i = \varphi_i$, $y\psi = \prod(y\varphi_i)$. So, $y\psi\alpha = \prod(y\varphi_i\alpha_i)$. Let $\prod b_i \in y\alpha_Y\psi^{\mathcal{P}}$ be given. Then it is enough to show that $\forall i \in I$, $b_i \in y\varphi_i\alpha_i$. Note that $b_i \in y\alpha_Y\psi^{\mathcal{P}}\pi_i^{\mathcal{P}} = y\alpha_Y(\psi\pi_i)^{\mathcal{P}}$. Since

$$\begin{aligned} y\alpha_Y(\psi\pi_i)^{\mathcal{P}} &= y\alpha_Y\varphi_i^{\mathcal{P}} \qquad (\mathcal{P} \text{ is a functor and } \psi\pi_i = \varphi_i) \\ &\subseteq y\varphi_i\alpha_i \qquad (\varphi_i \text{ is a lower } \mathcal{P}\text{-morphism}), \end{aligned}$$

$b_i \in y\varphi_i\alpha_i$. Therefore $\psi$ is a lower $\mathcal{P}$-morphism. □

**Proposition 4.4.3.** Let $f : (X, \alpha_X) \to (Z, \alpha_Z)$ and $g : (Y, \alpha_Y) \to (Z, \alpha_Z)$ be two lower $\mathcal{P}$-morphisms. Then there exists a pullback $((P, \alpha_P), (\pi_X, \pi_Y))$ in $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$ which is preserved by the underlying set functor, i.e. $P = \{(x, y) \in X \times Y \mid xf = yg\}$, and $\pi_X$ and $\pi_Y$ are the projection maps. Its structure map $\alpha_P$ is given by $(x, y)\alpha_P = P \bigcap[(x\alpha_X) \times (y\alpha_Y)]$

*Proof.* Let $(L, \alpha_L)$ be a $\mathcal{P}$-coalgebra and $q_X : L \to X$ and $q_Y : L \to Y$ be lower $\mathcal{P}$-morphisms such that $q_X f = q_Y g$. Let $P = \{(x, y) \in X \times Y \mid xf = yg\}$. Then there is a unique map $\tau : L \to P$ in **Set** with $\tau\pi_X = q_X$ and $\tau\pi_Y = q_Y$.




For given $l \in L$, since $\tau\pi_X = q_X$ and $\tau\pi_Y = q_Y$, $l\tau = (lq_X, lq_Y)$. So,

$$l\tau\alpha_P = P \bigcap [(lq_X\alpha_X) \times (lq_Y\alpha_Y)] .$$

Let $(a, b) \in l\alpha_L\tau^{\mathcal{P}}$ be given. Then $a \in l\alpha_L\tau^{\mathcal{P}}\pi_X^{\mathcal{P}}$ and $b \in l\alpha_L\tau^{\mathcal{P}}\pi_Y^{\mathcal{P}}$. Since

$$\begin{aligned} l\alpha_L\tau^{\mathcal{P}}\pi_X^{\mathcal{P}} &= l\alpha_L q_X^{\mathcal{P}} \qquad (\mathcal{P} \text{ is a functor and } \tau\pi_X = q_X) \\ &\subseteq lq_X\alpha_X \qquad (q_X \text{ is a lower } \mathcal{P}\text{-morphism}), \end{aligned}$$

$a \in lq_X\alpha_X$. Similarly, $b \in lq_Y\alpha_Y$. Therefore $\tau$ is a lower $\mathcal{P}$-morphism. $\square$

By Proposition 4.4.2 and 4.4.3, we obtain the following.

**Theorem 4.4.4.** $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$ is complete.

**Proposition 4.4.5.** $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$ (resp. $\underleftarrow{\mathbf{Set}}_{\mathcal{P}}$) is weakly $SI$-factorizable.

*Proof.* Let $\varphi : (X, \alpha_X) \to (Y, \alpha_Y)$ be a lower $\mathcal{P}$-morphism. For a given set $Z$, let $f : X \twoheadrightarrow Z$ and $g : Z \rightarrowtail Y$ be a $SI$-factorization with $fg = \varphi$. For a given $z \in Z$, we define a structure map $\alpha$ on $Z$ by

$$z\alpha = \bigcup_{xf=z} x\alpha_X f^{\mathcal{P}}.$$

Then it is easy to see that both $f : (X, \alpha_X) \twoheadrightarrow (Z, \alpha)$ and $g : (Z, \alpha) \rightarrowtail (Y, \alpha_Y)$ are lower $\mathcal{P}$-morphisms. The case of upper $\mathcal{P}$-morphisms can be done by defining a structure map $\alpha$ on $Z$ by

$$z\alpha = \bigcap_{xf=z} x\alpha_X f^{\mathcal{P}}.$$

□

By Theorem 4.2.6 and Proposition 4.4.5, we obtain the following.

**Corollary 4.4.6.** Every weak coquasivariety of $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$ is complete.

### 4.4.2 Cocompleteness

In this subsection, we show that $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$ is cocomplete. By Proposition 4.1.3, every sum exists in $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$. So, for the cocompleteness, it suffices to show the existence of every pushout.

**Lemma 4.4.7.** Let $f : (X, \alpha_X) \to (Y, \alpha_Y)$ and $g : (X, \alpha_X) \to (Z, \alpha_Z)$ be two lower $\mathcal{P}$-morphisms. Assume that a pushout $((P, \beta), (p_Y, p_Z))$ exists. Then the natural projection $\operatorname{nat}\theta : (Y + Z, \alpha_\Sigma) \to ((Y + Z)^\theta, \beta)$ is a lower $\mathcal{P}$-morphism, where $\theta$ is the smallest equivalence relation on $Y + Z$ containing all pairs $(xf, xg)$ with $x \in X$.

*Proof.* Since the pushout is preserved by the underlying set functor according to Lemma 4.4.1, $P = (Y + Z)^\theta$, where $\theta$ is the smallest equivalence relation on $Y + Z$ containing all pairs $(xf, xg)$ with $x \in X$. Furthermore, $p_Y = \iota_Y(\operatorname{nat}\theta)$ and $p_Z = \iota_Z(\operatorname{nat}\theta)$, where $\iota_Y$ and $\iota_Z$ are the insertion maps into the coproduct, and $\operatorname{nat}\theta$ is the natural projection. Note that $p_Y$ and $p_Z$ are lower $\mathcal{P}$-morphisms. By Proposition 4.1.3, $\iota_Y$ and $\iota_Z$ are $\mathcal{P}$-homomorphisms. For given $y \in Y$,

$$\begin{array}{ccccc}
Y & \xrightarrow{\iota_Y} & Y + Z & \xrightarrow{\operatorname{nat}\theta} & (Y + Z)^\theta \\
\downarrow{\scriptstyle \alpha_Y} & & \downarrow{\scriptstyle \alpha_\Sigma} & & \downarrow{\scriptstyle \beta} \\
Y\mathcal{P} & \xrightarrow[\iota_Y^{\mathcal{P}}]{} & (Y + Z)\mathcal{P} & \xrightarrow[(\operatorname{nat}\theta)^{\mathcal{P}}]{} & (Y + Z)^\theta\mathcal{P}
\end{array}$$

$$\begin{aligned} y\alpha_\Sigma(\operatorname{nat}\theta)^{\mathcal{P}} &= y\iota_Y\alpha_\Sigma(\operatorname{nat}\theta)^{\mathcal{P}} \\ &= y\alpha_Y\iota_Y^{\mathcal{P}}(\operatorname{nat}\theta)^{\mathcal{P}} \qquad (\iota_Y \text{ is a } \mathcal{P}\text{-homomorphism}) \\ &\subseteq y\iota_Y(\operatorname{nat}\theta)\beta \qquad (p_Y = \iota_Y(\operatorname{nat}\theta) \text{ is a lower } \mathcal{P}\text{-morphism}) \\ &= y(\operatorname{nat}\theta)\beta. \end{aligned}$$

Similarly, for given $z \in Z$, $z\alpha_\Sigma(\operatorname{nat}\theta)^{\mathcal{P}} \subseteq z(\operatorname{nat}\theta)\beta$. Therefore $\operatorname{nat}\theta$ is a lower $\mathcal{P}$-morphism.

□

By Lemma 4.4.7, it is natural to study the structure maps on $A^\theta$ such that $\operatorname{nat}\theta$ is a lower $\mathcal{P}$-morphism. Indeed, for an arbitrary equivalence relation $\theta$ on $A$, we can give a structure on $A^\theta$ so that $\operatorname{nat}\theta$ is a lower $\mathcal{P}$-morphism as follows.

**Definition 4.4.8.** Let $(A, \alpha)$ be a $\mathcal{P}$-coalgebra and let $\theta$ be an equivalence relation on $A$. We define a $\mathcal{P}$-coalgebra $(A^\theta, \alpha_\theta)$ by

$$a^\theta\alpha_\theta = \bigcup_{(a,b)\in\theta} b\alpha(\operatorname{nat}\theta).$$

It is easy to see that $\alpha_\theta$ is well-defined.

**Proposition 4.4.9.** Let $(A, \alpha)$ be a $\mathcal{P}$-coalgebra and let $\theta$ be an equivalence relation on $A$. Then, for each $a^\theta \in A^\theta$, $a^\theta\alpha_\theta$ is the smallest subset of $A^\theta$ so that $\operatorname{nat}\theta$ is a lower $\mathcal{P}$-morphism.

*Proof.*

$$\begin{array}{ccc} A & \xrightarrow{\operatorname{nat}\theta} & A^\theta \\ \alpha\downarrow & & \downarrow\alpha_\theta \\ A\mathcal{P} & \xrightarrow[(\operatorname{nat}\theta)^{\mathcal{P}}]{} & A^\theta\mathcal{P} \end{array}$$

Let $a \in A$ be given. Note that

$$a(\operatorname{nat}\theta)\alpha_\theta = a^\theta\alpha_\theta = \bigcup_{(a,b)\in\theta} b\alpha(\operatorname{nat}\theta).$$

Since $a\alpha(\operatorname{nat}\theta)^{\mathcal{P}} \subseteq \bigcup_{(a,b)\in\theta} b\alpha(\operatorname{nat}\theta)$, $\operatorname{nat}\theta$ is a lower $\mathcal{P}$-morphism. Now suppose that $\operatorname{nat}\theta$ is a lower $\mathcal{P}$-morphism with a structure map $\beta$ on $A^\theta$. Then, for given $a^\theta \in A^\theta$ and for any $b \in a^\theta$,

$$b\alpha(\operatorname{nat}\theta)^{\mathcal{P}} \subseteq b(\operatorname{nat}\theta)\beta = a(\operatorname{nat}\theta)\beta = a^\theta\beta.$$

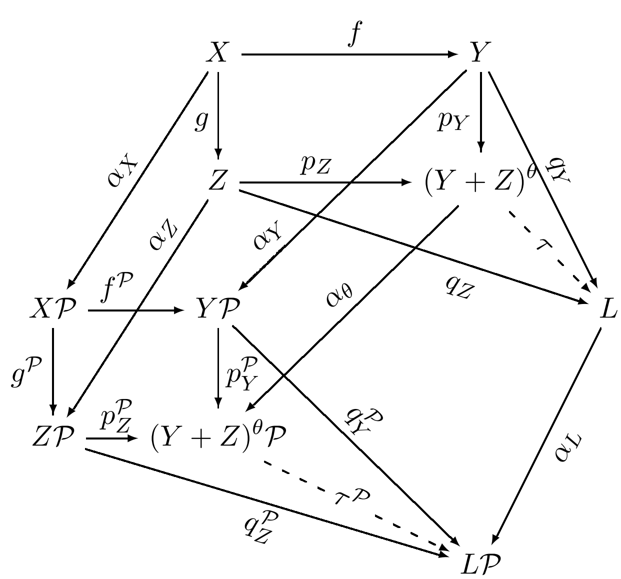
Thus $a^\theta \alpha_\theta \subseteq a^\theta \beta$. □

**Corollary 4.4.10.** In $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$, every equivalence relation on $(A, \alpha)$ is a weak congruence.

**Proposition 4.4.11.** Let $f : (X, \alpha_X) \to (Y, \alpha_Y)$ and $g : (X, \alpha_X) \to (Z, \alpha_Z)$ be two lower $\mathcal{P}$-morphisms. Then the pushout $((P, \alpha_P), (p_Y, p_Z))$ of $f$ and $g$ exists in $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$ and is preserved by the underlying set functor, i.e. $P = (Y + Z)^\theta$, $p_Y = \iota_Y(\operatorname{nat}\theta)$, and $p_Z = \iota_Z(\operatorname{nat}\theta)$, where $\theta$ is the smallest equivalence relation on $Y + Z$ containing all pairs $(xf, xg)$ with $x \in X$. The structure map $\alpha_P$ is $\alpha_\theta$.

*Proof.* Let $q_Y : Y \to L$ and $q_Z : Z \to L$ be lower $\mathcal{P}$-morphisms such that $fq_Y = gq_Z$. Then there is a unique map $\tau : (Y + Z)^\theta \to L$ such that $p_Y \tau = q_Y$ and $p_Z \tau = q_Z$. Let $\alpha_\Sigma$ be the structure map of the sum of $(Y, \alpha_Y)$ and $(Z, \alpha_Z)$ as in Proposition 4.1.3. Then $\alpha_\Sigma$ is the sum of $\alpha_Y$ and $\alpha_Z$. Let $a^\theta \in (Y + Z)^\theta$. W.l.o.g., we may assume that $a \in Y$. Let $b \in a^\theta$. If $b \in Y$, then

$$
\begin{aligned}
b\alpha_\Sigma(\operatorname{nat}\theta)\tau^{\mathcal{P}} &= b\alpha_\Sigma p_Y^{\mathcal{P}} \tau^{\mathcal{P}} \\
&= b\alpha_Y q_Y^{\mathcal{P}} \\
&\subseteq bq_Y \alpha_L \qquad (q_Y \text{ is a lower } \mathcal{P}\text{-morphism}) \\
&= aq_Y \alpha_L = a^\theta \tau \alpha_L.
\end{aligned}
$$



If $b \in Z$, then

$$\begin{aligned} b\alpha_\Sigma(\operatorname{nat}\theta)\tau^{\mathcal{P}} &= b\alpha_\Sigma p_Z^{\mathcal{P}}\tau^{\mathcal{P}} \\ &= b\alpha_Z q_Z^{\mathcal{P}} \\ &\subseteq bq_Z\alpha_L \qquad (q_Z \text{ is a lower } \mathcal{P}\text{-morphism}) \\ &= aq_Y\alpha_L = a^\theta\tau\alpha_L. \end{aligned}$$

Since $a^\theta\alpha_\theta = \bigcup_{(a,b)\in\theta} b\alpha_\Sigma(\operatorname{nat}\theta)$, $\tau$ is a lower $\mathcal{P}$-morphism. ☐

By Proposition 4.1.3 and 4.4.11, we obtain the following.

**Theorem 4.4.12.** $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$ is cocomplete.

## 4.5 Upper $\mathcal{P}$-morphisms

In Section 4.4, we have observed that $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$ is bicomplete although $\mathbf{Set}_{\mathcal{P}}$ is not complete. This is an example of the advantage obtained by weak homomorphisms. When it comes to upper $\mathcal{P}$-morphisms, the situation is reversed. By Theorem 2.3.4, $\mathbf{Set}_{\mathcal{P}}$ is cocomplete. In this section, we show that $\underleftarrow{\mathbf{Set}}_{\mathcal{P}}$ is not cocomplete.

**Lemma 4.5.1.** If a colimit exists in $\underleftarrow{\mathbf{Set}}_{\mathcal{P}}$, then it is preserved by the underlying set functor.

*Proof.* Let $I$ be a small category and let $D : I \to \underleftarrow{\mathbf{Set}}_{\mathcal{P}}$ be a functor. Let $(D_i, \alpha_i)$ denote $iD$. Assume that a colimit $((S, \beta), (\varepsilon_i)_{i\in I})$ of $D$ exists in $\underleftarrow{\mathbf{Set}}_{\mathcal{P}}$. Let $(S', (\varepsilon'_i)_{i\in I})$ be a cocone of $DU$. We define a structure map $\beta'$ on $S'$ by $x\beta' = \varnothing$ for each $x \in S'$. Then, for any $i \in I$, since for any $d \in D_i$, $d\alpha_i\varepsilon_i'^{\mathcal{P}} \supseteq \varnothing = d\varepsilon'_i\beta'$, $\varepsilon'_i$ is an upper $\mathcal{P}$-morphism. So, $((S', \beta'), (\varepsilon'_i)_{i\in I})$ is a cocone of $D$ and there exists a unique upper $\mathcal{P}$-morphism $\tau : S \to S'$ such that $\varepsilon'_i = \varepsilon_i\tau$. Therefore $(S, (\varepsilon_i)_{i\in I})$ is the colimit of $DU$ in $\mathbf{Set}$. ☐

**Lemma 4.5.2.** Let $f : (X, \alpha_X) \to (Y, \alpha_Y)$ and $g : (X, \alpha_X) \to (Z, \alpha_Z)$ be two upper $\mathcal{P}$-morphisms. Assume that a pushout $((P, \beta), (p_Y, p_Z))$ exists. Then the natural projection $\operatorname{nat}\theta : (Y+Z, \alpha_\Sigma) \to ((Y+Z)^\theta, \beta)$ is an upper $\mathcal{P}$-morphism, where $\theta$ is the smallest equivalence relation on $Y + Z$ containing all pairs $(xf, xg)$ with $x \in X$.

*Proof.* Reverse the direction of inclusions in Lemma 4.4.7. □

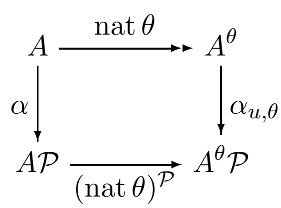
**Definition 4.5.3.** Let $(A, \alpha)$ be a $\mathcal{P}$-coalgebra and let $\theta$ be an equivalence relation on $A$. We define a $\mathcal{P}$-coalgebra $(A^\theta, \alpha_{u,\theta})$ by

$$a^\theta \alpha_{u,\theta} = \bigcap_{(a,b)\in\theta} b\alpha(\operatorname{nat}\theta).$$

It is easy to see that $\alpha_{u,\theta}$ is well-defined.

**Proposition 4.5.4.** Let $(A, \alpha)$ be a $\mathcal{P}$-coalgebra and let $\theta$ be an equivalence relation on $A$. Then, for each $a^\theta \in A^\theta$, $a^\theta \alpha_{u,\theta}$ is the largest subset of $A^\theta$ so that $\operatorname{nat}\theta$ is an upper $\mathcal{P}$-morphism.

*Proof.*




Let $a \in A$ be given. Note that

$$a(\operatorname{nat}\theta)\alpha_{u,\theta} = a^\theta \alpha_{u,\theta} = \bigcap_{(a,b)\in\theta} b\alpha(\operatorname{nat}\theta).$$

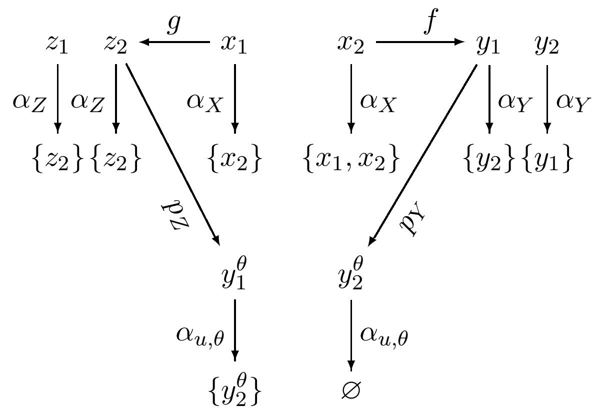
Since $a\alpha(\operatorname{nat}\theta)^{\mathcal{P}} \supseteq \bigcap_{(a,b)\in\theta} b\alpha(\operatorname{nat}\theta)$, $\operatorname{nat}\theta$ is an upper $\mathcal{P}$-morphism. Now suppose that $\operatorname{nat}\theta$ is an upper $\mathcal{P}$-morphism with a structure map $\beta$ on $A^\theta$. Then, for given $a^\theta \in A^\theta$ and for any $b \in a^\theta$,

$$b\alpha(\operatorname{nat}\theta)^{\mathcal{P}} \supseteq b(\operatorname{nat}\theta)\beta = a(\operatorname{nat}\theta)\beta = a^\theta\beta.$$

Therefore $a^\theta\alpha_{u,\theta} \supseteq a^\theta\beta$. □

**Theorem 4.5.5.** $\underleftarrow{\mathbf{Set}}_{\mathcal{P}}$ is not cocomplete.

*Proof.* One counterexample is enough. Let $(X = \{x_1, x_2\}, \alpha_X)$ be coalgebra such that $x_1\alpha_X = \{x_2\}$ and $x_2\alpha_X = X$. Let $(Y = \{y_1, y_2\}, \alpha_Y)$ and $(Z = \{z_1, z_2\}, \alpha_Z)$ be coalgebras such that $y_1\alpha_Y = \{y_2\}$, $y_2\alpha_Y = \{y_1\}$, $z_1\alpha_Z = \{z_2\}$, and $z_2\alpha_Z = \{z_2\}$. Let $f : X \to Y$ be a map defined by $x_i = y_i$ for $i = 1, 2$ and let $g : X \to Z$ be a map defined by $x_i = z_i$ for $i = 1, 2$. Then $f$ and $g$ are upper $\mathcal{P}$-morphisms.




Now assume that the pushout $((P, \alpha), (p_Y, p_Z))$ of $f$ and $g$ exists in $\underleftarrow{\mathbf{Set}}_{\mathcal{P}}$. Then by Lemma 4.5.1, $P = (Y + Z)^\theta$, where $\theta$ is the smallest equivalence relation on $Y + Z$ containing all pairs $(xf, xg)$ with $x \in X$. Furthermore, $p_Y = \iota_Y(\operatorname{nat}\theta)$ and $p_Z = \iota_Z(\operatorname{nat}\theta)$. It can be readily seen that $(Y + Z)^\theta = \{y_1^\theta, y_2^\theta\}$, where $y_1^\theta = \{y_1, z_1\}$ and $y_2^\theta = \{y_2, z_2\}$. Let $(L = \{0\}, \alpha_L)$ be a $\mathcal{P}$-coalgebra such that $0\alpha_L = \{0\}$. Let $q_Y : Y \to L$ and $q_Z : Z \to L$ be the constant functions. Then $q_Y$ and $q_Z$ are upper $\mathcal{P}$-morphisms such that $fq_Y = gq_Z$. There is a unique upper $\mathcal{P}$-morphism $\tau : (Y + Z)^\theta \to L$ such that $p_Y\tau = q_Y$ and $p_Z\tau = q_Z$. By Lemma 4.5.2 and Proposition 4.5.4, $\tau$ should be an upper $\mathcal{P}$-morphism with structure map $\alpha_{u,\theta}$ on $(Y + Z)^\theta$. However,

$$y_2^\theta \alpha_{u,\theta} \tau^{\mathcal{P}} = \varnothing \not\supseteq \{0\} = y_2^\theta \tau \alpha_L \,,$$

which is a contradiction. Therefore there is no pushout of $f$ and $g$ in $\underleftarrow{\mathbf{Set}}_{\mathcal{P}}$. $\square$

# CHAPTER 5. Graphic coalgebras with lower morphisms

In Chapter 3, we have observed that $N$-homomorphisms between graphic coalgebras are too strict, and that there is no simple construction for products. There is another aspect of the strictness of $N$-homomorphisms. Let $(X, \alpha) \in \mathcal{G}$ be connected, and let $S$ be a proper subset of $X$. Then by Proposition 3.2.4, there is no structure map $\alpha_S$ so that $(S, \alpha_S) \leq (X, \alpha)$. This implies that an induced subgraph may not be a subcoalgebra. These situations can be avoided by relaxing the condition for the definition of $N$-homomorphism. In this chapter, we suggest to use lower morphisms as a relaxation of $N$-homomorphisms.

## 5.1 Lower $N$-morphisms

### 5.1.1 Finite completeness

In this subsection, we discuss the finite completeness of $\underrightarrow{\mathbf{Set}}_N$. With the same proof of Lemma 4.4.1, we obtain the following.

**Lemma 5.1.1.** If a limit exists in $\underrightarrow{\mathbf{Set}}_N$, then it is preserved by the underlying set functor.

**Theorem 5.1.2.** $\underrightarrow{\mathbf{Set}}_N$ is not complete.

*Proof.* One counterexample is enough. For each natural number $n \in \mathbf{N}$, let $(X_n = \{a_n, b_n, c_n\}, \alpha_n)$ be an $N$-coalgebra such that $a_n\alpha_n = \{b_n, c_n\}$ and $b_n\alpha_n = c_n\alpha_n = \{a_n\}$. Assume that the product $((P, \alpha), (\pi_n)_{n\in\mathbf{N}})$ exists in $\underrightarrow{\mathbf{Set}}_N$. Then by Lemma 5.1.1, $P = \prod_{n\in\mathbf{N}} X_n$. Let $(\{0, 1\}, \beta)$ be a graphic $N$-coalgebra whose graph structure is the following;

$$0 \text{ —— } 1$$

For each $n$, we define a map $f_n : \{0, 1\} \to X_n$ by $0f_n = a_n$ and $1f_n = b_n$. A function $g_n : \{0, 1\} \to X_n$ is defined by $0g_n = a_n$ and $1g_n = c_n$. Then both $f_n$ and $g_n$ are lower

$N$-morphisms. Since $(P,\alpha)$ is the product, for each class of functions $(h_n)_{n\in\mathbf{N}}$ with $h_n = f_n$ or $g_n$, there is a unique lower $N$-morphism from $\{0,1\}$ to $P$. Therefore $|(\prod_{n\in\mathbf{N}} a_n)\alpha|$ is infinite, which is a contradiction. □

**Proposition 5.1.3.** For every family $(X_i,\alpha_i)_{i\in I}$ of finite number of $N$-coalgebras, there exists a product of $(X_i,\alpha_i)_{i\in I}$ in $\underrightarrow{\mathbf{Set}}_N$, which is preserved by the underlying set functor. If $I=\varnothing$, then the product is the terminal coalgebra $(\{*\},\alpha)$ with $*\alpha=\{*\}$. If $I\neq\varnothing$, then its structure map $\alpha$ is given by $(\prod_{i\in I} x_i)\alpha = \prod_{i\in I}(x_i\alpha_i)$. i.e., each projection map $\pi_i$ is a $N$-homomorphism.

**Proposition 5.1.4.** Let $f:(X,\alpha_X)\to(Z,\alpha_Z)$ and $g:(Y,\alpha_Y)\to(Z,\alpha_Z)$ be two lower $N$-morphisms. Then there exists a pullback $(P,\alpha_P)$ in $\underrightarrow{\mathbf{Set}}_N$ which is preserved by the underlying set functor. Its structure map $\alpha_P$ is given by $(x,y)\alpha_P = P\bigcap[(x\alpha_X)\times(y\alpha_Y)]$, i.e. the projection maps $\pi_X$ and $\pi_Y$ are $N$-homomorphisms.

By Proposition 5.1.3 and 5.1.4, we obtain the following.

**Theorem 5.1.5.** $\underrightarrow{\mathbf{Set}}_N$ is finitely complete.

**Proposition 5.1.6.** $\underrightarrow{\mathbf{Set}}_N$ is weakly $SI$-factorizable.

*Proof.* Let $\varphi:(X,\alpha_X)\to(Y,\alpha_Y)$ be a lower $N$-morphism. For a given set $Z$, let $f:X\twoheadrightarrow Z$ and $g:Z\rightarrowtail Y$ be a $SI$-factorization with $fg=\varphi$. For a given $z\in Z$, we define a structure map $\alpha$ on $Z$ by

$$z\alpha = \bigcup_{xf=z} x\alpha_X f^N.$$

Since $\varphi$ is a lower $N$-morphism, for each $x\in X$ with $xf=z$, $x\alpha_X\varphi^N\subseteq x\varphi\alpha_Y$. So

$$z\alpha g^N = (\bigcup_{xf=z} x\alpha_X f^N)g^N = \bigcup_{xf=z}(x\alpha_X f^N g^N) = \bigcup_{xf=z} x\alpha_X\varphi^N \subseteq x\varphi\alpha_Y.$$

Since $g$ is injective, $z\alpha$ is finite. Now it is easy to see that both $f:(X,\alpha_X)\twoheadrightarrow(Z,\alpha)$ and $g:(Z,\alpha)\rightarrowtail(Y,\alpha_Y)$ are lower $N$-morphisms. □

By Proposition 5.1.6 and Theorem 4.2.6, we obtain the following.

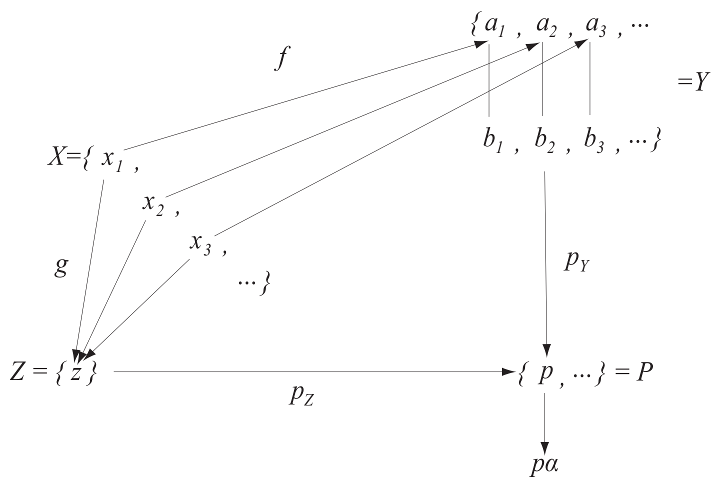
**Corollary 5.1.7.** Every weak co-quasivariety of $\underrightarrow{\mathbf{Set}}_N$ is finitely complete.

### 5.1.2 Non-cocompleteness

By Proposition 4.1.3, every sum exists in $\underrightarrow{\mathbf{Set}}_N$. So for the cocompleteness, we need to study the existence of every pushout.

**Theorem 5.1.8.** $\underrightarrow{\mathbf{Set}}_N$ is not cocomplete.

*Proof.* One counterexample is enough. Let $(X = \{x_n \mid n \in \mathbf{N}\}, \alpha_X)$, $(Y = \{a_n, b_n \mid n \in \mathbf{N}\}, \alpha_Y)$, and $(Z = \{z\}, \alpha_Z)$ be $N$-coalgebras such that $x_n\alpha_X = \varnothing$, $a_n\alpha_Y = \{b_n\}$, $b_n\alpha_Y = \{a_n\}$, and $z\alpha_Z = \varnothing$. Let $f : X \to Y$ be a map defined by $x_n f = a_n$. Let $g$ be the map from $X$ to $Z$. Then $f$ and $g$ are lower $N$-morphisms. Now assume that the pushout $((P, \alpha), (p_Y, p_Z))$ of $f$ and $g$ exists in $\underrightarrow{\mathbf{Set}}_N$. Then since $fp_Y = gp_Z$, there is an element $p \in P$ such that $zp_Z = p = a_n p_Y$ for any $n \in \mathbf{N}$.




Since $p\alpha$ is finite, we may let $p\alpha = \{c_1, c_2, \ldots, c_k\}$ for some $k \in \mathbf{N}$. Since $p_Y$ is a lower $N$-morphism, $b_n p_Y \subseteq p\alpha$ for any $n$. Since $\{b_n \mid n \in \mathbf{N}\}$ is infinite, there is some $i$ between 1 and $k$ such that $|\{b_n \in Y \mid b_n p_Y = c_i\}| \geq 2$. Without loss of generality, we may assume that $b_1 p_Y = b_2 p_Y = c_i$. Now let $(Q = \{q, c_1, c_2, \ldots, c_{k+1}\}, \beta)$ be an $N$-coalgebra such that $q\beta = \{c_1, \ldots, c_{k+1}\}$, and $c_j\beta = \{q\}$ for any $j$. We define a map $q_Z : Z \to Q$ by $zq_Z = q$. Let $q_Y : Y \to Q$ be a map defined by $a_n q_Y = q$ for any $n$, $b_1 q_Y = c_{k+1}$, and $b_n q_Y = b_n p_Y$ for any $n > 1$. Then $q_Z$ and $q_Y$ are lower $N$-morphisms with $fq_Y = gq_Z$. However there is no map

$\tau : P \to Q$ with $p_Y \tau = q_Y$, which is a contradiction. $\square$

Let $f : (X, \alpha_X) \to (Y, \alpha_Y)$ and $g : (X, \alpha_X) \to (Z, \alpha_Z)$ be two lower $N$-morphisms. When does the pushout of $f$ and $g$ exist? Let $\theta$ be the smallest equivalence relation on $Y + Z$ containing all pairs $(xf, xg)$ with $x \in X$. Assume that there is a structure map $\beta$ on $(Y + Z)^\theta$ with

$$\operatorname{nat}\theta : (Y + Z, \alpha_\Sigma) \to ((Y + Z)^\theta, \beta)$$

is a lower $N$-morphism, i.e. $\theta$ is a weak congruence. We define an $N$-coalgebra $((Y + Z)^\theta, \alpha_\theta)$ by

$$x^\theta \alpha_\theta = \bigcup_{(x,y)\in\theta} y\alpha_\Sigma(\operatorname{nat}\theta), \tag{5.1}$$

where $\operatorname{nat}\theta$ is the natural projection. It is easy to see that $\alpha_\theta$ is well-defined and

$$\operatorname{nat}\theta : (Y + Z, \alpha_\Sigma) \to ((Y + Z)^\theta, \alpha_\theta)$$

is a lower $N$-morphism.

**Proposition 5.1.9.** Let $f : (X, \alpha_X) \to (Y, \alpha_Y)$ and $g : (X, \alpha_X) \to (Z, \alpha_Z)$ be two lower $N$-morphisms. Assume that $\theta$ is a weak congruence, where $\theta$ is the smallest equivalence relation on $Y + Z$ containing all pairs $(xf, xg)$ with $x \in X$. Then the pushout $((P, \alpha_P), (p_Y, p_Z))$ of $f$ and $g$ exists in $\underrightarrow{\mathbf{Set}}_N$ and it is preserved by the underlying set functor, i.e. $P = (Y + Z)^\theta$, $p_Y = \iota_Y(\operatorname{nat}\theta)$, and $p_Z = \iota_Z(\operatorname{nat}\theta)$. The structure map $\alpha_P$ is $\alpha_\theta$.

## 5.2 Graphic coalgebras with lower morphisms

We denote the full subcategory of $\underrightarrow{\mathbf{Set}}_N$ with the object set $\mathcal{G}$ by $\underrightarrow{\mathbf{Gph}}$.

### 5.2.1 Lower $N$-morphisms on $\mathcal{G}$

By Definition 4.1.1, we obtain the following.

**Lemma 5.2.1.** A lower $N$-morphism preserves loops.

*Proof.* Let $f : (X, \alpha) \to (Y, \beta)$ be a lower $N$-morphism on $\mathcal{G}$. If $x \in X$ has a loop, then $x \in x\alpha$. Since $x\alpha f^N \subseteq xf\beta$, $xf \in xf\beta$. Therefore $xf$ has a loop. $\square$

From the above Lemma 5.2.1, the preimage of a $N$-coalgebra with no loops in $\mathcal{G}$ under a lower $N$-morphism has no loops.

**Theorem 5.2.2.** Let $f$ be a lower $N$-morphism from $(X, \alpha)$ to $(Y, \beta)$ on $\mathcal{G}$. Then $f$ is a graph homomorphism between corresponding graphs, i.e. $f$ preserves edges. Conversely, if $f$ is a graph homomorphism, then $f$ is a lower $N$-morphism between corresponding graphic $N$-coalgebras.

*Proof.* If $x' \in x\alpha$, then $x'f \in xf\beta$ since $x\alpha f^N \subseteq xf\beta$. So $f$ preserves edges. The other direction is clear. □

By Lemma 5.2.2, it is apparent that we have relaxed the homomorphism concept to a proper level. As a result, Proposition 3.2.4 might not hold in $\underrightarrow{\mathbf{Gph}}$. Also, the inverse image of 2 could be trivial. Indeed, bipartite graphs can be characterized as inverse images of 2. This is a nice example of the use of the coalgebraic language.

**Proposition 5.2.3.** Let $(X, \alpha) \in \mathcal{G}$. Then $(X, \alpha)$ is a bipartite graph if and only if there is a lower $N$-morphism $f : X \to 2$.

*Proof.* Suppose that $(X, \alpha)$ is a bipartite graph. Then there exists a partition $X_0$ and $X_1$ on $X$ such that $X_0$ and $X_1$ are independent. We define a function $f : X \to 2$ by $xf = 0$ for any $x \in X_0$, and $xf = 1$ for any $x \in X_1$. Then for any $x \in X_0$, $x\alpha f^N = \varnothing$ or $\{1\}$. So $x\alpha f^N \subseteq xf\alpha_2 = 0\alpha_2 = \{1\}$. Similarly, if $x \in X_1$, then $x\alpha f^N \subseteq xf\alpha_2$. Therefore $f$ is a lower $N$-morphism. Now suppose that $f : X \to 2$ is a lower $N$-morphism. By Lemma 5.2.1, $(X, \alpha)$ has no loop. If for any $x \in X$, $x\alpha = \varnothing$, then we are done. If $(X, \alpha)$ is not trivial, then the proof is same as the proof of Proposition 3.2.5. □

### 5.2.2 Finite completeness

In this subsection, we show the completeness of $\underrightarrow{\mathbf{Gph}}$ by use of the coalgebraic language. Note that $\underrightarrow{\mathbf{Set}}_N$ is finitely complete by Theorem 5.1.5. In order to show that $\underrightarrow{\mathbf{Gph}}$ is finitely complete, it is enough to prove that $\mathcal{G}$ is a weak co-quasivariety by Corollary 5.1.7.

Let $(\{0,1,2\},\alpha) \in \mathcal{G}$ with $0\alpha = \{1\}$, $1\alpha = \{0,2\}$, and $2\alpha = \{1\}$. The graph structure of $(\{0,1,2\},\alpha)$ is the following:

$$0 \text{ ——— } 1 \text{ ——— } 2$$

Let $(\{0,1\},\beta)$ be the $N$-coalgebra with $0\beta = \{1\}$ and $1\beta = \emptyset$. Now, we consider the canonical inclusion map $\iota : \{0,1\} \to \{0,1,2\}$. Then, $\iota$ becomes a lower morphism and $(\{0,1\},\beta) \leq (\{0,1,2\},\alpha)$. However, $(\{0,1\},\beta) \notin \mathcal{G}$. As a result, we have the following lemma.

**Lemma 5.2.4.** $\mathcal{G}$ is not closed under subcoalgebras.

Let $(X = \{0,1,2\},\alpha) \in \mathcal{G}$ such that $0\alpha = \{1\}$, $1\alpha = \{0\}$, and $2\alpha = \varnothing$. Let $(X,\beta) \in \underrightarrow{\mathbf{Set}}_N$ such that $0\beta = \{1\}$, $1\beta = \{0,2\}$, and $2\beta = \varnothing$. Then the identity map $id : X \to X$ is a surjective lower $N$-morphism. However, $(X,\beta) \notin \mathcal{G}$. So $\mathcal{G}$ is not a co-quasivariety. Note that every coproduct exists in $\underrightarrow{\mathbf{Set}}_N$ by Proposition 4.1.3.

**Proposition 5.2.5.** $\mathcal{G}$ forms a weak coquasivariety over $\underrightarrow{\mathbf{Set}}_N$.

*Proof.* (i) $\mathcal{G}$ is closed under $\Sigma$;

The proof is same as the proof of Proposition 3.2.6.

(ii) Let $f : (X,\alpha_X) \to (Y,\alpha_Y)$ be a surjective lower $N$-morphism where $(X,\alpha_X) \in \mathcal{G}$. We define a structure map $\alpha$ on $Y$ by $y\alpha = \bigcup_{xf=y} x\alpha_X f^N$ for $y \in Y$. For any $x \in X$, $x\alpha_X f^N \subseteq \bigcup_{x'f=xf} x'\alpha_X f^N = xf\alpha$. So, $f : (X,\alpha_X) \to (Y,\alpha)$ is a lower $N$-morphism. For any $y \in Y$, $y\alpha\iota^N \subseteq y\iota\alpha_Y$ since $\forall x \in X$ with $xf = y$, $x\alpha_X f^N \subseteq y\alpha_Y$. So $(Y,\alpha) \leq_w (Y,\alpha_Y)$. Now suppose that $y_1 \in y_2\alpha$ where $y_1, y_2 \in Y$. Then $y_1 \in a\alpha_X f^N$ for some $a \in X$ with $af = y_2$. So, $\exists b \in a\alpha_X$ such that $bf = y_1$. Since $(X,\alpha_X) \in \mathcal{G}$, $a \in b\alpha_X$. So $y_2 = af \in b\alpha_X f^N$. Hence $y_2 \in y_1\alpha$.

□

By Corollary 5.1.7, we obtain the following.

**Corollary 5.2.6.** $\underrightarrow{\mathbf{Gph}}$ is finitely complete.

Recall that although **Gph** is complete, there is still no simple construction for the products. However, $\underrightarrow{\mathbf{Set}}_N$ is finitely complete and we have simple constructions for the finite products and pullbacks by Proposition 5.1.3 and 5.1.4. Products and pullbacks of graphs were considered in [9], and they have simple constructions. We obtain an alternative proof using Proposition 5.1.3 and 5.1.4.

**Proposition 5.2.7.** For every family $(X_i, \alpha_i)_{i \in I}$ of finite number of graphic $N$-coalgebras, the product $(\prod X_i, \alpha)$ in Proposition 5.1.3 is also graphic.

*Proof.* The case $I = \varnothing$ is trivial. Suppose that $I \neq \varnothing$. Let $\prod x_i \in \prod X_i$ and $\prod y_i \in \prod X_i$ be given. We may assume that $\prod x_i \in (\prod y_i)\alpha$. Since $(\prod y_i)\alpha = \{\prod a_i \in \prod X_i | \forall i \in I, a_i \in y_i\alpha_i\}$, for all $i \in I$, $x_i \in y_i\alpha_i$. Since each $(X_i, \alpha_i) \in \mathsf{G}$, $y_i \in x_i\alpha_i$. So, $\prod y_i \in (\prod x_i)\alpha$. Hence $(\prod X_i, \alpha) \in \mathcal{G}$. □

By Proposition 5.2.7, the finite product of graphic $N$-coalgebras is constructed as in Proposition 5.1.3.

**Proposition 5.2.8.** Let $f : (X, \alpha_X) \to (Z, \alpha_Z)$ and $g : (Y, \alpha_Y) \to (Z, \alpha_Z)$ be two lower $N$-morphisms in $\mathcal{G}$. Then the pullback $(P, \alpha_P)$ in Proposition 5.1.4 is also graphic.

*Proof.* Suppose that $(x', y') \in (x, y)\alpha_P$, where $(x, y)$, $(x', y') \in P$. Then $x' \in x\alpha_X$ and $y' \in y\alpha_Y$. Since $(X, \alpha_X)$, $(Y, \alpha_Y) \in \mathcal{G}$, $x \in x' \in \alpha_X$ and $y \in y' \in \alpha_Y$. Therefore $(x, y) \in (x', y')\alpha_P$. □

By Proposition 5.2.8, the pullback of graphic $N$-coalgebras is constructed as in Proposition 5.1.4.

### 5.2.3 Non-cocompleteness

It is easy to see that every coproduct exists in $\underrightarrow{\mathbf{Gph}}$ from Proposition 4.1.3. $\underrightarrow{\mathbf{Gph}}$ is not cocomplete since the counterexample of Theorem 5.1.8 is considered in the category $\underrightarrow{\mathbf{Gph}}$.

**Lemma 5.2.9.** Let $(X, \alpha)$ be a graphic coalgebra and let $\theta$ be a weak congruence on $X$. Then $(X^\theta, \alpha_\theta)$ is a graphic coalgebra, where $\alpha_\theta$ is the structure map defined by

$$x^\theta \alpha_\theta = \bigcup_{(x,y)\in\theta} y\alpha(\operatorname{nat}\theta)\,.$$

*Proof.* Suppose that $y^\theta \in x^\theta \alpha_\theta$ where $x^\theta, y^\theta \in X^\theta$. Then there is $a \in x^\theta$ such that $y^\theta \in a\alpha(\operatorname{nat}\theta)$ by the definition of $\alpha_\theta$. So $b \in y^\theta$ for some $b \in a\alpha$. Since $(X, \alpha) \in \mathcal{G}$, $a \in b\alpha$. So $a^\theta \in b\alpha(\operatorname{nat}\theta)$. Hence,

$$x^\theta = a^\theta \in \bigcup_{(b,y)\in\theta} b\alpha(\operatorname{nat}\theta) = y^\theta \alpha_\theta\,.$$

□

By Lemma 5.2.9, we obtain the following.

**Proposition 5.2.10.** Let $f : (X, \alpha_X) \to (Y, \alpha_Y)$ and $g : (X, \alpha_X) \to (Z, \alpha_Z)$ be two lower $N$-morphisms in $\mathcal{G}$. Assume that $\theta$ is a weak congruence, where $\theta$ is the smallest equivalence relation on $Y + Z$ containing all pairs $(xf, xg)$ with $x \in X$. Then the pushout of $f$ and $g$ in Proposition 5.1.9 is also graphic.

# CHAPTER 6. Topological coalgebras with upper morphisms

In this chapter, we will work within the category $\underleftarrow{\mathbf{Set}}_{\mathcal{F}}$ of all coalgebras for the monotonic filter functor $\mathcal{F}$, with the upper morphisms as morphisms.

## 6.1 Upper $\mathcal{F}$-morphisms

**Lemma 6.1.1.** A map $f : (X, \alpha) \to (Y, \beta)$ is an upper $\mathcal{F}$-morphism if and only if for all $x \in X$ and all $V \subseteq Y$,

$$V \in xf\beta \Rightarrow f^{-1}(V) \in x\alpha.$$

*Proof.* $f$ is an upper $\mathcal{F}$-morphism

$$\text{iff } \forall x \in X,\ x\alpha(f^{\mathcal{F}}) \supseteq xf\beta$$

$$\text{iff } \forall x \in X, \forall V \subseteq Y,\ [V \in xf\beta \Rightarrow \exists U \subseteq X.\ U \in x\alpha,\ Uf \subseteq V]$$

$$\text{iff } \forall x \in X, \forall V \subseteq Y,\ [V \in xf\beta \Rightarrow \exists U \subseteq X.\ U \in x\alpha,\ U \subseteq f^{-1}(V)]$$

$$\text{iff } \forall x \in X, \forall V \subseteq Y,\ [V \in xf\beta \Rightarrow f^{-1}(V) \in x\alpha].$$

□

By using the proof of Proposition 3.1.4, we obtain the following.

**Lemma 6.1.2.** A subset $S \subseteq X$ is an $F$-subcoalgebra of $(X, \alpha)$ if $S \in s\alpha$ for each $s \in S$.

The following example shows that the converse of Lemma 6.1.2 may not hold. Let $(\{0,1\}, \alpha)$ be the $\mathcal{F}$-coalgebra where $0\alpha = 1\alpha = \{\{0,1\}\}$. Let $(\{0\}, \beta)$ be the $\mathcal{F}$-coalgebra with $0\beta = \{\{0\}\}$. Now we consider the canonical inclusion map $\iota : \{0\} \hookrightarrow \{0,1\}$. Then $\iota$ becomes an upper morphism and $(\{0\}, \beta) \leq_w (\{0,1\}, \alpha)$. However, $(\{0\}, \beta) \not\leq (\{0,1\}, \alpha)$.

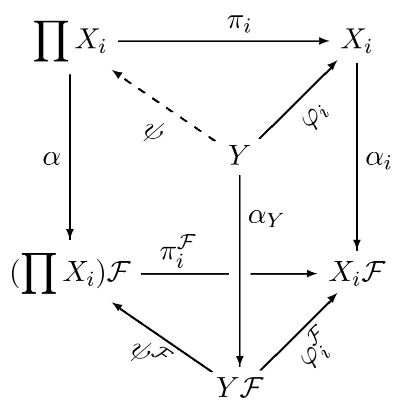
Note that for given families $(X_i)_{i\in I}$ of sets and $(H_i)_{i\in I}$, where each $H_i$ is a filter on $X_i$,

$$\widehat{\prod} H_i := \Big\{ \prod U_i \ \Big|\ U_i \in H_i,\ U_i = X_i \text{ for all but finitely many } i \Big\}$$

is a filter on $\prod X_i$.

**Proposition 6.1.3.** For a family $(X_i, \alpha_i)_{i\in I}$ of $\mathcal{F}$-coalgebras, there exists a product $\prod_{i\in I}(X_i, \alpha_i)$ in $\underleftarrow{\mathbf{Set}}_{\mathcal{F}}$ which is preserved by the underlying set functor. If $I = \varnothing$, then the product is the terminal coalgebra $(\{*\}, \alpha)$ with $*\alpha = \{\{*\}\}$. If $I \neq \varnothing$, then the structure map $\alpha$ is given by $(\prod x_i)\alpha = \widehat{\prod}(x_i\alpha_i)$.

*Proof.* It is easy to check that $(\{*\}, \alpha)$ is the terminal coalgebra. Suppose that $I \neq \varnothing$. Let $(Y, \alpha_Y)$ be an $\mathcal{F}$-coalgebra and $\varphi_i : Y \to X_i$ be an upper $\mathcal{F}$-morphism. Then there is a unique map $\psi : Y \to \prod X_i$ in **Set** with $\psi\pi_i = \varphi_i$.




Let $y \in Y$ be given. We want to show that $y\alpha_Y\psi^{\mathcal{F}} \supseteq y\psi\alpha$. Let $U \in y\psi\alpha$. Then $U = \prod U_i$, where $U_i \in y\varphi_i\alpha_i$ and $U_i \neq X_i$ for all but finitely many indices $i$. Clearly $\prod X_i \in y\alpha_Y\psi^{\mathcal{F}}$. Assume that $U \neq \prod X_i$. Then there is a positive integer $n$ such that:

$$I_n \subseteq I\,;$$

$$|I_n| = n\,;$$

$$\forall j \in I_n\,,\ U_j \neq X_j\,;$$

$$\forall i \in (I \setminus I_n)\,,\ U_i = X_i\,.$$

For each $j \in I_n$,

$$y\alpha_Y \psi^{\mathcal{F}} \pi_j^{\mathcal{F}} = y\alpha_Y \varphi_j^{\mathcal{F}} \quad (\mathcal{F} \text{ is a functor and } \psi\pi_j = \varphi_j)$$
$$\supseteq y\varphi_j \alpha_j \quad (\varphi_j \text{ is an upper } \mathcal{F}\text{-morphism}).$$

Thus for each $j \in I_n$, we have $U_j \in y\alpha_Y \psi^{\mathcal{F}} \pi_j^{\mathcal{F}}$. Note that if $H$ is a filter on $\prod X_i$, then $H\pi_i$ is a filter on $X_i$. So $U_j \in y\alpha_Y \psi^{\mathcal{F}} \pi_j^{\mathcal{F}} = y\alpha_Y \psi^{\mathcal{F}} \pi_j$. Then for each $j \in I_n$, there exists a $V_j \in y\alpha_Y \psi^{\mathcal{F}}$ such that $U_j = V_j \pi_j$. Since $U \supseteq \bigcap_{j \in I_n} V_j \in y\alpha_Y \psi^{\mathcal{F}}$, we have $U \in y\alpha_Y \psi^{\mathcal{F}}$. Therefore $\psi$ is an upper $\mathcal{F}$-morphism. □

**Proposition 6.1.4.** Consider two upper $\mathcal{F}$-morphisms $f : (X, \alpha_X) \to (Z, \alpha_Z)$ and $g : (Y, \alpha_Y) \to (Z, \alpha_Z)$. Then there exists a pullback $(P, \alpha_P)$ in $\underleftarrow{\mathbf{Set}}_{\mathcal{F}}$ which is preserved by the underlying set functor. Its structure map $\alpha_P$ is given by

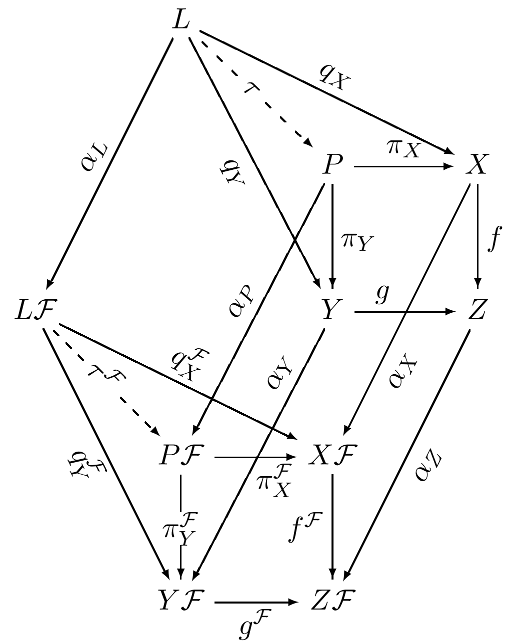
$$(x, y)\alpha_P = \uparrow \{(U \times V) \cap P \mid U \in x\alpha_X , V \in y\alpha_Y\} . \tag{6.1}$$

*Proof.* Let $P = \{(x, y) \in X \times Y \mid xf = yg\}$. We define a structure map $\alpha_P$ on $P$ by (6.1). First, we show that $\alpha_P$is well-defined. Put

$$S_{(x,y)} := \{(U \times V) \cap P \mid U \in x\alpha_X , V \in y\alpha_Y\} .$$

Since $P \in S_{(x,y)}$, the set $S_{(x,y)}$ is nonempty. For given $W_1$, $W_2 \in S_{(x,y)}$, there exist $U_1$, $U_2 \in x\alpha_X$ and $V_1$, $V_2 \in y\alpha_Y$ such that $W_1 = (U_1 \times V_1) \cap P$ and $W_2 = (U_2 \times V_2) \cap P$. Since $W_1 \cap W_2 = [(U_1 \cap U_2) \times (V_1 \cap V_2)] \cap P \in S_{(x,y)}$, the set $S_{(x,y)}$ is downward directed. Therefore $\alpha_P$ is well-defined. Now let $(L, \alpha_L)$ be an $\mathcal{F}$-coalgebra and $q_X : L \to X$ and $q_Y : L \to Y$ be upper $\mathcal{F}$-morphisms such that $q_X f = q_Y g$. Then there is a unique map $\tau : L \to P$ in **Set** with

$\tau\pi_X = q_X$ and $\tau\pi_Y = q_Y$.



For given $l \in L$, since $\tau\pi_X = q_X$ and $\tau\pi_Y = q_Y$, we have $l\tau = (lq_X, lq_Y)$. So

$$l\tau\alpha_P = \uparrow \{(U \times V) \cap P \mid U \in lq_X\alpha_X \,, V \in lq_Y\alpha_Y\} \,.$$

Let $W \in l\tau\alpha_P$. Then there exists $U \in lq_X\alpha_X$ and $V \in lq_Y\alpha_Y$ with $(U \times V) \cap P \subseteq W$. It suffices to show that $(U \times V) \cap P \in l\alpha_L\tau^{\mathcal{F}}$. Since $q_X$ is an upper $\mathcal{F}$-morphism,

$$U \in lq_X\alpha_X \subseteq l\alpha_L q_X^{\mathcal{F}} = l\alpha_L\tau^{\mathcal{F}}\pi_X^{\mathcal{F}} \,.$$

Since $l\alpha_L\tau^{\mathcal{F}}$ is a filter on $P$, there exists $W_1 \in l\alpha_L\tau^{\mathcal{F}}$ such that $W_1\pi_X \subseteq U$. Similarly, there exists $W_2 \in l\alpha_L\tau^{\mathcal{F}}$ such that $W_2\pi_Y \subseteq V$. Since

$$W_1 \cap W_2 \subseteq (U \times V) \cap P \,,$$

we have $(U \times V) \cap P \in l\alpha_L\tau^{\mathcal{F}}$. Therefore $\tau$ is an upper $\mathcal{F}$-morphism. □

By Propositions 6.1.3 and 6.1.4, we obtain the following.

**Theorem 6.1.5.** $\underleftarrow{\textbf{Set}}_{\mathcal{F}}$ is complete.

**Lemma 6.1.6.** Let $g : X \rightarrowtail Y$ be an injective map and let $(G_i)_{i\in I}$ be a class of filters on $X$. Then

$$\bigcap_{i\in I} \uparrow (G_i g) \;\subseteq\; \uparrow\Big(\bigcap_{i\in I} G_i g\Big) \,.$$

*Proof.* Since $g$ is injective, $G_i g$ is a filter on $Xg$ for each $i \in I$. For a given element $U$ of $\bigcap_{i\in I} \uparrow(G_i g)$, there is $V_i \in G_i g$ such that $V_i \subseteq U \cap Xg$ for each $i \in I$. Since $G_i g$ is a filter on $Xg$, we have $U \cap Xg \in \bigcap_{i\in I} G_i g$. Thus $U \in \uparrow\left(\bigcap_{i\in I} G_i g\right)$. □

**Proposition 6.1.7.** $\underleftarrow{\mathbf{Set}}_{\mathcal{F}}$ is weakly $SI$-factorizable.

*Proof.* Let $\varphi : (X, \alpha_X) \to (Y, \alpha_Y)$ be an upper $\mathcal{F}$-morphism. For a given set $Z$, let $f : X \twoheadrightarrow Z$ and $g : Z \rightarrowtail Y$ be an $SI$-factorization with $fg = \varphi$. For a given $z \in Z$, we define a structure map $\alpha$ on $Z$ by

$$z\alpha = \bigcap_{xf=z} x\alpha_X f^{\mathcal{F}} .$$

Then $\forall x \in X$, $xf\alpha \subseteq x\alpha_X f^{\mathcal{F}}$. Thus $f$ is an upper $\mathcal{F}$-morphism. Now let $z \in Z$. Since $\varphi$ is an upper $\mathcal{F}$-morphism, for every $x \in X$ with $xf = z$,

$$zg\alpha_Y = xfg\alpha_Y \subseteq x\alpha_X \varphi^{\mathcal{F}} = x\alpha_X f^{\mathcal{F}} g^{\mathcal{F}} \ldots$$

So $zg\alpha_Y \subseteq \bigcap_{xf=z}(x\alpha_X f^{\mathcal{F}} g^{\mathcal{F}}) = \bigcap_{xf=z} \uparrow (x\alpha_X f^{\mathcal{F}} g)$. By Lemma 6.1.6,

$$\begin{aligned}\bigcap_{xf=z} \uparrow (x\alpha_X f^{\mathcal{F}} g) &\subseteq \uparrow \Big[\bigcap_{xf=z} (x\alpha_X f^{\mathcal{F}} g)\Big] \\ &= \uparrow \Big[\Big(\bigcap_{xf=z} x\alpha_X f^{\mathcal{F}}\Big) g\Big] \qquad (g \text{ is injective}) \\ &= \Big(\bigcap_{xf=z} x\alpha_X f^{\mathcal{F}}\Big) g^{\mathcal{F}} = z\alpha g^{\mathcal{F}} .\end{aligned}$$

Therefore $g$ is an upper $\mathcal{F}$-morphism. □

By Theorem 4.2.6, Theorem 6.1.5, and Proposition 6.1.7, we obtain the following.

**Corollary 6.1.8.** Each weak coquasivariety of $\underleftarrow{\mathbf{Set}}_{\mathcal{F}}$ is complete.

## 6.2 Topological coalgebras with upper morphisms

Denote the full subcategory of $\underleftarrow{\mathbf{Set}}_{\mathcal{F}}$ with the object class $\mathcal{T}$ by $\underleftarrow{\mathbf{Tp}}$. We show that weak coalgebra homomorphisms coincide with continuous maps. Based on this, we derive the equivalence between **Top** and $\underleftarrow{\mathbf{Tp}}$. Here is the main theorem.

**Theorem 6.2.1.** Suppose that $(X, \tau_X)$ and $(Y, \tau_Y)$ are topological spaces. Consider a function $f : X \to Y$. Then $f : (X, U_{\tau_X}) \to (Y, U_{\tau_Y})$ is an upper $\mathcal{F}$-morphism if and only if $f : (X, \tau_X) \to (Y, \tau_Y)$ is continuous.

*Proof.* Suppose that $f : (X, \tau_X) \to (Y, \tau_Y)$ is continuous. Let $x \in X$ and let $V \subseteq Y$. If $V \in xfU_{\tau_Y}$, then there exists an open set $U \in \tau_Y$ such that $xf \in U \subseteq V$. Since $f$ is continuous, $x \in f^{-1}(U) \in \tau_X$. Since $f^{-1}(U) \subseteq f^{-1}(V)$, we have $f^{-1}(V) \in xU_{\tau_X}$. Therefore $f$ is an upper $\mathcal{F}$-morphism by Lemma 6.1.1. Now suppose that $f$ is an upper $\mathcal{F}$-morphism. Let $V \in \tau_Y$. For any $x \in f^{-1}(V)$, there exists $U_x \in \tau_X$ such that $x \in U_x \subseteq f^{-1}(V)$ by Lemma 6.1.1. Since $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$, we have $f^{-1}(V) \in \tau_X$. Hence $f$ is continuous. □

**Corollary 6.2.2.** Let $(X, \tau_X)$ be a topological space, with corresponding coalgebra $(X, U_{\tau_X})$. If $S$ is a subset of $X$, then it is a subcoalgebra.

*Proof.* Let $\tau_S = \{S \cap U \mid U \in \tau_X\}$ be the subspace topology. Then by Theorem 6.2.1, $(S, U_{\tau_S}) \leq_w (X, U_{\tau_X})$. □

Theorem 6.2.1 indicates that we have appropriately relaxed the homomorphism concept. The bicompleteness of $\underleftarrow{\mathbf{Tp}}$ will follow. It is easy to check that

$$G : \mathbf{Top} \to \underleftarrow{\mathbf{Tp}}, \tag{6.2}$$

defined by $(X, \tau)G = (X, U_\tau)$ for $(X, \tau) \in \mathrm{Ob}(\mathbf{Top})$ and $f^G = f$ for $f \in \mathrm{Mor}(\mathbf{Top})$, is a functor. Indeed $G$ is an equivalence.

**Proposition 6.2.3.** The functor $G : \mathbf{Top} \to \underleftarrow{\mathbf{Tp}}$ of (6.2) provides an equivalence between the category of continuous maps and the category of upper $\mathcal{F}$-morphisms for topological coalgebras.

*Proof.* Clearly, $G$ is faithful and dense. By Theorem 6.2.1, $G$ is full. Therefore $G$ is an equivalence. □

Note that **Top** is bicomplete. By Proposition 6.2.3, we obtain the following.

**Corollary 6.2.4.** $\underleftarrow{\mathbf{Tp}}$ is bicomplete.

## 6.3 Closure properties

Recall that a weak coquasivariety of a complete weak category is again complete. Since $\underleftarrow{\mathbf{Set}}_{\mathcal{F}}$ is complete, it is natural to ask whether $\mathcal{T}$ forms a weak coquasivariety. First we consider the closure property under subcoalgebras. Let $(\{0,1\},\tau)$ be the topological space with $\tau = \{\varnothing, \{0,1\}\}$. Then we obtain the $\mathcal{F}$-coalgebra $(\{0,1\}, U_\tau) \in \mathcal{T}$, with

$$0U_\tau = 1U_\tau = \{\{0,1\}\}\,.$$

Let $(\{0,1\},\beta)$ be the $\mathcal{F}$-coalgebra with $0\beta = \{\{1\},\{0,1\}\}$ and $1\beta = \{\{0\},\{0,1\}\}$. Then $(\{0,1\},\beta) \leq_w (\{0,1\},U_\tau)$. However, $(\{0,1\},\beta) \notin \mathcal{T}$. As a result, we obtain the following proposition.

**Proposition 6.3.1.** The class $\mathcal{T}$ is not closed under subcoalgebras. Hence it does not form a covariety over $\underleftarrow{\mathbf{Set}}_{\mathcal{F}}$.

Consider $(X = \{1,2,3,4\}, U_\tau) \in \mathcal{T}$ with the topology

$$\tau = \{\varnothing, \{1\}, \{1,3\}, \{1,2,4\}, X\}\,.$$

Let $(Y = \{a,b,c\},\beta) \in \underleftarrow{\mathbf{Set}}_{\mathcal{F}}$ such that $a\beta = \{\{a,c\},Y\}$, $b\beta = \{\{a,b\},Y\}$, and $c\beta = \{Y\}$. Then the function $f : X \to Y$ defined by $1f = 2f = a$, $3f = b$, and $4f = c$ is a surjective upper $\mathcal{F}$-morphism. However, $(Y,\beta) \notin \mathcal{T}$. Thus $\mathcal{T}$ is not closed under $\mathsf{H}$, and is not a coquasivariety. Now we define a topology $\tau_Y$ on $Y$ by

$$\{V \subseteq Y \ \mid\ Vf^{-1} \in \tau\} = \{\varnothing, \{a,c\}, Y\}\,.$$

Then it is easy to see that the topology $\tau_Y$ is the finest topology so that $f$ becomes a continuous function. However, $(Y, U_{\tau_Y}) \nleq_w (Y,\beta)$. As a result, we obtain the following proposition.

**Proposition 6.3.2.** $\mathcal{T}$ does not form a weak coquasivariety over $\underleftarrow{\mathbf{Set}}_{\mathcal{F}}$.

# CHAPTER 7. Future research problems

## 7.1 Almost homomorphisms

For given sets $A$ and $B$, if $A \subseteq B$ and $|B \setminus A| < \infty$, then we denote this containment by $A \subseteq_{cofin} B$. Denote the symmetric difference between $A$ and $B$ by $A \triangle B$, i.e.

$$A \triangle B = (A \cup B) \setminus (A \cap B)\,.$$

Let $T : \mathbf{Set} \to \mathbf{Set}$ be a monotonic endofunctor. Then $T$ is said to be a *strictly monotonic endofunctor* if for a given map $f : X \to Y$ and for given sets $A$, $B \in XT$, the containment $A \subseteq_{cofin} B$ implies $Af^T \subseteq_{cofin} Bf^T$.

**Definition 7.1.1.** Let $(X, \alpha)$ and $(Y, \beta)$ be $T$-coalgebras for a strictly monotonic endofunctor $T : \mathbf{Set} \to \mathbf{Set}$. Then an *almost $T$-homomorphism* from $(X, \alpha)$ to $(Y, \beta)$ is a map $f : X \to Y$ such that for each $x \in X$, we have

$$|x\alpha f^T \triangle x f \beta| < \infty\,.$$

It can be readily seen that the class of all $T$-coalgebras forms a category denoted by $\underleftrightarrow{\mathbf{Set}}_T$ with almost $T$-homomorphisms. Furthermore, we have the following result.

**Theorem 7.1.2.** For a strictly monotonic endofunctor $T$ on the category of sets, the class of all almost $T$-homomorphisms is weakly closed.

In Chapter 5 and 6, we have seen how lower and upper morphisms work nicely in mathematical structures. It would be interesting and worth to find some mathematical structures in which almost homomorphisms give a nice relationship between the corresponding coalgebras. Another interesting question would be an interpretation of almost homomorphisms into transition systems or automata.

## 7.2 Generalization of graph minors

A *quasi-order* $(Q, \leq)$ is a class $Q$ together with a transitive and reflexive relation $\leq$.

**Definition 7.2.1.** Let $(Q, \leq)$ be a quasi-order and let $H$ be a subclass of $Q$. Then $H$ is called *hereditary* if $y \leq x$ implies $y \in H$ for every $x \in H$. If $x \leq y$ implies $y \in H$ for every $x \in H$, then $H$ is said to be *monotone*.

For a given subclass $H$ of a quasi-order $(Q, \leq)$, $H^c := Q \setminus H$.

**Proposition 7.2.2.** Let $H$ be a subclass of a quasi-order $(Q, \leq)$. Then $H$ is hereditary if and only if $H^c$ is monotone.

*Proof.* If $H = Q$, then $H^c = \varnothing$ is monotone and $H$ is hereditary. So assume that $H$ is a proper subclass of $Q$. Suppose that $H$ is hereditary. Let $x \in H^c$ and let $x \leq y$. If $y \notin H^c$, then $y \in H$. Since $H$ is hereditary, $x \in H$, which is a contradiction. Therefore $y \in H^c$ and $H^c$ is monotone. Now assume that $H^c$ is monotone. Let $x \in H$ and let $y \leq x$. If $y \notin H$, then $y \in H^c$. Since $H^c$ is monotone, $x \in H^c$, which is a contradiction. Therefore $H$ is hereditary. □

Let $H$ be a subclass of a quasi-order $(Q, \leq)$. We define

$$\uparrow H := \{q \in Q \mid \exists x \in H \,\bullet\, x \leq q\}.$$

**Definition 7.2.3.** Let $H$ be a subclass of a quasi-order $(Q, \leq)$. A *basis* for $H$ is a subclass $\mathcal{B}$ of $H$ satisfying $\uparrow \mathcal{B} = H$.

It is natural to ask when a given subclass of a quasi-order has a basis.

**Proposition 7.2.4.** Let $H$ be a subclass of a quasi-order $(Q, \leq)$. Then $H$ is monotone if and only if $H$ has a basis.

*Proof.* Suppose that $H$ is monotone. Then $\uparrow H = H$. Now suppose that $H$ has a basis $\mathcal{B}$. Let $x \in H$ and let $x \leq y$. Then there exists $b \in \mathcal{B}$ such that $b \leq x \leq y$. By the definition of basis, $y \in H$. Therefore $H$ is monotone. □

Let $H$ be a subclass of a quasi-order $(Q, \leq)$. If $H$ is hereditary, then $H$ can be characterized by excluding some members of $Q$ by Proposition 7.2.2 and 7.2.4. $H$ is called an *antichain* if for any distinct $a$, $b \in H$, $a \not\leq b$ and $a \not\leq b$. If $H$ has a basis, then we are interested in finding a basis as small as possible. In particular, if we have a finite basis $\mathcal{B}$, then $\mathcal{B}$ can be reduced to an antichain basis for $H$. In 1952, G. Higman gave a necessary and sufficient condition when a basis can be reduced to a finite basis [11]. We state this result in the rest of this section.

A quasi-order $(Q, \leq)$ is a *well-quasi-order* if for every countable sequence $q_1, q_2, \ldots$ of members of $Q$, there exist $1 \leq i < j$ such that $q_i \leq q_j$.

**Definition 7.2.5.** A quasi-order $(Q, \leq)$ is *well-founded* if and only if there is no infinitely descending chain.

**Theorem 7.2.6.** [11] Let $(Q, \leq)$ be a quasi-order. Then the following are equivalent.

1. $Q$ is well-quasi-ordered.

2. $Q$ is well-founded and has no infinite antichains.

3. For any subclass $H$ of $Q$, $\uparrow H$ has a finite basis.

In Mathematics, to find a well-quasi-order having nice structure relationship or to determine if a given quasi-order is well-quasi-ordered have been important but not easy problems. The minor order on the finite undirected graphs is a good example. It was known that the minor order is a qusi-order. The statement that the minor order is a well-quasi-order was formulated as a conjecture by Klaus Wagner in 1937, and was called Wagner's conjecture until it was proved by Neil Robertson and Paul D. Seymour, who published its proof in a series of twenty papers from 1983 to 2004.

As the class of all lower morphisms contains the class of the finite undirected graphs, it would be an important future research problem to find a generalization of graph minor order and its purely categorical description.

## BIBLIOGRAPHY


[1] M. Barr, *Terminal coalgebras in well-founded set theory*, Theoretical Computer Science, (1993), no. 114(2), 299-315.

[2] K.O. Chung, *Weak Homomorphisms of Coalgebras*, Preprint, 2006.

[3] K.O. Chung, *Graphic Coalgebras*, In Preparation.

[4] K.O. Chung and J.D.H. Smith, *Homomorphisms for topological coalgebras*, In Preparation.

[5] A. Corradini, M. Gross-Rhode, R. Heckel, *A coalgebraic presentation of structured Transition systems*, Theoretical Computer Science, (2001), no. 260(1-2), 27-55.

[6] H.P. Gumm, *Elements of the general theory of coalgebras*, LUATCS 99, Rand Afrikaans University, Johannesburg, South Africa, 1999.

[7] H.P. Gumm, *Functors for Coalgebras*, Algebra Universalis, 45 (2001), 135-147.

[8] H.P. Gumm and T. Schröder, *Products of Coalgebras*, Algebra Universalis, 46 (2001), 163-185.

[9] P. Hell, *Rétractions de graphes*, Ph.D. thesis, Université de Montréal, 1972.

[10] H. Herrlich, G.E. Strecker, *Category Theory*, Heldermann Verlag, Berlin, 2nd edition, 1979.

[11] G. Higman, *Ordering by divisibility in abstract algebra*, Proc. London Math. Soc., 2 (1952), 326-336.

[12] J. Hughes and B. Jacobs, *Simulations in Coalgebra*, Theoretical Computer Science, (2004), no. 327(1-2), 71-108.

[13] A.B. Romanowska, J.D.H. Smith, *Modal Theory*, Heldermann Verlag, Berlin, 1985.

[14] J.D.H. Smith and A.B. Romanowska, *Modes*, World Scientific, Singapore, 2002.

## ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to my advisor, Prof. Jonathan D.H. Smith for his expert advice and guidance throughout this research and its writing. I would also like to thank my committee members for their time and efforts reading this thesis: Dr. Fritz Keinert, Dr. Ling Long, Dr. Roger Maddux, and Dr. Sung-Yell Song. I would additionally like to thank Prof. Hong Goo Park for his moral support and encouragement throughout my graduate career.

I am grateful to my parents, sister, wife's family members, especially parents-in-law for their encouragement and love to me. Special thanks are due to my lovely wife, Ju Hye Lee. I want to express my deepest thanks to her for her love and support throughout my career in graduate school. My son, Byungseo has been the hope and bliss of my life. I appreciate their endless love for me.