Nonseparable multivariate wavelets

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To my daughter Aastha

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ABSTRACT

We review the one-dimensional setting of wavelet theory and generalize it to nonseparable multivariate wavelets. This process presents significant technical difficulties. Some techniques of the one-dimensional setting carry over in a more or less straightforward way; some do not generalize at all.

The main results include the following: an algorithm for computing the moments for multivariate multiwavelets; a necessary and sufficient condition for the approximation order; the lifting scheme for multivariate wavelets; and a generalization of the method of Lai [12] for the biorthogonal completion of a polyphase matrix under suitable conditions.

One-dimensional techniques which cannot be generalized include the factorization of the polyphase matrix, and a general solution to the completion problem.

CHAPTER 1. Introduction

Wavelet bases have proved useful for a number of applications in signal processing, numerical analysis, operator theory, and other fields like physics, engineering etc. [9]. In the one-dimensional setting, a refinable function is the solution of the two scale recursion relation

$$\phi(x) = \sqrt{2}\sum_k h_k \phi(2x-k),$$

where h_k are called the filter coefficients. We assume that only finitely many coefficients are nonzero; this produces ϕ with compact support. Under some additional conditions, ϕ gives rise to a multiresolution approximation and a wavelet function ψ. The wavelet function ψ has the property that the family of functions

$$\psi_{j,k}(x) = 2^{-j/2}\psi(2^j x - k)$$

forms an orthonormal basis for $L^2(\mathbb{R})$. Orthogonality, compactness of support, approximation order, vanishing moments, symmetry, smoothness, and decay are some of the important properties of a wavelet. In the one-dimensional setting, it is possible to construct wavelets with desirable specific properties. For example, the Daubechies family of wavelets [9] provides a very good example of wavelets with compact support and arbitrary regularity.

There are several applications that require a higher-dimensional setting, for example, image processing. A natural approach to generalize the idea to higher dimensions is through tensor products of one dimensional wavelets. This approach, however, has a major drawback: it favors the horizontal and vertical directions. The most general approach is to dilate by using an expansive integer matrix M which maps the integer lattice $\Gamma = \mathbb{Z}^d$ into a sublattice [1], [3]. We have the following recursion relation in the higher dimensional setting.

$$\boldsymbol{\phi}(\mathbf{x}) = \sqrt{m}\sum_{\mathbf{k}} h_{\mathbf{k}} \boldsymbol{\phi}(M\mathbf{x} - \mathbf{k}),$$

where $h_{\mathbf{k}}$ are $r\times r$ matrices and $\boldsymbol{\phi}:\mathbb{R}^d\to\mathbb{C}^r$. It is known that $(m-1)r$ wavelets are needed to generate $L^2(\mathbb{R}^d)$ [8], [1] where $m=|\det M|$. An orthonormal wavelet set associated with the dilation matrix M is a finite set $\psi^{(i)},\ i=1,2,\ldots,(m-1)r$, such that $m^{-j/2}\psi^{(i)}(M^j\mathbf{x}-\mathbf{k})$, $i=1,\ldots,(m-1)r,\ j\in Z,\ \mathbf{k}\in\mathbb{Z}^d$ forms an orthogonal basis for $L^2(\mathbb{R}^d)$. If $r=1$, we call it a scalar scaling function or a scalar wavelet.

Unfortunately, the techniques of the one-dimensional setting cannot be applied to the higher dimensional setting. The first reason is that given a $d-$variable scaling function it is difficult to construct $d-$variable wavelets. The second reason is that one cannot necessarily factor a multivariate trigonometric polynomial. The third reason is that it is hard to generalize the Fejer-Riesz lemma to multivatiate trigonometric polynomials. Also, we lack tools to investigate their properties.

Cohen and Daubechies [8] constructed nonseparable orthonormal wavelets for the class of dilation with $m=|\det M|=2$ with an arbitrary number of vanishing moments. These turned out to be discontinuous. In the same paper, however, they present an example of arbitrarily smooth biorthogonal nonseparable wavelet basis for the quincunx dilation. Kovacevic and Vetterli in [3] constructed a continuous, compactly supported scaling function (K-V scaling function) in $\mathbb{R}^2$ using the standard quincunx as the dilation matrix. Madych and Gröchenig in [5] constructed several nonseparable Haar-type scaling functions in $\mathbb{R}^d$ which are discontinuous characteristic functions of compact sets. It has also been shown by Belogay and Wang in [20] that there exists a family of compactly supported scaling function on $\mathbb{R}^2$ with arbitrary smoothness that are refinable with respect to a matrix that gives the column lattice, see section (3.1). He and Lai in [13] have several other examples. Although many special multivariate nonseparable wavelets have been constructed, it is still an open problem how to construct multivariate compactly supported orthogonal wavelets for any given compactly supported scaling function.

The construction of wavelets can be put in terms of a matrix completion problem [11], where the first row is given and we seek a paraunitary completion of it. In particular, if a wavelet basis exists, it is required to have $(m-1)r$ wavelet functions. It has been proved that under certain conditions the completion can be done [21], [17] but no constructive method has be suggested so far. Even under these additional conditions the completion does not necessarily preserve the orthogonality, compactness of the support, or the regularity. In [17], Judith and Marc showed that, in general, it is not always possible to obtain wavelets that correspond to a given scaling function with desirable properties. They started with a particular continuous scaling function and showed that there is no continuous wavelet that goes with it.

Similarly, methods for computing moments and approximation orders in higher dimensions are not very clear. In the one-dimensional case, a wavelet has p vanishing moments iff the corresponding scaling function has approximation order p, and the moments are fairly easy to compute. In the multivariate case, the calculation of moments becomes much more complicated. Using the notations of C. Heil from [2],[1], we have worked out an algorithm for computing moments in chapter 5. The connection between vanishing moments and approximation order is likewise much more complex in the multivariate case. We have established theorems corresponding to the scalar results.

Lifting, a procedure for constructing wavelets with desired properties, such as approximation order and symmetry, from simpler wavelets, is well studied in the one-dimensional case [15], [16], and Keinert generalized this idea to the case of multiwavelets in his paper "Raising multiwavelet approximation order through lifting" [7]. The idea of lifting wavelets [15] for higher approximation order does apply to the higher dimensional setting (chapter 6). We have generalized the lifting procedure to multivariate wavelets, and derived the conditions necessary to raise the approximation order.

In the scalar case, the polyphase matrix of any orthogonal wavelet can be factored into easier terms, either lifting factors or projection factors. Keinert [6] recently wrote a book "Wavelets and Multiwavelets" [6] where lifting and the projection factors are described in the multiwavelet setting. This does not seem to be possible in general in the multivariate case. While such factorizations may not be possible in general, it may be possible under some conditions on the dilation matrix and on the placement of the filter coefficients. This case needs to be investigated.

Generalizing the method of Lai [12], under some extra conditions, we show (chapter 8) that the completion can be done so as to have compactly supported wavelets. The completion we obtain gives us wavelets with the same regularity as the scaling function, and the support remains compact, although it gets bigger than the support of the scaling function.

CHAPTER 2. Scalar Wavelet Theory

Definition 1 A *refinable function* is a function $\phi : \mathbb{R} \to \mathbb{C}$ which satisfies a *two-scale refinement equation* or *recursion relation* of the form

$$\phi(x) = \sqrt{2} \sum_k h_k \phi(2x - k),$$

where $\{h_k\}_{k\in\mathbb{Z}} \in l^2(\mathbb{Z})$ are called the *recursion coefficients.*

Our special interest are the functions that have compact support, which implies that the $\{h_k\}$ are finitely supported. The refinable function ϕ is called *orthogonal* if

$$< \phi(x), \phi(x-k) >= \delta_{0,k},\ \ k \in \mathbb{Z}.$$

Two refinable functions ϕ and $\tilde{\phi}$ are called *biorthogonal* if

$$< \phi(x), \tilde{\phi}(x-k) >= \delta_{0,k},\ \ k \in \mathbb{Z}.$$

We also call $\tilde{\phi}$ the *dual* of ϕ.

Example 1: Haar scaling function h:

$$\phi(x) = \begin{cases} 1 & \text{if } 0 \le x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

The recursion coefficients for h are $h_0 = h_1 = 1/\sqrt{2}$.

Example 2: Scaling function for the Daubechies wavelet with two vanishing moment $db2$. The recursion coefficients are given by

$$h_0 = \frac{1+\sqrt{3}}{4\sqrt{2}},\ h_1 = \frac{3+\sqrt{3}}{4\sqrt{2}},\ h_2 = \frac{3-\sqrt{3}}{4\sqrt{2}},\ h_3 = \frac{1-\sqrt{3}}{4\sqrt{2}}.$$

Theorem 1 A necessary condition for orthogonality is

$$\sum_k h_k h_{k-2l}^* = \delta_{0,l}, \tag{2.1}$$

where the * denotes the complex conjugate. Similarly, a necessary condition for biorthogonality is

$$\sum_k h_k \tilde{h}_{k-2l}^* = \delta_{0,l}.$$

Proof: This is proved in [6].

However, this condition is not sufficient to ensure orthogonality. For example

$$\phi(x) = \phi(2x) + \phi(2x-3)$$

satisfies the orthogonality condition (2.1), but its solution $\phi(x) = \chi_{[0,3)}$ does not have orthogonal integer translates. There are several sufficient conditions, for example *the convergence of cascade algorithm* [6],[9] (see next section).

2.1 Computing Point Values

The *cascade algorithm* is fixed point iteration applied to the refinement equation. It can be used to find approximate point values.

Definition 2 The cascade algorithm consists of selecting a suitable starting function $\phi^{(0)}(x) \in L_2$, and then producing a sequence of functions

$$\phi^{(n+1)}(x) = \sqrt{2} \sum_k h_k \phi^{(n)}(2x-k).$$

Theorem 2 If the cascade algorithm converges for both ϕ and $\tilde{\phi}$, then the necessary condition for the orthogonality is also the sufficient condition.

Proof: See [10].

The point values of the scaling function can also be obtained by solving an eigenvalue problem. It usually works for continuous ϕ, but may fail in some cases. The refinement equation for integer points is equivalent to an eigenvalue problem

$$\phi = T\phi.$$

Example: The Daubechies scaling function $db2$ has support $[0,3]$.

The recursion relation for the integer points in the support leads to

$$\begin{bmatrix} \phi(0) \\ \phi(1) \\ \phi(2) \\ \phi(3) \end{bmatrix} = \sqrt{2} \begin{bmatrix} h_0 & 0 & 0 & 0 \\ h_2 & h_1 & h_0 & 0 \\ 0 & h_3 & h_2 & h_1 \\ 0 & 0 & 0 & h_3 \end{bmatrix} \begin{bmatrix} \phi(0) \\ \phi(1) \\ \phi(2) \\ \phi(3) \end{bmatrix}.$$

Since h_0, h_3 are not $1/\sqrt{2}$, we know that $\phi(0) = \phi(3) = 0$, and the above problem reduces to

$$\begin{bmatrix} \phi(1) \\ \phi(2) \end{bmatrix} = \sqrt{2} \begin{bmatrix} h_1 & h_0 \\ h_3 & h_2 \end{bmatrix} \begin{bmatrix} \phi(1) \\ \phi(2) \end{bmatrix}.$$

The solution, normalized to $\phi(1) + \phi(2) = 1$, is

$$\phi(1) = \frac{1+\sqrt{3}}{2}, \quad \phi(2) = \frac{1-\sqrt{3}}{2}.$$

Then, we can use the refinement equation to obtain values at half integers, quarter integers, and so on.

2.2 Multiresolution Analysis

Multiresolution analysis (MRA) forms the most important concept for the construction of the scaling function, wavelets and the development of the algorithms. Multiresolution analysis can be viewed as a sequence of approximations of a given function at different resolutions.

Definition 3 A *multiresolution analysis* on $\mathbb{R}$ is a doubly infinite nested sequence of subspaces $\{V_j\}$ of $L_2(\mathbb{R})$

$$\cdots V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots$$

with properties

(i) $\mathrm{clos}_{L_2}(\bigcup_{j\in\mathbb{Z}} V_j) = L_2(\mathbb{R})$.

(ii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.

(iii) $\phi(x) \in V_j$ if and only if $\phi(2x) \in V_{j+1}$.

(iv) $\phi(x) \in V_j \Rightarrow \phi(x - 2^{-j}k) \in V_j$, for all j and $k \in \mathbb{Z}$.

(v) There exists a function $\phi(x) \in L_2(\mathbb{R})$ called the *scaling function* such that $\{\phi(x-k), k \in \mathbb{Z}\}$

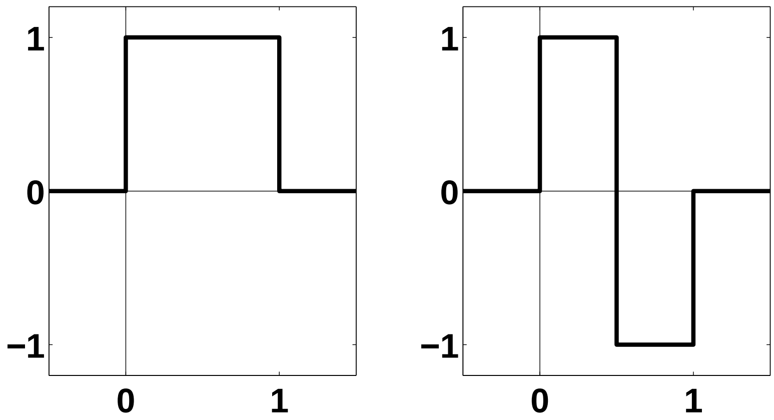

Figure 2.1 Haar scaling function and Haar wavelet

forms a *Riesz basis* for V_0. That is, for every $f \in V_0$ there exists a unique sequence $\{\alpha_n\}_{n\in\mathbb{Z}}$ such that

$$f(x) = \sum_{n\in\mathbb{Z}} \alpha_n \phi(x-n)$$

with convergence in $L_2(\mathbb{R})$, and

$$A \sum_{n\in\mathbb{Z}} |\alpha_n|^2 \le || \sum_{n\in\mathbb{Z}} \alpha_n \phi(x-n)||^2 \le B \sum_{n\in\mathbb{Z}} |\alpha_n|^2$$

with $0 < A \le B < \infty$ constants independent of $f \in V_0$.

Notation: $\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)$.

Since $V_0 \subset V_1$, and since $\{\phi_{1,k}(x)\}_{k\in\mathbb{Z}}$ is a basis for V_1,

$$\phi(x) = \sqrt{2}\sum_k h_k \phi(2x-k)$$

for some h_k. That is, ϕ is refinable.

Example: The Haar scaling function h and the Daubechies scaling function $db2$ both define MRAs and have compact support.

Suppose that $\phi \in L_2(\mathbb{R})$ is a scaling function which generates an MRA $\{V_j\}$. One can show, under some mild conditions, that there exists a *dual scaling function* $\tilde{\phi} \in L_2(\mathbb{R})$, which

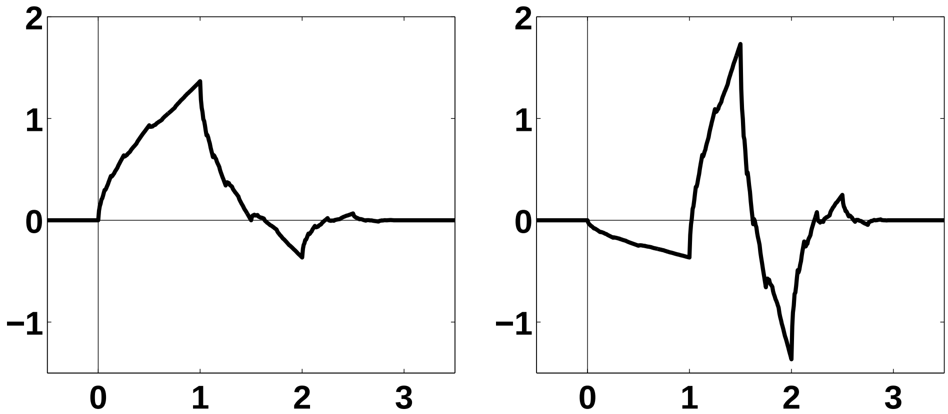

Figure 2.2 Scaling function for db2 wavelet and db2 wavelet

satisfies the biorthogonality relations

$$< \phi(x), \tilde{\phi}(x-k) >= \delta_{0,k}, \ k \in \mathbb{Z},$$

and which generates a dual MRA $\{\tilde{V}_j\}$.

The wavelet space W_0 (resp. $\tilde{W}_0$) is the complement of V_0 (resp. $\tilde{V}_0$) in V_1 (resp. $\tilde{V}_1$) such that

$$V_0 \cap W_0 = \{0\}, \ \ V_1 = V_0 \oplus W_0, \ \text{ and } \ \tilde{V}_0 \cap \tilde{W}_0 = \{0\}, \ \ \tilde{V}_1 = \tilde{V}_0 \oplus \tilde{W}_0.$$

Under mild conditions, the space W_0 (resp. $\tilde{W}_0$) is generated by the integer translates of a function $\psi \in L_2(\mathbb{R})$ (resp. $\tilde{\psi} \in L_2(\mathbb{R})$). ψ (resp. $\tilde{\psi}$) is called the wavelet (dual wavelet) function. It satisfies a two scale relation

$$\psi(x) = \sqrt{2} \sum_k g_k \phi(2x - k)$$

or

$$\tilde{\psi}(x) = \sqrt{2} \sum_k \tilde{g}_k \tilde{\phi}(2x - k)$$

for some g_k or $\tilde{g}_k$.

Note: We also call $\{h_k\}$ the *scaling filter* or the *lowpass filter* and $\{g_k\}$ the *wavelet filter* or the *highpass filter*.

Example: Haar wavelet h.

$$\psi(x) = \begin{cases} 1 & \text{if } 0 \le x < 1/2, \\ -1 & \text{if } 1/2 \le x < 1. \end{cases}$$

This obeys the following two scale relations

$$\phi(x) = \phi(2x) + \phi(2x-1),$$

$$\psi(x) = \phi(2x) - \phi(2x-1).$$

Here $h_1 = \frac{1}{\sqrt{2}}, h_2 = \frac{1}{\sqrt{2}}, g_1 = \frac{1}{\sqrt{2}}$ and $g_2 = -\frac{1}{\sqrt{2}}$.

We summarize the properties of orthogonal wavelets as follows.

Theorem 3 Let $\{V_j\}$ be an orthogonal MRA with *scaling filter* h_k and *wavelet filter* g_k. Then

(i) $\sum\limits_k h_k = \sqrt{2}$;

(ii) $\sum\limits_k g_k = 0$;

(iii) $\sum\limits_k h_k h^*_{k-2n} = \sum\limits_k g_k g^*_{k-2n} = \delta_n$;

(iv) $\sum\limits_k g_k h^*_{k-2n} = 0, \quad \text{for all } n \in \mathbb{Z}$;

(v) $\sum\limits_k h^*_{m-2k} h_{n-2k} + \sum\limits_k g^*_{m-2k} g_{n-2k} = \delta_{n-m}$.

Note: Condition (i) is referred to as a *normalization condition.* Condition (iii) and (iv) are referred to as *orthogonality conditions.* Condition (v) is referred to as the *perfect reconstruction condition.*

Proof: This is proved in [14].

2.3 Decomposition and Reconstruction

Given a function $f(x) \in L_2(\mathbb{R})$, define for $k, j \in \mathbb{Z}$,

$$c_{j,k} = < f, \phi_{j,k} > \quad \text{and} \quad d_{j,k} = < f, \psi_{j,k} > .$$

Since we have $V_1 = V_0 \oplus W_0$, at level n we have,

$$\begin{aligned} V_n &= V_{n-1} \oplus W_{n-1} \\ &= V_{n-2} \oplus W_{n-2} \oplus W_{n-1} \\ &= \cdots \\ &= V_0 \oplus W_1 \oplus \cdots \oplus W_{n-1}. \end{aligned}$$

We have the following theorem.

Theorem 4

$$V_n = \bigoplus_{k=-\infty}^{n-1} W_k,$$

which implies that

$$L_2(\mathbb{R}) = \bigoplus_{k=-\infty}^{\infty} W_k.$$

Proof: This is proved in [6].

Theorem 5 Let V_j be an orthogonal MRA with scaling function ϕ and wavelet ψ. Let h_k and g_k be the corresponding recursion coefficients. Then the decomposition relations are given by

$$c_{j-1,k} = \sum_n c_{j,n} h^*_{n-2k},$$
$$d_{j-1,k} = \sum_n c_{j,n} g^*_{n-2k},$$

and the reconstruction relations are given by

$$c_{j,k} = \sum_n c_{j-1,n} h_{k-2n} + \sum_n d_{j-1,n} g_{k-2n}.$$

Proof: This is proved in [14].

Definition 4 Let c_k be the signal. The *downsampling operator* $\downarrow 2$ is defined by

$$(\downarrow 2c)_k = c_{2k}.$$

The *upsampling operator* $\uparrow 2$ is defined by

$$(\uparrow 2c)_k = \begin{cases} c_{k/2} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

The downsampling is obtained by removing every odd term in c_k, and the upsampling is obtained by inserting a zero between adjacent entries of c_k. Decomposition can be thought of as *convolution* with filters h^*_{-n},and g^*_{-n} respectively, followed by downsampling, that is

$$c_{0,k} = \downarrow 2(c_{1,n} * h^*_{-n})_k,$$

and

$$d_{0,k} = \downarrow 2(c_{1,n} * g^*_{-n})_k,$$

and the reconstruction can be thought of as upsampling followed by convolution with h_k and g_k

$$c_{1,k} = (\uparrow 2c)_{0,n} * h_k + (\uparrow 2d)_{0,n} * g_k.$$

2.4 Symbol, Modulation Matrix and Polyphase Matrix

Definition 5 The *symbol* of a refinable function is the trigonometric polynomial

$$H(\xi) = \frac{1}{\sqrt{2}} \sum_k h_k e^{-ik\xi}.$$

By theorem 3(i), $H(0) = 1$. The orthogonality conditions (theorem 3) are equivalent to

$$\begin{aligned} &|H(\xi)|^2 + |H(\xi+\pi)|^2 = 1, \\ &|G(\xi)|^2 + |G(\xi+\pi)|^2 = 1, \\ &H(\xi)G^*(\xi) + H(\xi+\pi)G^*(\xi+\pi) = 0. \end{aligned} \tag{2.2}$$

Together, these conditions are known as *quadrature mirror filter conditions* (QMF).

The biorthogonality condition turns out to be

$$\begin{aligned} &H(\xi)\tilde{H}^*(\xi) + H(\xi+\pi)\tilde{H}^*(\xi+\pi) = 1, \\ &G(\xi)\tilde{G}^*(\xi) + G(\xi+\pi)\tilde{G}^*(\xi+\pi) = 1, \\ &H(\xi)\tilde{G}^*(\xi) + H(\xi+\pi)\tilde{G}^*(\xi+\pi) = 0. \end{aligned} \tag{2.3}$$

The recursion relations in the frequency domain can be written as

$$\widehat{\phi}(\xi) = H(\xi/2)\widehat{\phi}(\xi/2),$$

where

$$\widehat{\phi}(\xi) = \int_{-\infty}^{\infty} \phi(x) e^{-ix\xi} dx$$

is the Fourier transform of ϕ. Likewise,

$$\widehat{\psi}(\xi) = G(\xi/2)\widehat{\phi}(\xi/2),$$

where

$$G(\xi) = \frac{1}{\sqrt{2}} \sum_k g_k e^{-ik\xi}.$$

$H(\xi)$ and $G(\xi)$ are the the symbols for the scaling function and the wavelet, respectively.

Definition 6 The matrix

$$M(\xi) = \begin{bmatrix} H(\xi) & H(\xi+\pi) \\ G(\xi) & G(\xi+\pi) \end{bmatrix}$$

is called the *modulation matrix.*

Definition 7 We define the *polyphase symbols* as

$$H_0(z) = \sum_k h_{2k} z^k$$

and

$$H_1(z) = \sum_k h_{2k+1} z^k.$$

Note that

$$H(z) = \frac{H_0(z^2) + zH_1(z^2)}{\sqrt{2}}$$

where $z = e^{-i\xi}$. The polyphase symbols of the corresponding wavelet are defined similarly.

Definition 8 The matrix

$$P(\xi) = \begin{bmatrix} H_0(\xi) & H_1(\xi) \\ G_0(\xi) & G_1(\xi) \end{bmatrix}$$

is called the *polyphase matrix.*

Conditions (2.2) are equivalent to $M(\xi)M(\xi)^* = P(\xi)P(\xi)^* = I$.

Definition 9 A *trigonometric matrix polynomial* $A(\xi)$ is called *paraunitary* if

$$A(\xi)A(\xi)^* = A(\xi)^* A(\xi) = I.$$

Note: M being paraunitary is equivalent to P being paraunitary.

2.5 Moments

Definition 10 The j^{th} *continuous moments* of ϕ and ψ are defined by

$$\mu_j = \int_{-\infty}^{\infty} x^j \phi(x) dx,$$

$$\nu_j = \int_{-\infty}^{\infty} x^j \psi(x) dx.$$

Any wavelet ψ that comes from an MRA must satisfy

$$\nu_0 = \int_{-\infty}^{\infty} \psi(x) dx = 0.$$

This is the zeroth moment of ψ.

Definition 11 The j^{th} *discrete moments* are defined by

$$m_j = \frac{1}{\sqrt{2}} \sum_k k^j h_k,$$

$$n_j = \frac{1}{\sqrt{2}} \sum_k k^j g_k.$$

Note: $m_0 = 1$.

One can show that

$$\mu_j = 2^{-j} \sum_{s=0}^{j} \binom{j}{s} m_{j-s} \mu_s \tag{2.4}$$

and

$$\nu_j = 2^{-j} \sum_{s=0}^{j} \binom{j}{s} n_{j-s} \mu_s.$$

In particular, we can choose μ_0 to be an arbitrary nonzero number. The remaining μ_j, ν_j are then uniquely defined.

2.6 Approximation Order and Accuracy

We say that a refinable function has approximation order k if for $0 \le k \le N-1$, the polynomial x^k can be reproduced exactly as a linear combination of its integer shifts. It turns out that if the dual wavelet $\tilde{\psi}$ has N vanishing moments, then the scaling function ϕ has approximation order N, as given by the following theorem.

Theorem 6 Let $\phi(x)$ be a compactly supported scaling function associated with an MRA. Let $\tilde{\psi}(x)$ be the dual wavelet. Then for each N the following are equivalent.

(i) $\int_{\mathbb{R}} x^k \tilde{\psi}(x)dx = 0$ for $0 \leq k \leq N-1$.

(ii) $\sum_n \tilde{g}_n n^k = 0$ for $0 \leq k \leq N-1$.

(iii) $H(\xi)$ can be factored as

$$H(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^N L(\xi)$$

for some 2π - periodic trigonometric polynomial $L(\xi)$.

(iv) $H^{(k)}(\pi) = 0$, for $0 \leq k \leq N-1$.

Note: The conditions in this theorem are necessary conditions for approximation order N. They are sufficient if the cascade algorithm converges.

Proof: This is proved in [6].

Example: h and $db2$ as explained earlier have approximation orders zero and one, respectively. The symbol for the scaling function h factors as

$$H(\xi) = \frac{1+e^{-i\xi}}{2},$$

where $L(\xi) = 1$, and the symbol for the Daubechies scaling function $db2$, factors as

$$H(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^2 \left(\frac{1+\sqrt{3}}{2} + \frac{1-\sqrt{3}}{2}e^{-i\xi}\right).$$

2.7 Lifting

Using the *lifting scheme* one can start with a very simple or trivial multiresolution analysis, and gradually work one's way up to a multiresolution analysis with particular properties. It is one of the most elegant ways of generating a biorthogonal MRA.

Definition 12 A filter pair H, G is *complementary* if the corresponding polyphase matrix $P(z)$ has determinant 1.

Theorem 7 Let (H, G) be complementary, then any other finite filter G^{new} complementary to H is of the form

$$G^{new} = G(z) + H(z)s(z^2),$$

where $s(z)$ is a Laurent polynomial. Conversely, any filter of this form is complementary to G.

Proof: This is proved in [15].

The new polyphase matrix can be written as

$$P^{new}(z) = \begin{bmatrix} 1 & 0 \\ s(z) & 1 \end{bmatrix} P(z).$$

This creates a new dual scaling function whose filter is given by the dual polyphase matrix

$$\tilde{P}^{new}(z) = \begin{bmatrix} 1 & -s(z^{-1}) \\ 0 & 1 \end{bmatrix} \tilde{P}(z),$$

which implies that the new wavelet, the dual scaling function and wavelets are given by

$$\begin{aligned} G^{new}(z) &= G(z) + H(z)s(z^2), \\ \tilde{H}^{new}(z) &= \tilde{H}(z) - \tilde{G}(z)s(z^{-2}), \\ \tilde{G}^{new}(z) &= \tilde{G}(z). \end{aligned}$$

The scaling function here does not change at all, while the wavelet and the dual scaling function change according to the above relations. The dual wavelet also changes, but in a less fundamental way than the wavelet and the scaling function. More precisely, the dual wavelet changes because the dual scaling function from which it is built changes, while the filter coefficients remain exactly the same.

The power behind the lifting scheme is that through $s(z)$ we have full control over all wavelets and dual functions that can be built from a particular scaling function. This means that we can start from a simple or trivial set of biorthogonal functions and use suitable $s(z)$ so that after lifting the wavelet has desirable properties. The lifting scheme can also be used for the dual scaling function and the wavelet in a similar way; this is called *dual lifting*. Lifting and dual lifting can be iterated to get an MRA with desired properties.

Example: Lifting the Haar wavelet h.

We start from the Haar wavelet and try to use the lifting scheme to increase the number of vanishing moments of the wavelet from one to two. Initially

$$\widetilde{H}(z) = H(z) = \frac{1+z}{2}$$

and

$$\widetilde{G}(z) = G(z) = \frac{1-z}{2}.$$

After lifting we get

$$G^{new}(z) = G(z) + H(z)s(z^2).$$

We need $G^{new}(0) = 0$ for one vanishing moment, which implies that $s(0) = 0$. For two vanishing moments we have to have $G^{new}(0) = G^{new'}(0) = 0$. Let's work with $z = e^{-i\xi}$ for simplicity. Thus

$$G^{new'}(0) = G'(0) + H'(0)s(0) + 2H(0)s'(0) = 0,$$

or

$$s'(0) = -\frac{G'(0)}{2H(0)} = -\frac{i}{4}.$$

We can choose (for symmetry) $s(\xi) = (-i/4)\sin\xi$. Thus the new wavelet symbol under lifting can be written as

$$\begin{aligned} G^{new}(\xi) &= \frac{1 - e^{-i\xi} + s(2\xi)(1 + e^{-i\xi})}{2} \\ &= \frac{1}{2} - \frac{e^{-i\xi}}{2} - \frac{i\sin 2\xi}{8} - \frac{ie^{-i\xi}\sin 2\xi}{8} \\ &= \frac{1}{2} - \frac{e^{-i\xi}}{2} - \frac{e^{2i\xi} - e^{-2i\xi}}{16} - \frac{e^{i\xi} - e^{-3i\xi}}{16}, \end{aligned}$$

or

$$G^{new}(z) = \frac{1}{16}z^2 - \frac{1}{16}z^{-2} - \frac{1}{2} + \frac{1}{2}z + \frac{1}{16}z^3 - \frac{1}{16}z^{-1}.$$

The corresponding dual scaling symbol $\tilde{H}^{new}$ can be written as

$$\tilde{H}^{new}(z) = \frac{1}{2} + \frac{1}{2}z + \frac{1}{16}z^{-2} - \frac{1}{16}z^{-1} - \frac{1}{16}z^3 + \frac{1}{16}z^{-1}.$$

2.8 Factorization

Various techniques to factor existing wavelet filters into basic building blocks are known. For example, it is known that every polyphase matrix in one dimension factors into lifting factors, viz.

$$P(z) = \prod_{i=1}^{m} \begin{bmatrix} 1 & s_i(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t_i(z) & 1 \end{bmatrix} \begin{bmatrix} k & 0 \\ 0 & 1/k \end{bmatrix},$$

where $s_i(z)$ and $t_i(z)$ are Laurent polynomials and k is a constant [15].

It is also known that the polyphase matrix of every orthogonal wavelet can be factored [6] in the form

$$P(z) = QF_1(z), \dots, F_n(z),$$

where Q is a constant orthogonal matrix and each F_k is a *projection factor* of the form

$$F(z) = (I - \mathbf{u}\mathbf{u}^*) + \mathbf{u}\mathbf{u}^* z$$

for some unit vector $\mathbf{u}$.

2.9 Completion

A completion problem for a wavelet is the problem of finding the corresponding wavelet given the scaling function. This is equivalent to completing the polyphase matrix to a paraunitary matrix when the first row is given. Let Δ be the determinant of P. Since $PP^* = I$,

$$P^* = \begin{bmatrix} H_0^* & G_0^* \\ H_1^* & G_1^* \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} G_1 & -H_1 \\ -G_0 & H_0 \end{bmatrix} = P^{-1},$$

so

$$G_0 = -\Delta H_1^*, \;\; G_1 = \Delta H_0^*.$$

Since the determinant must be a monomial for compactly supported wavelets, we have $\Delta = \alpha z^{\mathbf{k}}, |\alpha| = 1$.

Thus the matrix completion problem in the one dimensional case can always be solved. In short, given H_0, H_1 we can always find G_0, G_1 a such that P is paraunitary.

Example: Daubechies scaling function $db2$.

Given the scaling function and its symbol

$$H(\xi) = \frac{1}{\sqrt{2}}(h_0 + h_1 z + h_2 z^2 + h_3 z^3),$$

the polyphase matrix has the following paraunitary completion.

$$P = \begin{bmatrix} h_0 + h_2 z & h_1 + h_3 z \\ -(h_1 + h_3 z^{-1}) & h_0 + h_2 z^{-1} \end{bmatrix},$$

or equivalently the following paraunitary completion which keeps the same support.

$$P = \left[\begin{array}{cc} h_0 + h_2 z & h_1 + h_3 z \\ -h_1 z - h_3 & h_0 z + h_2 \end{array} \right].$$

CHAPTER 3. The Multivariate Setting

3.1 Lattices

A matrix M is said to be a *dilation matrix* if it has integer entries and all of its eigenvalues are greater than one in absolute value. *A linear combination* of N vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$ is an expression of the form $\sum_{i=1}^{N} c_i\mathbf{x}_i$ where c_i are real numbers. The c_i are called *coefficients.* The set of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$ is said to be *linearly independent* if $\sum_{i=1}^{N} c_i\mathbf{x}_i = 0$ implies $c_i = 0$ for all i. If the set of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_d$ is linearly independent, then the totality of the vectors of the form $\{\sum_{i=1}^{d} n_i\mathbf{x}_i \mid n_i \in \mathbb{Z}\}$ is called a *d-dimensional lattice.* We denote it by Γ. In such a case, $M = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_d]$ is called the *sampling matrix* [19], and is said to generate the lattice. If M is the identity matrix, then each $\mathbf{x}_i$ is a unit vector pointing in the i^{th} direction, and the resulting lattice Γ is $\mathbb{Z}^d$.

Note: Note that several different dilation matrices M may produce the same lattice.

Definition 13 Given a dilation matrix M, the *fundamental parallelepiped* of the lattice Γ is defined by

$$F(M) = \{\mathbf{y} \in \mathbb{R}^d \mid \mathbf{y} = \mathrm{M}\mathbf{x} \text{ for some } \mathbf{x} \in [0,1)^d\}.$$

The fundamental parallelepiped depends on the matrix M, not just on the lattice Γ.

Let Γ be some lattice and consider the order m group $\frac{\Gamma}{M\Gamma}$, where $m = |\det M|$. Its complete set of representatives is called the *digit set* and denoted by $D = \{\mathbf{d}_0, \mathbf{d}_1, \ldots, \mathbf{d}_{m-1}\}$. More precisely we take $D = \Gamma \cap F$ and $\cup_{\mathbf{d}\in D}(\Gamma + \mathbf{d}) = \mathbb{Z}^d$.

Example 1: Standard quincunx. The dilation matrix

$$M = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

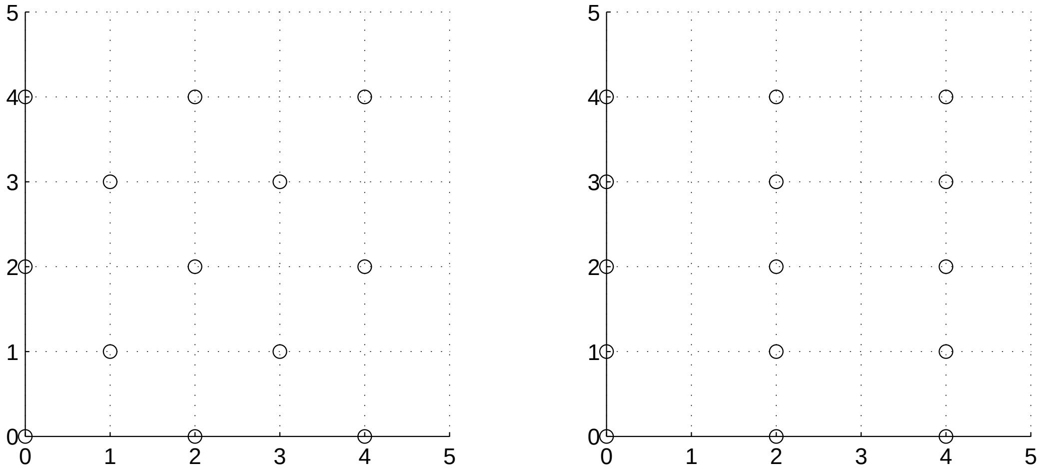

Figure 3.1 Standard and nonstandard quincunx and column lattices

gives rise to the quincunx lattice shown in fig. 3.1 (left), with digit set $\{(0,0)^t, (1,0)^t\}$.

Example 2: Nonstandard quincunx. The dilation matrix

$$M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

gives rise to the same lattice as before, but the digit set is $\{(0,0)^t, (0,1)^t\}$.

Example 3: Column lattice. The dilation matrix

$$M = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

gives rise to the column sublattice shown in fig. 3.1 (right), with digit set $\{(0,0)^t, (1,0)^t\}$.

3.2 Refinable Vector Functions

Let M be a fixed dilation matrix associated with the lattice $\Gamma \subset \mathbb{Z}^d$. The equation

$$\boldsymbol{\phi}(\mathbf{x}) = \sqrt{m} \sum_{\mathbf{k} \in \Lambda} h_{\mathbf{k}} \, \boldsymbol{\phi}\,(M\mathbf{x} - \mathbf{k}), \;\; \mathbf{x} \in \mathbb{R}^d, \tag{3.1}$$

where Λ is some finite subset of Γ and $h_{\mathbf{k}}$ are fixed $r \times r$ matrices, is called the *refinement equation.* A solution of the refinement equation is called a *vector scaling function* or *refinable vector function.* r is called the multiplicity of $\boldsymbol{\phi}$. If the matrix $\Delta = \dfrac{1}{\sqrt{m}} \sum_{\mathbf{k} \in \Lambda} h_{\mathbf{k}}$ has eigenvalues

$\lambda_1 = 1$ and $|\lambda_2|, \ldots, |\lambda_r| < 1$, then there exists a non-zero compactly supported distribution $\boldsymbol{\phi}(\mathbf{x}) = (\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), \ldots, \phi_r(\mathbf{x}))^t$ satisfying the refinement equation (3.1) [1]. Furthermore, $\hat{\boldsymbol{\phi}}(\xi)$ is a continuous vector function and $\hat{\boldsymbol{\phi}}(0) \neq \mathbf{0}$, where $\hat{\boldsymbol{\phi}}$ is the Fourier transform of $\boldsymbol{\phi}$. When $r = 1$, we say that (3.1) is a single function refinement equation. $\boldsymbol{\phi}$ is *orthogonal* if

$$< \boldsymbol{\phi}(\mathbf{x} - \mathbf{j}), \boldsymbol{\phi}(\mathbf{x} - \mathbf{k}) >= \delta_{\mathbf{jk}} I,$$

where I is the $r \times r$ identity matrix. If $\mathbf{f}$ and $\mathbf{g}$ are $r \times 1$ vector functions, the inner product is defined as the following:

$$< \mathbf{f}, \mathbf{g} >= \int \mathbf{f}\mathbf{g}^* = \begin{pmatrix} \int f_1 g_1^* & \cdots & \int f_1 g_r^* \\ \int f_2 g_1^* & \cdots & \int f_2 g_r^* \\ \int f_3 g_1^* & \cdots & \int f_3 g_r^* \\ \vdots & \cdots & \vdots \\ \int f_r g_1^* & \cdots & \int f_r g_r^* \end{pmatrix}.$$

The * denotes the complex conjugate transpose.

Let $\boldsymbol{\phi}(\mathbf{x})$ be a refinable vector function. Using the notation $\boldsymbol{\phi}_{n,\mathbf{j}}(\mathbf{x}) = m^{\frac{n}{2}} \boldsymbol{\phi}(M^n \mathbf{x} - \mathbf{j})$, we get

$$\begin{aligned} \boldsymbol{\phi}(M^n \mathbf{x} - \mathbf{j}) &= m^{\frac{1}{2}} \sum_{\mathbf{k} \in \Lambda} h_{\mathbf{k}} \ \boldsymbol{\phi}(M(M^n \mathbf{x} - \mathbf{j}) - \mathbf{k}) \\ &= m^{\frac{1}{2}} \sum_{\mathbf{k} \in \Lambda} h_{\mathbf{k}} \boldsymbol{\phi}(M^{n+1} \mathbf{x} - M\mathbf{j} - \mathbf{k}). \end{aligned}$$

Therefore

$$\begin{aligned} \boldsymbol{\phi}_{n,\mathbf{j}}(\mathbf{x}) &= m^{\frac{n+1}{2}} \sum_{\mathbf{k} \in \Lambda} h_{\mathbf{k}} \ \boldsymbol{\phi}(M^{n+1} \mathbf{x} - M\mathbf{j} - \mathbf{k}) \\ &= \sum_{\mathbf{k} \in \Lambda} h_{\mathbf{k}} \ \boldsymbol{\phi}_{n+1, M\mathbf{j}+\mathbf{k}}(\mathbf{x}). \end{aligned}$$

Let $M\mathbf{j} + \mathbf{k} = \mathbf{p}$ and define $h_{\mathbf{p}} = 0$ for $\mathbf{p} \notin \Lambda$. Then

$$\boldsymbol{\phi}_{n,\mathbf{j}}(\mathbf{x}) = \sum_{\mathbf{p} \in \Gamma} h_{\mathbf{p} - M\mathbf{j}} \ \boldsymbol{\phi}_{n+1,\mathbf{p}}(\mathbf{x}). \tag{3.2}$$

Lemma 1 If $\boldsymbol{\phi}$ is refinable and orthogonal, we have

$$\delta_{\mathbf{jk}} I = \sum_{\mathbf{p}} h_{\mathbf{p} - M\mathbf{j}} h^*_{\mathbf{p} - M\mathbf{k}}. \tag{3.3}$$

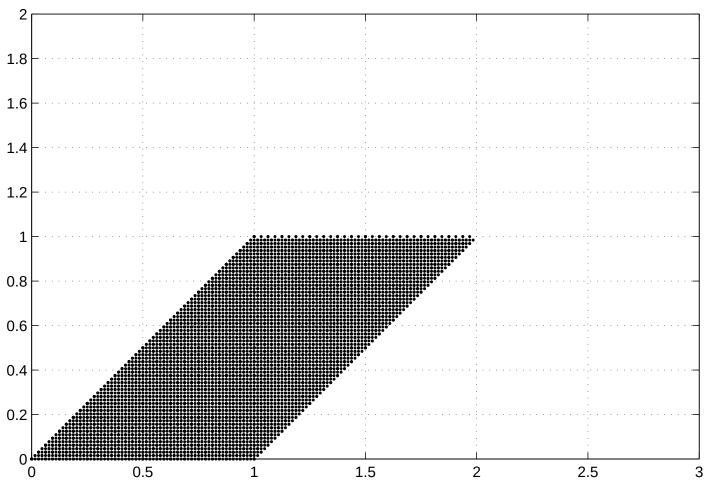

Figure 3.2 K_Λ corresponding to standard quincunx

Proof: Using orthogonality and the relation (3.2)

$$\begin{aligned} \delta_{\mathbf{jk}} I &= < \boldsymbol{\phi}(\mathbf{x}-\mathbf{j}), \boldsymbol{\phi}(\mathbf{x}-\mathbf{k}) > \\ &= < \sum_{\mathbf{p}} h_{\mathbf{p}-M\mathbf{j}} \boldsymbol{\phi}_{1,\mathbf{p}}, \sum_{\mathbf{q}} h_{\mathbf{q}-M\mathbf{k}} \boldsymbol{\phi}_{1,\mathbf{q}} > \\ &= \sum_{\mathbf{p},\mathbf{q}} h_{\mathbf{p}-M\mathbf{j}} < \boldsymbol{\phi}_{1,\mathbf{p}}, \boldsymbol{\phi}_{1,\mathbf{q}} > h^*_{\mathbf{q}-M\mathbf{k}} \\ &= \sum_{\mathbf{p}} h_{\mathbf{p}-M\mathbf{j}} h^*_{\mathbf{p}-M\mathbf{k}}. \end{aligned}$$

However, these are only necessary conditions. Sufficient conditions can be found in [22] and [23].

Example: Let H denote the scaling function corresponding to

$$M = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \ \Lambda = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\},$$

with the coefficients of Haar scaling function $h_0 = h_1 = \frac{1}{\sqrt{2}}$. The scaling function is two-dimensional Haar scaling function $\phi = \chi_s$, where S is as shown in fig 3.2.

Example: Let M be the standard quincunx and let

$$\Lambda = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right\},$$

and let unspecified coefficients h_0, h_1, h_2, h_3 be placed at the corresponding positions. Then the orthogonality conditions are

$$h_0^2 + h_1^2 + h_2^2 + h_3^2 = 1,$$

$$h_0\, h_2 + h_1\, h_3 = 0.$$

In this case the coefficients of the scalar Daubechies wavelet with two vanishing moments will work. Let DB2 denote the scaling function corresponding to

$$M = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \Lambda = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right\},$$

with the coefficients of Daubechies scaling function $db2$. The corresponding scaling function is shown in fig 3.4.

Example: Let M be the standard quincunx and let

$$\Lambda = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\},$$

where the unspecified coefficients h_i are placed as shown:

$$\begin{array}{cccc} & h_6 & h_7 & \\ h_2 & h_3 & h_4 & h_5 \\ & h_0 & h_1 & \end{array}$$

The orthogonality conditions are

$$\begin{aligned} h_0^2 + h_1^2 + h_2^2 + h_3^2 + h_4^2 + h_5^2 + h_6^2 + h_7^2 &= 1, \\ h_0h_2 + h_1h_3 + h_4h_6 + h_5h_7 &= 0, \\ h_0h_4 + h_1h_5 + h_2h_6 + h_3h_7 &= 0, \\ h_2h_4 + h_3h_5 &= 0, \\ h_0h_6 + h_1h_7 &= 0. \end{aligned}$$

The two-particular family of solutions of the above equations is

$$
\begin{aligned}
h_0 &= \frac{\mu(1-\nu+\mu+\mu\nu)}{(\mu^2+1)(\nu^2+1)}, \\
h_1 &= \frac{-\mu(1+\nu-\mu+\mu\nu)}{(\mu^2+1)(\nu^2+1)}, \\
h_2 &= \frac{\mu\nu(1+\nu-\mu+\mu\nu)}{(\mu^2+1)(\nu^2+1)}, \\
h_3 &= \frac{\mu\nu(1-\nu+\mu+\mu\nu)}{(\mu^2+1)(\nu^2+1)}, \\
h_4 &= \frac{-\nu(1-\nu+\mu+\mu\nu)}{(\mu^2+1)(\nu^2+1)}, \\
h_5 &= \frac{\nu(1+\nu-\mu+\mu\nu)}{(\mu^2+1)(\nu^2+1)}, \\
h_6 &= \frac{(1+\nu-\mu+\mu\nu)}{(\mu^2+1)(\nu^2+1)}, \\
h_7 &= \frac{(1-\nu+\mu+\mu\nu)}{(\mu^2+1)(\nu^2+1)}.
\end{aligned}
$$

The corresponding scaling functions aew known as K-V scaling functions after Kovacević-Vetterli [3].

3.3 Support

Our interest is in compactly supported scaling functions, which implies that $\{h_{\mathbf{k}}\}_{\mathbf{k}\in\Lambda}$ are finitely supported. For $\mathbf{k} \in \mathbb{Z}^d$, let $w_{\mathbf{k}} : \mathbb{R}^d \to \mathbb{R}^d$ be the affine map

$$w_{\mathbf{k}}(\mathbf{x}) = M^{-1}(\mathbf{x}+\mathbf{k}).$$

Let $\mathcal{H}(\mathbb{R}^d)$ be the space of all non-empty, compact subsets of $\mathbb{R}^d$. The Hausdorff metric $h(.,.)$ is defined by

$$h(B, C) = \inf\{\epsilon > 0 : B \subset C_\epsilon \text{ and } C \subset B_\epsilon\},$$

where

$$B_\epsilon = \{\mathbf{x} \in \mathbb{R}^n : \mathrm{dist}(\mathbf{x}, B) < \epsilon\}.$$

Under the Hausdorff metric, $\mathcal{H}(\mathbb{R}^d)$ is a complete metric space. For a finite set $H \subset \mathbb{Z}^d$, define the iterated function system (IFS)

$$\begin{aligned} w_H(B) &= \bigcup_{\mathbf{k} \in H} w_{\mathbf{k}}(B) \\ &= M^{-1}(B+H), \end{aligned}$$

where $B \in \mathcal{H}(\mathbb{R}^d)$. Since M is expansive, there exists a matrix norm with $||M^{-1}|| < 1$, and therefore w_H is a contractive map on $\mathbb{R}^d$. By the contraction mapping theorem, there exists a unique compact set K_H such that $w_H(K_H) = K_H$. K_H is called the *attractor* of the IFS generated by H and can be expressed as

$$K_H = \text{clos}(\sum_{j=1}^{\infty} M^{-j}(H)), \tag{3.4}$$

with convergence in the Hausdorff norm. Our interest will be the attractor associated with the IFS generated by the support of the refinement coefficients, Λ. The following theorem estimates the support of the scaling function.

Theorem 8 *If a function $\boldsymbol{\phi} : \mathbb{R}^n \to \mathbb{C}^r$ is a compactly supported solution of the refinement equation, then* $\text{supp}(\boldsymbol{\phi}) \subset K_\Lambda$.

Proof: If $\mathbf{x} \in \text{supp}(\boldsymbol{\phi})$, then $M\mathbf{x} - \mathbf{k} \in \text{supp}(\boldsymbol{\phi})$ for some $\mathbf{k} \in \Lambda$. Therefore, $\mathbf{x} \in M^{-1}\text{supp}(f) + M^{-1}\Lambda$, or $\text{supp}(\boldsymbol{\phi}) \subset M^{-1}\Lambda + M^{-1}\text{supp}(\boldsymbol{\phi})$. Iterating this gives

$$\text{supp}(\boldsymbol{\phi}) \subset M^{-1}\Lambda + M^{-2}\Lambda + M^{-3}\Lambda + \cdots + M^{-j}\Lambda + M^{-j}\text{supp}(\boldsymbol{\phi}).$$

Since M^{-1} is a contraction, this gives

$$\text{supp}(\boldsymbol{\phi}) \subset \sum_{j=1}^{\infty} M^{-j}(\Lambda) + \epsilon, \ \text{ for all } \epsilon.$$

Thus

$$\text{supp}(\boldsymbol{\phi}) \subset K_\Lambda, \ \text{ by (3.4)}.$$

3.4 Estimating the Support

In many cases, $M^n = cI$ for some $n \in \mathbb{N}$, $c \in \mathbb{R}$. In those cases, we can use (3.4) to estimate the support of $\boldsymbol{\phi}$.

Example 1:

$$M = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \ \Lambda = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

Here $M^2 = 2I$. We have the following table.

k	$M^{-k}\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$M^{-k}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
1	$(0,0)^t$	$(1/2,1/2)^t$
2	$(0,0)^t$	$(1/2,0)^t$
3	$(0,0)^t$	$(1/4,1/4)^t$
4	$(0,0)^t$	$(1/4,0)^t$
5	$(0,0)^t$	$(1/8,1/8)^t$
6	$(0,0)^t$	$(1/8,0)^t$
7	$(0,0)^t$	$(1/16,0)^t$
$\vdots$	$\vdots$	$\vdots$

If $(x,y) \in K_\Lambda$, then

$$0 \leq x \leq 1/2 + 1/2 + 1/4 + 1/4 + 1/8 + 1/8 + \cdots = 2,$$

$$0 \leq y \leq 1/2 + 1/4 + 1/8 + 1/16 + 1/32 + \cdots = 1,$$

so $\operatorname{supp} \boldsymbol{\phi} \subset [0,2] \times [0,1]$. The actual support is the set shown in fig 3.2.

Example 2:

$$M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \ \Lambda = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

Here $M^4 = -4I$.

k	$M^{-k}\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$M^{-k}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
1	$(0,0)^t$	$(1/2,1/2)^t$
2	$(0,0)^t$	$(1/2,0)^t$
3	$(0,0)^t$	$(1/4,-1/4)^t$
4	$(0,0)^t$	$(0,-1/4)^t$
5	$(0,0)^t$	$(-1/8,-1/8)^t$
6	$(0,0)^t$	$(-1/8,0)^t$
7	$(0,0)^t$	$(-1/16,1/16)^t$
8	$(0,0)^t$	$(0,1/16)^t$
9	$(0,0)^t$	$(1/32,1/32)^t$
10	$(0,0)^t$	$(1/64,1/0)^t$
$\vdots$	$\vdots$	$\vdots$

If $(x,y) \in K_\Lambda$, then

$$x \leq 1/2 + 1/2 + 1/4 + 1/32 + 1/32 + 1/64 + \cdots$$

$$= (1/2 + 1/2 + 1/4)[1 + 1/16 + 1/16^2 + \cdots] = 4/3,$$

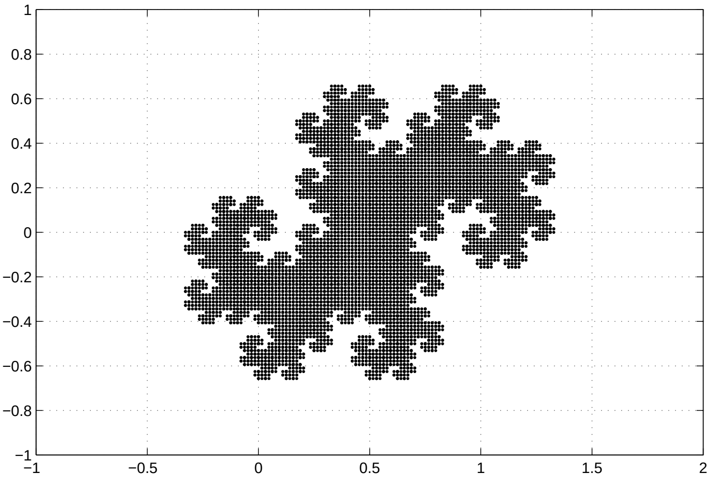

Figure 3.3 K_Λ corresponding to non-standard quincunx

$$x \geq -[1/8 + 1/8 + 1/16 + \cdots] = -1/3,$$

$$y \leq 1/2 + 1/16 + 1/16 + 1/32 + \cdots = 2/3,$$

$$y \geq -[1/4 + 1/4 + 1/8 + \cdots] = -2/3,$$

so supp $\boldsymbol{\phi} \subset [-1/3, 4/3] \times [-2/3, 2/3]$. The actual support is the set shown in fig 3.3.

Theorem 9

$$\begin{aligned} S(\Lambda) + \mathbf{j} &= S(\Lambda + (M - I)\mathbf{j}), \\ S(\Lambda + \mathbf{p}) &= S(\Lambda + (M - I)^{-1}\mathbf{p}). \end{aligned}$$

In words: shifting the support of $\boldsymbol{\phi}$ by $\mathbf{j}$ can be achieved by shifting the position of the coefficients $h_\mathbf{k}$ by $(M - I)\mathbf{j}$. Conversely, shifting the position of the coefficients $h_\mathbf{k}$ by $\mathbf{p}$ shifts the support of $\boldsymbol{\phi}$ by $(M - I)^{-1}\mathbf{p}$.

Proof: Let

$$S = \cup_\mathbf{k} M^{-1}(S + \mathbf{k}).$$

Then

$$
\begin{aligned}
S+\mathbf{j} &= \cup_{\mathbf{k}} M^{-1}(S+\mathbf{k})+\mathbf{j} \\
&= \cup_{\mathbf{k}} M^{-1}(S+\mathbf{k}+M\mathbf{j}) \\
&= \cup_{\mathbf{k}} M^{-1}(S+\mathbf{k}+M\mathbf{j}+\mathbf{j}-\mathbf{j}) \\
&= \cup_{\mathbf{k}} M^{-1}(S+\mathbf{j}+\mathbf{k}+(M-I)\mathbf{j}).
\end{aligned}
$$

The second equality is obtained by setting $\mathbf{j}=(M-I)^{-1}\mathbf{p}$.

Theorem 10 Assume A is a nonsingular matrix which commutes with M. Then

$$S(A\Lambda)=AS(\Lambda)$$

In words: replacing Λ by $A\Lambda$ replaces S by AS.

Proof: $AM=MA \Rightarrow AM^{-1}=M^{-1}A$. Then

$$
\begin{aligned}
AS &= \cup_{\mathbf{k}} AM^{-1}(S+\mathbf{k}) \\
&= \cup_{\mathbf{k}} M^{-1}A(S+\mathbf{k}) \\
&= \cup_{\mathbf{k}} M^{-1}(AS+A\mathbf{k}) \\
&= \cup_{\mathbf{p}\in A\Lambda} M^{-1}(AS+\mathbf{p}).
\end{aligned}
$$

3.5 Computing Point Values

The point values of the refinable function can be approximated by using the following cascade algorithm, as in the univariate case.

Definition 14 The cascade algorithm consists of selecting a suitable starting function $\boldsymbol{\phi}^{(0)}(x)\in L_2$, and then producing a sequence of functions

$$\boldsymbol{\phi}^{(n+1)}(\mathbf{x})=\sqrt{m}\sum_{\mathbf{k}} h_{\mathbf{k}}\boldsymbol{\phi}^{(n)}(M\mathbf{x}-\mathbf{k}).$$

Theorem 11 If the cascade algorithm converges for both $\boldsymbol{\phi}$ and $\tilde{\boldsymbol{\phi}}$, then the necessary condition for the orthogonality is also the sufficient condition.

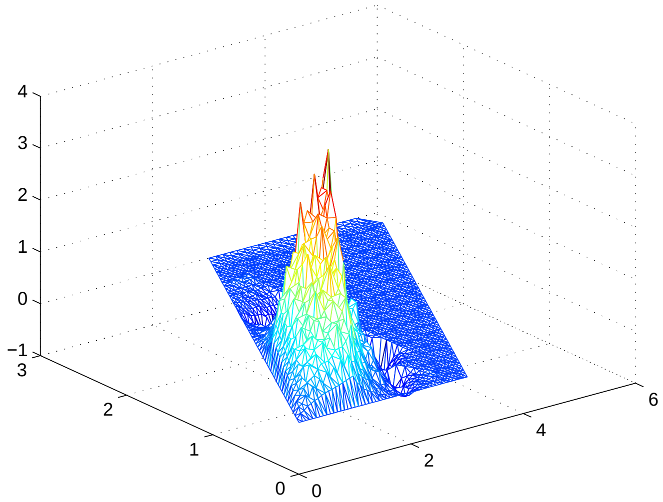

Figure 3.4 DB2 Scaling function and its support

Proof: This is proved in [22].

The point values of the scaling function can also be obtained by solving an eigenvalue problem, as in the scalar case. Let $\mathbf{x}_1, \ldots \mathbf{x}_l$ be the points with integer coordinates inside the support of $\boldsymbol{\phi}$. Let

$$\boldsymbol{\phi} = \begin{pmatrix} \boldsymbol{\phi}(\mathbf{x}_1) \\ \boldsymbol{\phi}(\mathbf{x}_2) \\ \vdots \\ \boldsymbol{\phi}(\mathbf{x}_l) \end{pmatrix}.$$

Writing out the recursion relation for each $\boldsymbol{\phi}(\mathbf{x}_j)$ we get an eigenvalue problem

$$\boldsymbol{\phi} = T\boldsymbol{\phi}.$$

Example 1: Two-dimensional Haar function H. The support is shown in fig (3.2). There are no interior points with integer coordinater inside the support, and the method requires $\boldsymbol{\phi}= 0$ on the boundary, so this is not going to work.

Example 2: Two-dimensional Daubechies wavelet DB2. The support is shown in fig 3.4.

There are 4 interior points, namely $\{(2,1)^t, (3,1)^t, (3,2)^t, (4,2)^t\}$.

$$T = \sqrt{2}\begin{pmatrix} h_1 & h_0 & 0 & 0 \\ 0 & 0 & h_1 & h_0 \\ h_3 & h_2 & 0 & 0 \\ 0 & 0 & h_3 & h_3 \end{pmatrix}.$$

We normalized the point values so as to have the sum equal to 1.

$$\begin{array}{rcl} \phi(2,1) & = & \frac{1}{2}(2+\sqrt{3}) \\ \phi(3,1) & = & -\frac{1}{2} \\ \phi(3,2) & = & -\frac{1}{2} \\ \phi(4,2) & = & \frac{1}{2}(2-\sqrt{3}) \end{array}.$$

Example 3 Kovacević-Vetterli (K-V). There are 18 interior points.

CHAPTER 4. Multivariate Wavelet Theory

4.1 Multiresolution Analysis, Multiwavelets

Definition 15 *A multiresolution approximation (MRA) of* $L_2(\mathbb{R}^d)$ is a nested sequence of subspaces

$$\cdots \subset \ V_{-1}^{(0)} \ \subset \ V_0^{(0)} \ \subset \ V_1^{(0)} \ \subset \ V_2^{(0)} \ \subset \cdots$$

satisfying

(i) $\operatorname{clos}(\bigcup_j V_j^{(0)}) = L^2(\mathbb{R}^d)$

(ii) $\bigcap_j V_j^{(0)} = \{0\}$

(iii) $f(\mathbf{x}) \in V_j^{(0)} \iff f(M\mathbf{x}) \in V_{j+1}^{(0)}$

(iv) $f(\mathbf{x}) \in V_j^{(0)} \iff f(\mathbf{x} + M^{-j}\mathbf{k}) \in V_j^{(0)} \ \forall\, \mathbf{k} \in \ \mathbb{Z}^d$

(v) There exists a refinable vector function $\boldsymbol{\phi}^{(0)}$ so that

$$\{\phi_j^{(0)}(\mathbf{x} + \mathbf{k}) \ : j = 1, \dots, r, \mathbf{k} \in \mathbb{Z}^d\}$$

forms a Riesz basis of $V_0^{(0)}$. $\boldsymbol{\phi}^{(0)}$ is a called a *multiscaling function.*

The MRA is *orthogonal* if $\boldsymbol{\phi}^{(0)}$ is orthogonal. Assuming orthogonality, (iii) implies that $\{\sqrt{m}\phi_i^{(0)}(M\mathbf{x}-\mathbf{k})\}_{i=1,\dots,r,\mathbf{k}\in\mathbb{Z}^d}$ is an orthonormal basis for the space $V_1^{(0)}$, and since $V_0^{(0)} \subset V_1^{(0)}$, we have

$$\boldsymbol{\phi}^{(0)}(\mathbf{x}) = \sqrt{m} \sum_{\mathbf{k}\in\Lambda} h_{\mathbf{k}}^{(0)}\, \boldsymbol{\phi}^{(0)}\,(M\mathbf{x} - \mathbf{k}), \quad \mathbf{x} \in \mathbb{R}^d, \tag{4.1}$$

for some $h_{\mathbf{k}}^{(0)}$, where $\sqrt{m}$ is the normalizing factor that preserves the L^2-norm. This shows that any scaling function $\boldsymbol{\phi}$ is a refinable function. However, not every refinable function generates an MRA.

Now let us consider the subspace W_0, the orthogonal complement of $V_0^{(0)}$ in $V_1^{(0)}$. Since the determinant of M is $m, m-1$ wavelets are needed to characterize the wavelet space W_0 [8]. Let us assume the existence of $m-1$ wavelets $\boldsymbol{\phi}^{(1)}, \ldots, \boldsymbol{\phi}^{(m-1)}$ and let

$$W_0 = V_0^{(1)} \oplus V_0^{(2)} \ldots \oplus V_0^{(m-1)},$$

where $V_0^{(i)} = \text{span}(\boldsymbol{\phi}^{(i)}(\mathbf{x}-\mathbf{k})), \mathbf{k} \in \mathbb{Z}^d,\ i = 1, 2, \ldots, m-1$. Let

$$W_j = \{\boldsymbol{\phi}(M^j\mathbf{x}) : \boldsymbol{\phi}(\mathbf{x}) \in W_0\}.$$

The sequence of spaces $\{W_j\}$ satisfies conditions similar to conditions (i) through (v) of an MRA [6]. We have the following lemma, similar to the scalar case.

Lemma 2

$$V_n^{(0)} = \bigoplus_{k=-\infty}^{n-1} W_k,$$

which implies that

$$L_2(\mathbb{R}^d) = \text{clos}(\bigoplus_{k=-\infty}^{\infty} W_k).$$

Proof: The proof is similar to the one presented in [6].

The spaces W_j are called the *wavelet spaces.* Let us assume the existence of the wavelets and let the scaling function span $V_j^{(0)}$. Then

$$V_1^{(0)} = V_0^{(0)} \oplus V_0^{(1)} \oplus V_0^{(2)} \oplus \cdots \oplus V_0^{(m-1)},$$

which gives the following recursion relations

$$\boldsymbol{\phi}^{(0)}(\mathbf{x}) = \sqrt{m} \sum_{\mathbf{k} \in \Lambda} h_{\mathbf{k}}^{(0)} \boldsymbol{\phi}^{(0)}(M\mathbf{x}-\mathbf{k}), \tag{4.2}$$

$$\boldsymbol{\phi}^{(\mu)}(\mathbf{x}) = \sqrt{m} \sum_{\mathbf{k} \in \Lambda} h_{\mathbf{k}}^{(\mu)} \boldsymbol{\phi}^{(0)}(M\mathbf{x}-\mathbf{k}), \quad \mu = 1, 2 \ldots, m-1. \tag{4.3}$$

Definition 16 The wavelets are called orthogonal if

$$< \boldsymbol{\phi}^{(\mu)}(\mathbf{x}-\mathbf{j}), \boldsymbol{\phi}^{(\nu)}(\mathbf{x}-\mathbf{k}) >= \delta_{\mu\nu}\delta_{\mathbf{jk}} I. \tag{4.4}$$

As in the proof of lemma 1, this implies that the necessary condition for orthogonality is

$$\delta_{\mu\nu}\delta_{\mathbf{jk}} I = \sum_{\mathbf{p}} h_{\mathbf{p}-M\mathbf{j}}^{(\mu)} h_{\mathbf{p}-M\mathbf{k}}^{*(\nu)}.$$

Definition 17 The wavelets are called *biorthogonal* if

$$< \boldsymbol{\phi}^{(\mu)}(\mathbf{x}-\mathbf{j}), \tilde{\boldsymbol{\phi}}^{(\nu)}(\mathbf{x}-\mathbf{k}) >= \delta_{\mu\nu}\delta_{\mathbf{jk}} I. \tag{4.5}$$

$\tilde{\boldsymbol{\phi}}$ is called the *dual* of $\boldsymbol{\phi}$. In such a case the MRA generated by $\tilde{\boldsymbol{\phi}}^{(0)}$ is called the dual MRA and the corresponding wavelets $\tilde{\boldsymbol{\phi}}^{(i)}$, $i = 1, 2, \ldots, m-1$, are called the dual wavelets. If $\boldsymbol{\phi}$ is orthogonal, then $\boldsymbol{\phi} = \tilde{\boldsymbol{\phi}}$.

4.2 Decomposition and Reconstruction

Let us take an arbitrary $f(\mathbf{x}) \in V_n^{(0)}$ for some n. According to the decomposition

$$V_n^{(0)} = V_{n-1}^{(0)} \oplus V_{n-1}^{(1)} \oplus V_{n-1}^{(2)} \oplus \cdots \oplus V_{n-1}^{(m-1)},$$

we have

$$\begin{aligned} f(\mathbf{x}) &= \sum_{\mathbf{j}} \mathbf{f}_{n,\mathbf{j}}^{*(0)} \boldsymbol{\phi}_{n,\mathbf{j}}^{(0)}(\mathbf{x}) \\ &= \sum_{\mu,\ \mathbf{k}} \mathbf{f}_{n-1,\mathbf{k}}^{*(\mu)}\ \boldsymbol{\phi}_{n-1,\mathbf{k}}^{(\mu)}(\mathbf{x}), \\ \text{where} \qquad \mathbf{f}_{n,\mathbf{k}}^{*(\mu)} &= < f\ ,\ \tilde{\boldsymbol{\phi}}_{n,\mathbf{k}}^{(\mu)} > . \end{aligned}$$

It is enough to compute the inner product at one level. We have

$$\begin{aligned} < \boldsymbol{\phi}_{n,\mathbf{j}}^{(0)}\ ,\ \tilde{\boldsymbol{\phi}}_{n-1,\mathbf{k}}^{(\mu)} > &= < \boldsymbol{\phi}_{n,\mathbf{j}}^{(0)}\ ,\ \sum_{\mathbf{i}} \tilde{h}_{\mathbf{i}-M\mathbf{k}}^{(\mu)}\, \tilde{\boldsymbol{\phi}}_{n,\mathbf{i}}^{(0)} > \\ &= \tilde{h}_{\mathbf{j}-M\mathbf{k}}^{*(\mu)}, \\ < \boldsymbol{\phi}_{n-1,\mathbf{k}}^{(\mu)}\ ,\ \tilde{\boldsymbol{\phi}}_{n,\mathbf{j}}^{(0)} > &= h_{\mathbf{j}-M\mathbf{k}}^{(\mu)}. \end{aligned}$$

Thus, the decomposition and the reconstruction equations can be written as

$$\mathbf{f}_{n-1,\mathbf{k}}^{(\mu)*} = \sum_{\mathbf{j}} \mathbf{f}_{n,\mathbf{j}}^{(0)*}\, \tilde{h}_{\mathbf{j}-M\mathbf{k}}^{*(\mu)}, \tag{4.6}$$

$$\mathbf{f}_{n,\mathbf{j}}^{(0)*} = \sum_{\mu,\mathbf{k}} \mathbf{f}_{n-1,\mathbf{k}}^{(\mu)*}\, h_{\mathbf{j}-M\mathbf{k}}^{*(\mu)}. \tag{4.7}$$

4.3 Symbol and Modulation Matrix

Definition 18 Given a complex valued vector $\mathbf{r} = [r_1, r_2, \ldots r_n]^t$, an integer valued vector $\mathbf{s} = [s_1, s_2 \ldots s_n]^t$ and an integer-valued matrix $L = [L_0, L_1 \ldots L_n]$, where L_i is the i^{th} column of L. Then, the vector $\mathbf{r}$ raised to the vector $\mathbf{s}$ power is a scalar and is defined to be

$$\mathbf{r}^{\mathbf{s}} = r_1^{s_1} r_2^{s_2} \ldots r_n^{s_n},$$

and $\mathbf{r}$ raised to the power L is a row vector defined as

$$\mathbf{r}^L = [\mathbf{r}^{L_1}, \mathbf{r}^{L_2} \cdots \mathbf{r}^{L_n}].$$

In particular, if $\mathbf{z} = (z_1, z_2, \ldots, z_d)$ and $\mathbf{k} = (k_1, k_2, \ldots, k_d)$, then $\mathbf{z}^{\mathbf{k}} = z_1^{k_1} \ldots z_d^{k_d},\ z_j = e^{-i\xi_j}$.

Definition 19 The *symbols* of the scaling function and wavelet coefficients are defined by

$$H^{(\mu)}(\boldsymbol{\xi}) = \frac{1}{\sqrt{m}} \sum_{\mathbf{k}} h_{\mathbf{k}}^{(\mu)}\ e^{-i<\mathbf{k},\, \boldsymbol{\xi}>},$$

or in $\mathbf{z}$ notation,

$$H^{(\mu)}(\mathbf{z}) = \frac{1}{\sqrt{m}} \sum_{\mathbf{k}} h_{\mathbf{k}}^{(\mu)}\ \mathbf{z}^{\mathbf{k}}.$$

Since

$$\boldsymbol{\phi}^{(\mu)}(\mathbf{x}) = m^{\frac{1}{2}} \sum_{\mathbf{k}\in\boldsymbol{\Lambda}} h_{\mathbf{k}}^{(\mu)} \boldsymbol{\phi}^{(0)}(M\mathbf{x} - \mathbf{k}),$$

we have

$$\hat{\boldsymbol{\phi}}^{(\mu)}(\boldsymbol{\xi}) = H^{(\mu)}(M^{-t}\boldsymbol{\xi})\hat{\boldsymbol{\phi}}^{(0)}(M^{-t}\boldsymbol{\xi}).$$

Lemma 3 Let $\Gamma = M\mathbb{Z}^d$ be the lattice and D be the digit set. Then for $\mathbf{k} \in \mathbb{Z}^d$,

$$\sum_{\mathbf{d}\in D} e^{-2\pi i<\mathbf{k}\,,\, M^{-T}\mathbf{d}>} = \begin{cases} m & \text{if} \quad \mathbf{k} \in \Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: This is proved in [4].

Theorem 12 The orthogonality conditions in terms of symbols H are

$$\sum_{\mathbf{d}\in D} H^{(\mu)}(\boldsymbol{\xi} + 2\pi M^{-T}\mathbf{d}) H^{(\nu)*}(\boldsymbol{\xi} + 2\pi M^{-T}\mathbf{d}) = \delta_{\mu\nu} I. \tag{4.8}$$

Proof: Since

$$H^{(\mu)}(\boldsymbol{\xi}) = \frac{1}{\sqrt{m}} \sum_{\mathbf{k}} \ h_{\mathbf{k}}^{(\mu)} \ e^{-i<\mathbf{k},\ \boldsymbol{\xi}>},$$

we find

$$\begin{aligned} H^{(\mu)}(\boldsymbol{\xi} + 2\pi M^{-T}\mathbf{d}) &= \frac{1}{\sqrt{m}} \sum_{\mathbf{k}} \ h_{\mathbf{k}}^{(\mu)} \ e^{-i<\mathbf{k},\ \boldsymbol{\xi}+2\pi M^{-T}\mathbf{d}>} \\ &= \frac{1}{\sqrt{m}} \sum_{\mathbf{k}} \ h_{\mathbf{k}}^{(\mu)} \ e^{-i<\mathbf{k},\ \boldsymbol{\xi}>} e^{-2\pi i<\mathbf{k}, M^{-T}\mathbf{d}>}. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{\mathbf{d}} H^{(\mu)}(\boldsymbol{\xi} + 2\pi M^{-T}\mathbf{d}) H^{(\nu)*}(\boldsymbol{\xi} + 2\pi M^{-T}\mathbf{d}) \\ &\qquad = \frac{1}{m} \sum_{\mathbf{d}} \sum_{\mathbf{j}} \sum_{\mathbf{k}} h_{\mathbf{j}}^{(\mu)} h_{\mathbf{k}}^{(\nu)*} e^{-i<\mathbf{k}-\mathbf{j}, \boldsymbol{\xi}>} e^{-2\pi i<\mathbf{k}-\mathbf{j}, M^{-T}\mathbf{d}>} \\ &\qquad = \sum_{\mathbf{j}} \sum_{\mathbf{k}} h_{\mathbf{j}}^{(\mu)} h_{\mathbf{k}}^{(\nu)*} e^{-i<\mathbf{k}-\mathbf{j}, \boldsymbol{\xi}>} \frac{1}{m} \sum_{\mathbf{d}} e^{-2\pi i<\mathbf{k}-\mathbf{j}, M^{-T}\mathbf{d}>}. \end{aligned}$$

Setting $\mathbf{j} = \mathbf{k} - M\,\mathbf{l}$ and using lemma 2, this equals

$$\sum_{\mathbf{k},\mathbf{j}} h_{\mathbf{k}-M\mathbf{l}}^{(\mu)} h_{\mathbf{k}}^{(\nu)*} e^{-i<M\mathbf{l}, \boldsymbol{\xi}>} = \sum_{\mathbf{l}} \left\{ \sum_{\mathbf{k}} h_{\mathbf{k}-M\mathbf{l}}^{(\mu)} h_{\mathbf{k}}^{(\nu)*} \right\} e^{-i<M\mathbf{l}, \boldsymbol{\xi}>} = \delta_{\mu\nu} I$$

by the orthogonality condition.

Example 1: Standard quincunx

$$M = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

which has $\{(0,0)^t, (1,0)^t\}$ as its digit set. The orthogonality condition turns out to be

$$|H(\ \boldsymbol{\xi}\ )|^2 + |H(\boldsymbol{\xi}\ + (\pi,\pi)^t)|^2 = 1.$$

Example 2: Non-standard quincunx

$$M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

The orthogonality condition is

$$|H(\boldsymbol{\xi})|^2 + |H(\boldsymbol{\xi} + (\pi, -\pi)^t)|^2 = 1.$$

Example 3: Column lattice

$$M = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}.$$

The orthogonality condition turns out to be

$$|H(\boldsymbol{\xi})|^2 + |H(\boldsymbol{\xi} + (\pi, 0)^t)|^2 = 1.$$

Definition 20 The *modulation matrix* is

$$M(\boldsymbol{\xi}) = \begin{bmatrix} H^{(0)}(\boldsymbol{\xi}) & H^{(0)}(\boldsymbol{\xi} + 2\pi M^{-t}\mathbf{d}_1) & \cdots & H^{(0)}(\boldsymbol{\xi} + 2\pi M^{-t}\mathbf{d}_{m-1}) \\ H^{(1)}(\boldsymbol{\xi}) & H^{(1)}(\boldsymbol{\xi} + 2\pi M^{-t}\mathbf{d}_1) & \cdots & H^{(1)}(\boldsymbol{\xi} + 2\pi M^{-t}\mathbf{d}_{m-1}) \\ H^{(2)}(\boldsymbol{\xi}) & H^{(2)}(\boldsymbol{\xi} + 2\pi M^{-t}\mathbf{d}_1) & \cdots & H^{(2)}(\boldsymbol{\xi} + 2\pi M^{-t}\mathbf{d}_{m-1}) \\ \vdots & \vdots & \cdots & \vdots \\ H^{(m-1)}(\boldsymbol{\xi}) & H^{(m-1)}(\boldsymbol{\xi} + 2\pi M^{-t}\mathbf{d}_1) & \cdots & H^{(m-1)}(\boldsymbol{\xi} + 2\pi M^{-t}\mathbf{d}_{m-1}) \end{bmatrix},$$

where $\mathbf{d}_0, \mathbf{d}_1 \ldots \mathbf{d}_{m-1}$ is any ordering of the digits with $\mathbf{d}_0 = \mathbf{0}$.

The modulation matrices in the standard quincunx, non-standard quincunx and the column cases are respectively

$$M(\boldsymbol{\xi}) = \begin{bmatrix} H^{(0)}(\boldsymbol{\xi}) & H^{(0)}(\boldsymbol{\xi} + (\pi, \pi)^t) \\ H^{(1)}(\boldsymbol{\xi}) & H^{(1)}(\boldsymbol{\xi} + (\pi, \pi)^t) \end{bmatrix},$$

$$M(\boldsymbol{\xi}) = \begin{bmatrix} H^{(0)}(\boldsymbol{\xi}) & H^{(0)}(\boldsymbol{\xi} + (\pi, -\pi)^t) \\ H^{(1)}(\boldsymbol{\xi}) & H^{(1)}(\boldsymbol{\xi} + (\pi, -\pi)^t) \end{bmatrix},$$

and

$$M(\boldsymbol{\xi}) = \begin{bmatrix} H^{(0)}(\boldsymbol{\xi}) & H^{(0)}(\boldsymbol{\xi} + (\pi, 0)^t) \\ H^{(1)}(\boldsymbol{\xi}) & H^{(1)}(\boldsymbol{\xi} + (\pi, 0)^t) \end{bmatrix}.$$

4.4 Polyphase Components and Polyphase Matrix

An arbitrary $\mathbf{k} \in \mathbb{Z}^d$ can be written uniquely as

$$\mathbf{k} = M\mathbf{j} + \mathbf{d},$$

where $M\mathbf{j} \in \Gamma$ and $\mathbf{d} \in D$. To make the notation easier, we fix some ordering $\mathbf{d}_0, \ldots, \mathbf{d}_{m-1}$ with $\mathbf{d}_0 = \mathbf{0}$.

Notation. Let $h_{\mathbf{j}}^{(\mu,k)} = h_{M\mathbf{j}+\mathbf{d}_k}^{(\mu)}$.

Definition 21 The *polyphase symbols* are

$$H^{(\mu,k)}(\boldsymbol{\xi}\ ) = \sum_{\mathbf{j}} h_{M\mathbf{j}+\mathbf{d}_k}^{(\mu)} e^{-i<\mathbf{j},\, \boldsymbol{\xi}>} = \sum_{\mathbf{j}} h_{\mathbf{j}}^{(\mu,k)} e^{-i<\mathbf{j},\boldsymbol{\xi}>}.$$

The symbol can be expressed in terms of polyphase components as

$$H^{(\mu)}(z) = \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} \mathbf{z}^{-\mathbf{d}_j} H^{(\mu,j)}(\mathbf{z}^M),$$

where $j = 1, 2, \ldots, m-1$.

The decomposition and reconstruction relations can be written in polyphase form as

$$\begin{aligned}
\mathbf{f}_{n-1,\mathbf{k}}^{(\mu)*} &= \sum_{\mathbf{j}} \mathbf{f}_{n,\mathbf{j}}^{(0)*} \tilde{h}_{\mathbf{j}-M\mathbf{k}}^{(\mu)*} \\
&= \sum_{k} \sum_{\mathbf{l}} \mathbf{f}_{n,M\mathbf{l}+\mathbf{d}_k}^{(0)*} \tilde{h}_{M\mathbf{l}-M\mathbf{k}+\mathbf{d}_k}^{(\mu)*} \\
&= \sum_{k} \sum_{\mathbf{l}} \mathbf{f}_{n,\mathbf{l}}^{(0,k)*} \tilde{h}_{\mathbf{l}-\mathbf{k}}^{(\mu,k)*},
\end{aligned} \tag{4.9}$$

and

$$\mathbf{f}_{n,\mathbf{l}}^{(\mu,k)*} = \sum_{k} \sum_{\mu} \sum_{\mathbf{k}} \mathbf{f}_{n-1,\mathbf{k}}^{(\mu)*} \tilde{h}_{\mathbf{l}-\mathbf{k}}^{(\mu,k)}, \tag{4.10}$$

which are the sums of convolutions.

Definition 22 The *polyphase matrix* is

$$P(\boldsymbol{\xi}) = \begin{bmatrix} H^{(0,0)}(\boldsymbol{\xi}) & H^{(0,1)}(\boldsymbol{\xi}) & \cdots & H^{(0,m-1)}(\boldsymbol{\xi}) \\ H^{(1,0)}(\boldsymbol{\xi}) & H^{(1,1)}(\boldsymbol{\xi}) & \cdots & H^{(1,m-1)}(\boldsymbol{\xi}) \\ H^{(2,0)}(\boldsymbol{\xi}) & H^{(2,1)}(\boldsymbol{\xi}) & \cdots & H^{(2,m-1)}(\boldsymbol{\xi}) \\ \vdots & \vdots & \cdots & \vdots \\ H^{(m-1,0)}(\boldsymbol{\xi}) & H^{(m-1,1)}(\boldsymbol{\xi}) & \cdots & H^{(m-1,m-1)}(\boldsymbol{\xi}) \end{bmatrix}.$$

The orthogonality condition is

$$P(\boldsymbol{\xi})P(\boldsymbol{\xi})^* = I. \tag{4.11}$$

Example 1: The Haar wavelet H. The symbol is

$$H^{(0)}(\mathbf{z}) = \frac{1+z_1}{2}.$$

Thus,

$$H^{(0)}(\mathbf{z})^* = \frac{1+z_1^{-1}}{2}.$$

$$H^{(0)}(\mathbf{z})H^{(0)}(\mathbf{z})^* + H^{(0)}(-\mathbf{z})H^{(0)}(-\mathbf{z})^* = \frac{1}{4}[(1+z_1)(1+\frac{1}{z_1}) + (1-z_1)(1-\frac{1}{z_1})] = 1,$$

where $\frac{1}{\mathbf{z}} = (\frac{1}{z_1}, \frac{1}{z_2})$, The orthogonality condition is satisfied. The polyphase symbols are

$$H^{(0,0)}(\mathbf{z}) = H^{(0,1)}(\mathbf{z}) = \frac{1}{\sqrt{2}}.$$

The scaling function is the two-dimensional Haar scaling function $\boldsymbol{\phi} = \chi_S$, where S is as shown in fig 3.2.

Example 2: The Daubechies wavelet DB2. The symbol is

$$H^{(0)}(\mathbf{z}) = \frac{1}{\sqrt{2}}(h_0 + h_1 z_1 + h_2 z_1^2 + h_3 z_1^3).$$

The polyphase symbols are

$$\begin{aligned} H^{(0,0)}(\mathbf{z}) &= \sum_{\mathbf{j}} h_{M\mathbf{j}} \mathbf{z}^{\mathbf{j}} \\ &= h_0 + h_2 z_1 z_2, \end{aligned}$$

and

$$H^{(0,1)}(\mathbf{z}) = h_1 + h_3 z_1 z_2.$$

The corresponding scaling function is shown in fig 3.4.

Example 3: K-V scaling function. The symbol is

$$H^{(0)}(\mathbf{z}) = \frac{1}{\sqrt{2}}(h_2 + h_3 z_1 + h_4 z_1^2 + h_5 z_1^3 + h_0 z_1 z_2^{-1} + h_1 z_1^2 z_2^{-1} + h_6 z_1 z_2 + h_7 z_1^2 z_2).$$

The polyphase symbols are

$$H^{(0,0)}(z) = h_2 + h_6 z_1 + h_0 z_2 + h_4 z_1 z_2$$

$$H^{(1,1)}(z) = h_3 + h_7 z_1 + h_1 z_2 + h_5 z_1 z_2.$$

4.5 Necessary and Sufficient Conditions for the Existence of Wavelets

Now we state the necessary and sufficient conditions such that the lattice translates of $\{\boldsymbol{\phi}^{(j)} : j = 1, 2....m-1\}$ will form an orthonormal basis for W_0.

Theorem 13 *Let* $\{V_j^{(0)}\}$ *be an orthogonal MRA for* $L_2(\mathbb{R}^d)$*. Then the following are equivalent.*

(i) $\{\boldsymbol{\phi}^{(j)}(\mathbf{x}-\mathbf{k}) : j = 1, 2....m-1,\ \mathbf{k} \in \Gamma\}$ *will form an orthonormal basis for* W_0.

(ii) $P(\boldsymbol{\xi})$ *is paraunitary a.e.*

(iii) $M(\boldsymbol{\xi})$ *is paraunitary a.e.*

(iv) The recursion coefficients $\{h_{\mathbf{k}}^{(\mu)} : j = 0, 1......m-1\ \}$ *satisfy*

$$\delta_{\mu,\nu}\delta_{\mathbf{jk}} = \sum_{\mathbf{p}} h_{\mathbf{p}-M\mathbf{j}}^{(\mu)} h_{\mathbf{p}-M\mathbf{k}}^{*(\nu)}.$$

Proof: This is proved in [11].

Thus, once an MRA has been found, we can construct a wavelet basis for $L_2(\mathbb{R}^d)$ if we can complete a paraunitary matrix, namely P. We will consider the completion problem in section 8.1.

CHAPTER 5. Computing Moments and Approximation Order

5.1 Moments

We will use the standard multi-index notation $\mathbf{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ is a d-tuple of non-negative integers. The degree of $\boldsymbol{\alpha}$ is

$$|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \cdots + \alpha_d.$$

The number of different multi-indices $\boldsymbol{\alpha}$ of degree s is

$$d_s = \begin{pmatrix} s+d-1 \\ d-1 \end{pmatrix}.$$

Let

$$D^{\boldsymbol{\alpha}} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}.$$

Definition 23 The *discrete moments* are defined by

$$m_{\boldsymbol{\alpha}}^{(j)} = \frac{1}{\sqrt{m}} \sum_{\mathbf{k}} \mathbf{k}^{\boldsymbol{\alpha}} \, h_{\mathbf{k}}^{(j)} = i^{|\boldsymbol{\alpha}|} D^{\boldsymbol{\alpha}} H^{(j)}(\mathbf{0}),$$

$j = 0, 1, 2, \ldots, m-1.$

Definition 24 The *continuous moments* are defined by

$$\boldsymbol{\mu}_{\boldsymbol{\alpha}}^{(j)} = \int \mathbf{x}^{\boldsymbol{\alpha}} \boldsymbol{\phi}^{(j)}(\mathbf{x}) d\mathbf{x} = i^{|\boldsymbol{\alpha}|} D^{\boldsymbol{\alpha}} \hat{\boldsymbol{\phi}}^{(j)}(\mathbf{0})$$

$j = 0, 1, 2, \ldots, m-1.$

Note: $m_{\boldsymbol{\alpha}}^{(j)}$ is an $r \times r$ matrix and $\boldsymbol{\mu}_{\boldsymbol{\alpha}}^{(j)}$ is an $r \times 1$ vector.

In sections (5.1) and (5.2), we will drop the subscript, and always consider $\boldsymbol{\phi} = \boldsymbol{\phi}^{(0)}$, $m_{\boldsymbol{\alpha}} = m_{\boldsymbol{\alpha}}^{(0)}$. For a particular s choose any fixed ordering of all $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}| = s$. This ordering will be fixed from now on. Let $\boldsymbol{\alpha}_j$ be the j^{th} index. We collect the d_s monomials $\mathbf{x}^{\boldsymbol{\alpha}}$ in the chosen order to form a column vector of monomials

$$X^{[s]}(\mathbf{x}) = \begin{pmatrix} \mathbf{x}^{\boldsymbol{\alpha}_1} \\ \mathbf{x}^{\boldsymbol{\alpha}_2} \\ \vdots \\ \mathbf{x}^{\boldsymbol{\alpha}_{d_s}} \end{pmatrix}, \quad \mathbf{x} \in \mathbb{R}^d.$$

Notation: We also use the notation $[\mathbf{x}^{\boldsymbol{\alpha}}]_{|\boldsymbol{\alpha}|=s}$ to represent a column vector of monomials in the given order. Given any expansive $d \times d$ matrix $M = [m_{ij}]_{i,j=1,2,\ldots,d}$ with scalar entries, let $M^{[s]} = [m^{[s]}_{i,j}]$ be the $d_s \times d_s$ matrix whose scalar entries $m^{[s]}_{i,j}$ are defined by the equation

$$(M\mathbf{x})^{\boldsymbol{\alpha}_i} = \sum_{j=1}^{d_s} m^{[s]}_{i,j} \mathbf{x}^{\boldsymbol{\alpha}_j}.$$

Dilation of $X^{[s]}(x)$ by M obeys the rule

$$X^{[s]}(Mx) = M^{[s]} X^{[s]}(x).$$

If $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d)^t$ is the vector of eigenvalues of M, then $(\boldsymbol{\lambda}^{\boldsymbol{\alpha}_1}, \ldots, \boldsymbol{\lambda}^{\boldsymbol{\alpha}_{d_s}})^t$ is the vector of all eigenvalues of $M^{[s]}$ [2].

Note: Since the eigenvalues of $M^{[s]}$ are of the form $\boldsymbol{\lambda}^{\boldsymbol{\alpha}_j}$, $M^{[s]}$ is invertible, since all of the eigenvalues of M are strictly greater than 1 in absolute value. It is easy to show that $(M^{[s]})^{-1} = (M^{-1})^{[s]}$. Consider

$$\boldsymbol{\mu}^{[s]}_i = \int \mathbf{x}^{\boldsymbol{\alpha}_i} \boldsymbol{\phi}(\mathbf{x}) d\mathbf{x},$$

and

$$\boldsymbol{\mu}^{[s]} = \begin{pmatrix} \boldsymbol{\mu}^{[s]}_1 \\ \boldsymbol{\mu}^{[s]}_2 \\ \vdots \\ \boldsymbol{\mu}^{[s]}_{d_s} \end{pmatrix}.$$

$\boldsymbol{\mu}^{[s]}$ is a column vector of size $rd_s \times 1$, where each $\boldsymbol{\mu}^{[s]}_i$ is of size $r \times 1$, $i = 1, 2, \ldots d_s$.

Let

$$\mathbf{m}^{[s]} = \begin{pmatrix} m_{\boldsymbol{\alpha}_1} \\ m_{\boldsymbol{\alpha}_2} \\ \vdots \\ m_{\boldsymbol{\alpha}_{d_s}} \end{pmatrix}.$$

The size of $\mathbf{m}^{[s]}$ is $d_s r \times r$.

Let I be the $r \times r$ identity matrix.

$$
\begin{aligned}
(M^{[s]} \otimes I)\boldsymbol{\mu}^{[\boldsymbol{s}]} &= \begin{pmatrix} m_{11}^{[s]}I & m_{12}^{[s]}I & \cdots & m_{1,d_s}^{[s]}I \\ m_{21}^{[s]}I & m_{22}^{[s]}I & \cdots & m_{2,d_s}^{[s]}I \\ \vdots & \vdots & \cdots & \vdots \\ m_{d_s,1}^{[s]}I & m_{d_s,2}^{[s]}I & \cdots & m_{d_s,d_s}^{[s]}I \end{pmatrix} \begin{pmatrix} \boldsymbol{\mu}_1^{[s]} \\ \boldsymbol{\mu}_2^{[s]} \\ \vdots \\ \boldsymbol{\mu}_{d_s}^{[s]} \end{pmatrix} \\
&= \begin{pmatrix} m_{11}^{[s]}\boldsymbol{\mu}_1^{[s]} + \cdots + m_{1,d_s}^{[s]}\boldsymbol{\mu}_{d_s}^{[s]} \\ m_{21}^{[s]}\boldsymbol{\mu}_1^{[s]} + \cdots + m_{2,d_s}^{[s]}\boldsymbol{\mu}_{d_s}^{[s]} \\ \vdots \\ m_{d_s,1}^{[s]}\boldsymbol{\mu}_1^{[s]} + \cdots + m_{d_s,d_s}^{[s]}\boldsymbol{\mu}_{d_s}^{[s]} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= \begin{pmatrix} \int(m_{11}^{[s]}\mathbf{x}^{\boldsymbol{\alpha}_1} + \cdots + m_{1,d_s}^{[s]}\mathbf{x}^{\boldsymbol{\alpha}_{d_s}})\boldsymbol{\phi}(\mathbf{x})d\mathbf{x} \\ \int(m_{21}^{[s]}\mathbf{x}^{\boldsymbol{\alpha}_1} + \cdots + m_{2,d_s}^{[s]}\mathbf{x}^{\boldsymbol{\alpha}_{d_s}})\boldsymbol{\phi}(\mathbf{x})d\mathbf{x} \\ \vdots \\ \int(m_{d_s,1}^{[s]}\mathbf{x}^{\boldsymbol{\alpha}_1} + \cdots + m_{d_s,d_s}^{[s]}\mathbf{x}^{\boldsymbol{\alpha}_{d_s}})\boldsymbol{\phi}(\mathbf{x})d\mathbf{x} \end{pmatrix} \\
&= \sqrt{m} \begin{pmatrix} \sum_{\mathbf{k}} \int(m_{11}^{[s]}\mathbf{x}^{\boldsymbol{\alpha}_1} + \cdots + m_{1,d_s}^{[s]}\mathbf{x}^{\boldsymbol{\alpha}_{d_s}})h_{\mathbf{k}}\boldsymbol{\phi}(M\mathbf{x}-\mathbf{k})d\mathbf{x} \\ \sum_{\mathbf{k}} \int(m_{21}^{[s]}\mathbf{x}^{\boldsymbol{\alpha}_1} + \cdots + m_{2,d_s}^{[s]}\mathbf{x}^{\boldsymbol{\alpha}_{d_s}})h_{\mathbf{k}}\boldsymbol{\phi}(M\mathbf{x}-\mathbf{k})d\mathbf{x} \\ \vdots \\ \sum_{\mathbf{k}} \int(m_{d_s,1}^{[s]}\mathbf{x}^{\boldsymbol{\alpha}_1} + \cdots + m_{d_s,d_s}^{[s]}\mathbf{x}^{\boldsymbol{\alpha}_{d_s}})h_{\mathbf{k}}\boldsymbol{\phi}(M\mathbf{x}-\mathbf{k})d\mathbf{x} \end{pmatrix} \\
&= \sqrt{m} \begin{pmatrix} \sum_{\mathbf{k}} \int X^{[s]}(M\mathbf{x})_1 h_{\mathbf{k}}\boldsymbol{\phi}(M\mathbf{x}-\mathbf{k})d\mathbf{x} \\ \sum_{\mathbf{k}} \int X^{[s]}(M\mathbf{x})_2 h_{\mathbf{k}}\boldsymbol{\phi}(M\mathbf{x}-\mathbf{k})d\mathbf{x} \\ \vdots \\ \sum_{\mathbf{k}} \int X^{[s]}(M\mathbf{x})_{d_s} h_{\mathbf{k}}\boldsymbol{\phi}(M\mathbf{x}-\mathbf{k})d\mathbf{x} \end{pmatrix} \\
&= \frac{1}{\sqrt{m}} \begin{pmatrix} \sum_{\mathbf{k}} \int X^{[s]}(\mathbf{y})_1 h_{\mathbf{k}}\boldsymbol{\phi}(\mathbf{y}-\mathbf{k})d\mathbf{y} \\ \sum_{\mathbf{k}} \int X^{[s]}(\mathbf{y})_2 h_{\mathbf{k}}\boldsymbol{\phi}(\mathbf{y}-\mathbf{k})d\mathbf{y} \\ \vdots \\ \sum_{\mathbf{k}} \int X^{[s]}(\mathbf{y})_{d_s} h_{\mathbf{k}}\boldsymbol{\phi}(\mathbf{y}-\mathbf{k})d\mathbf{y} \end{pmatrix} \\
&= \frac{1}{\sqrt{m}} \begin{pmatrix} \sum_{\mathbf{k}} \int X^{[s]}(\mathbf{x}+\mathbf{k})_1 h_{\mathbf{k}}\boldsymbol{\phi}(\mathbf{x})d\mathbf{x} \\ \sum_{\mathbf{k}} \int X^{[s]}(\mathbf{x}+\mathbf{k})_2 h_{\mathbf{k}}\boldsymbol{\phi}(\mathbf{x})d\mathbf{x} \\ \vdots \\ \sum_{\mathbf{k}} \int X^{[s]}(\mathbf{x}+\mathbf{k})_{d_s} h_{\mathbf{k}}\boldsymbol{\phi}(\mathbf{x})d\mathbf{x} \end{pmatrix} \\
&= \frac{1}{\sqrt{m}} \left[\sum \int \sum_{0 \le \boldsymbol{\beta} \le \boldsymbol{\alpha}} \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} \mathbf{x}^{\boldsymbol{\beta}} \ \mathbf{k}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} \ h_{\mathbf{k}}\boldsymbol{\phi}(\mathbf{x})d\mathbf{x} \right]_{|\boldsymbol{\alpha}\ |=s} \\
&= \left[\sum_{0 \le \boldsymbol{\beta} \le \boldsymbol{\alpha}} \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} m_{\boldsymbol{\alpha}-\boldsymbol{\beta}} \ \mu_{\boldsymbol{\beta}} \right]_{|\boldsymbol{\alpha}\ |=s},
\end{aligned}
$$

where

$$\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} = \begin{cases} \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \cdots \begin{pmatrix} \alpha_r \\ \beta_r \end{pmatrix} & \text{if } \beta_i \le \alpha_i, \text{for each i}, \\ 0, & \text{otherwise.} \end{cases}$$

This can be used to compute $\boldsymbol{\mu}^{[s]}$ recursively. Note that $\boldsymbol{\mu}^{[0]}$ is an eigenvector of $m^{[0]}$, which is defined uniquely up to a constant. The rest of them are then uniquely defined.

Some special cases:

Case I : $d = 1, M = 2$, so $M^{[s]} = 2^s$.

$$\mu^{[s]} = \mu_s = 2^{-s} \sum_{\beta=0}^{s} \begin{pmatrix} s \\ \beta \end{pmatrix} m_{s-\beta} \mu_\beta.$$

This is leads to (2.4) in section 2.5.

Case II: Arbitrary d, and $|\boldsymbol{\alpha}| = 0$, so $M^{[0]} = 1$. Thus

$$\boldsymbol{\mu}^{[0]} = m_0 \boldsymbol{\mu}^{[0]}.$$

In the scalar case $r = 1$, $\mu_0 = m_0 \mu_0,\ m_0 = 1$, therefore we can take $\mu_0 = 1$.

Case III: Arbitrary d and $|\boldsymbol{\alpha}| = 1$. We have $M^{[1]} = M$. Therefore

$$(M \otimes I)\boldsymbol{\mu}^{[1]} = \mathbf{m}^{[1]} \boldsymbol{\mu}_0 + (I \otimes m_0)\boldsymbol{\mu}^{[1]},$$

so

$$\boldsymbol{\mu}^{[1]} = (M \otimes I - I \otimes m_0)^{-1} m^{[1]} \boldsymbol{\mu}_0.$$

Again for the scalar case $r = 1$, $m_0 = 1$, $\mu_0 = 1$, and the above reduces to

$$M\boldsymbol{\mu}^{[1]} = \mathbf{m}^{[1]} \boldsymbol{\mu}_0 + \boldsymbol{\mu}^{[1]}.$$

therefore

$$\boldsymbol{\mu}^{[1]} = (M - I)^{-1} \mathbf{m}^{[1]}.$$

Example: Haar wavelet H.

$$\begin{aligned} m_0 &= 1, \\ m_{(1,0)} &= \frac{1}{2}, \\ m_{(0,1)} &= 0, \end{aligned}$$

therefore

$$
\begin{aligned}
\mathbf{m}^{[1]} &= \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \\
\mu_0 &= 1, \\
\boldsymbol{\mu}^{[\mathbf{1}]} &= (M-I)^{-1}\begin{pmatrix} 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}.
\end{aligned}
$$

Verify:

$$
\begin{aligned}
\boldsymbol{\mu}_{(1,0)} &= \int\int_S x_1 dx_1 dx_2 \\
&= \int_0^1 \int_{x_2}^{x_2+1} x_1 dx_1 dx_2 \\
&= \int_0^1 \left[\frac{1}{2}x_1^2\right]_{x_2}^{x_2+1} dx_2 \\
&= \int_0^1 (x_2 + \frac{1}{2}) dx_2 \\
&= \left[\frac{1}{2}x_2^2 + \frac{1}{2}x_2\right]_0^1 \\
&= 1,
\end{aligned}
$$

likewise

$$
\begin{aligned}
\boldsymbol{\mu}_{(0,1)} &= \int_0^1 \int_{x_2}^{x_2+1} x_2 dx_1 dx_2 \\
&= \int_0^1 [x_2 x_1 dx_2]_{x_2}^{x_2+1} \\
&= \frac{1}{2}.
\end{aligned}
$$

Example: Daubechies wavelet DB2.

$$
\begin{aligned}
m_0 &= 1, \\
m_{(1,0)} &= \frac{1}{\sqrt{2}}(h_1 + 2h_2 + 3h_3) = \frac{3-\sqrt{3}}{2} \\
m_{(0,1)} &= 0,
\end{aligned}
$$

therefore

$$\begin{aligned}
\mathbf{m}^{[1]} &= \begin{pmatrix} \frac{3-\sqrt{3}}{2}, \\ 0 \end{pmatrix}, \\
\boldsymbol{\mu}_0 &= 1, \\
\boldsymbol{\mu}^{[\mathbf{1}]} &= (M-I)^{-1} \begin{pmatrix} \frac{3-\sqrt{3}}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{3-\sqrt{3}}{2} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 3-\sqrt{3} \\ \frac{3-\sqrt{3}}{2} \end{pmatrix}.
\end{aligned}$$

5.2 Approximation Order and Accuracy

The *accuracy* of a refinable function $\mathbf{f}: \mathbb{R}^d \to \mathbb{C}^r$ is the largest integer k such that every polynomial $q(x)$ with degree $< k$ can be reproduced from linear combinations of the translates of f along Γ. The precise definition is given below.

Definition 25 A function $\mathbf{f}: \mathbb{R}^d \to \mathbb{C}^r$ has accuracy k if for every polynomial $q(x)$ with degree $< k$ there exist $1 \times r$ vectors $\{c^*_{\mathbf{k}}\}_{\mathbf{k}\in\Gamma}$ such that

$$q(\mathbf{x}) = \sum_{\mathbf{k}\in\Gamma} c^*_{\mathbf{k}} \mathbf{f}(\mathbf{x}+\mathbf{k}).$$

We use the terms that we used when computing moments. Let us consider the approximation order p. For all $s < p$, we have

$$X^{[s]}(\mathbf{x}) = \sum_{\mathbf{k}} g_{\mathbf{k}} \boldsymbol{\phi}(\mathbf{x}-\mathbf{k}), \tag{5.1}$$

where each $g_{\mathbf{k}}$ is a $d_s \times r$ matrix. Using the two scale refinement equation,

$$\begin{aligned}
X^{[s]}(\mathbf{x}) &= \sqrt{m} \sum_{\mathbf{k},\mathbf{l}} g_{\mathbf{k}} h_{\mathbf{l}} \boldsymbol{\phi}\ (M\mathbf{x} - M\mathbf{k} - \mathbf{l}) \\
&= \sqrt{m} \sum_{\mathbf{k},\mathbf{p}} g_{\mathbf{k}} h_{\mathbf{p}-M\mathbf{k}} \boldsymbol{\phi}\ (M\mathbf{x} - \mathbf{p}).
\end{aligned}$$

Letting M$\mathbf{x} = \mathbf{y}$,

$$(M^{-1})^{[s]} X^{[s]}(\mathbf{y}) = \sqrt{m} \sum_{\mathbf{k},\mathbf{p}} g_{\mathbf{k}} h_{\mathbf{p}-M\mathbf{k}} \boldsymbol{\phi}\ (\mathbf{y}-\mathbf{p}),$$

$$X^{[s]}(\mathbf{y}) = M^{[s]} \sqrt{m} \sum_{\mathbf{k},\mathbf{p}} g_{\mathbf{k}} h_{\mathbf{p}-M\mathbf{k}} \boldsymbol{\phi}\ (\mathbf{y}-\mathbf{p}). \tag{5.2}$$

From (5.1 and 5.2), we have

$$g_{\mathbf{p}} = \sqrt{m} M^{[s]} \sum_{\mathbf{k}} g_{\mathbf{k}} h_{\mathbf{p}-M\mathbf{k}} \quad \text{for all } \mathbf{p}.$$

Now

$$X^{[s]}(\mathbf{x}) = \sum_{\mathbf{k}} \begin{pmatrix} g_{1,1}, g_{1,2}, \ldots, g_{1,r} \\ g_{2,1}, g_{2,2}, \ldots, g_{2,r} \\ \vdots \\ g_{ds,1}, g_{ds,2}, \ldots, g_{ds,r} \end{pmatrix}_{\mathbf{k}} \begin{pmatrix} \phi_1(\mathbf{x}-\mathbf{k}) \\ \phi_2(\mathbf{x}-\mathbf{k}) \\ \vdots \\ \phi_r(\mathbf{x}-\mathbf{k}) \end{pmatrix}.$$

We note that

$$< X^{[s]}(\mathbf{x}), \tilde{\boldsymbol{\phi}}(\mathbf{x}-\mathbf{k}) >= [\mathbf{y}^*_{\boldsymbol{\alpha}}(\mathbf{k})]_{|\boldsymbol{\alpha}|=s},$$

where

$$\mathbf{y}^*_{\boldsymbol{\alpha}}(\mathbf{k}) = \sum_{0 \le \boldsymbol{\beta} \le \boldsymbol{\alpha}} \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} \tilde{\boldsymbol{\mu}}^*_{\boldsymbol{\beta}} \, \mathbf{k}^{\boldsymbol{\alpha} - \boldsymbol{\beta}},$$

thus we get

$$g_{\mathbf{k}} = [\mathbf{y}^*_{\boldsymbol{\alpha}}(\mathbf{k})]_{|\boldsymbol{\alpha}|=s}\,.$$

We have the following theorem.

Theorem 14 A necessary condition for a refinable function to have approximation order s is

$$[y^*_{\boldsymbol{\alpha}}(\mathbf{p})]_{|\boldsymbol{\alpha}|=s} = \sqrt{m} M \sum_{\mathbf{k}} [y^*_{\boldsymbol{\alpha}}(\mathbf{k})]_{|\boldsymbol{\alpha}|=s}\, h_{\mathbf{p}-M\mathbf{k}}, \text{ for all } \mathbf{p}. \tag{5.3}$$

The condition is also sufficient if the cascade algorithm converges.

Special case: $|\boldsymbol{\alpha}| = 0$. Here

$$\mathbf{y}^*_{\mathbf{0}}(\mathbf{p}) = \tilde{\boldsymbol{\mu}}^*_{\mathbf{0}}.$$

Therefore, the above condition reduces to

$$\tilde{\boldsymbol{\mu}}^*_{\mathbf{0}} = \sqrt{m} M \sum_{\mathbf{k}} \tilde{\boldsymbol{\mu}}^*_{\mathbf{0}} h_{\mathbf{p}-M\mathbf{k}}, \text{ for all } \mathbf{p}.$$

Special case: $|\boldsymbol{\alpha}| = 1$. Here

$$g_{\mathbf{k}} = \begin{bmatrix} \mathbf{y}^*_{(1,0,\ldots,0)}(\mathbf{k}) \\ \mathbf{y}^*_{(0,1,\ldots,0)}(\mathbf{k}) \\ \vdots \\ \mathbf{y}^*_{(0,0,\ldots,1)}(\mathbf{k}) \end{bmatrix}.$$

Let $\mathbf{e}_i$ be the i^{th} standard unit vector in $\mathbb{R}^d$ and let $\mathbf{k} = (k_1, \ldots, k_d)$, then

$$\mathbf{y}_{\mathbf{e}_i}(\mathbf{k}) = \tilde{\boldsymbol{\mu}}^*_{\mathbf{e}_i} + k_i \tilde{\boldsymbol{\mu}}^*_{\mathbf{0}}.$$

Thus

$$g_{\mathbf{k}} = \begin{bmatrix} \mathbf{y}^*_{(1,0,\ldots,0)}(\mathbf{k}) \\ \mathbf{y}^*_{(0,1,\ldots,0)}(\mathbf{k}) \\ \vdots \\ \mathbf{y}^*_{(0,0,\ldots,1)}(\mathbf{k}) \end{bmatrix} = \tilde{\boldsymbol{\mu}}^{[1]*} + \mathbf{k}\tilde{\boldsymbol{\mu}}^*_{\mathbf{0}}.$$

Therefore, for approximation order two we have to have

$$\tilde{\boldsymbol{\mu}}^{[1]*} + \mathbf{p}\tilde{\boldsymbol{\mu}}^*_{\mathbf{0}} = \sqrt{m}M \sum_{\mathbf{k}} (\tilde{\boldsymbol{\mu}}^{[1]*} + \mathbf{k}\tilde{\boldsymbol{\mu}}^*_{\mathbf{0}}) h_{\mathbf{p}-M\mathbf{k}}, \text{ for all } \mathbf{p}.$$

Examples 1: Haar wavelet H.

Order one:

$$H^{(0)}(\mathbf{0}) = H^{(0)}(\mathbf{0}) = \frac{1}{\sqrt{2}},$$

so the approximation order condition for order one is satisfied.

Order two: For $\mathbf{l} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

$$\boldsymbol{\mu}^{[1]*} + \mathbf{l}\boldsymbol{\mu}^*_0 = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}.$$

For $\mathbf{l} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

$$\boldsymbol{\mu}^{[1]*} + \mathbf{l}\boldsymbol{\mu}^*_0 = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1/2 \end{pmatrix}.$$

But

$$\sqrt{2}\, M \sum_{\mathbf{k}} (\boldsymbol{\mu}^{[1]*} + \mathbf{k}\boldsymbol{\mu}^*_0) h_{\mathbf{l}-M\mathbf{k}} = \sqrt{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} \frac{1}{\sqrt{2}} \quad = \quad \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$$

and

$$\sqrt{2}\, M \sum_{\mathbf{k}} (\boldsymbol{\mu}^{[1]*} + \mathbf{k}\boldsymbol{\mu}^*_0) h_{\mathbf{l}-M\mathbf{k}} = \sqrt{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} \frac{1}{\sqrt{2}} \quad = \quad \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$$

for $\mathbf{l} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and for $\mathbf{l} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ respectively, so no approximation order two holds.

Example 2: Daubechies wavelet DB2.

Order one:

$$H^{(0)}(\mathbf{0}) = h_0 + h_2 = \frac{1}{\sqrt{2}} \text{ and } H^{(1)}(\mathbf{0}) = h_1 + h_3 = \frac{1}{\sqrt{2}},$$

so the approximation condition for order one is satisfied.

Order 2: For $\mathbf{l} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

$$\boldsymbol{\mu}^{[1]*} + \mathbf{l}\boldsymbol{\mu}_0^* = \begin{pmatrix} 3-\sqrt{3} \\ (3-\sqrt{3})/2 \end{pmatrix},$$

and for $\mathbf{l} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

$$\boldsymbol{\mu}^{[1]*} + \mathbf{l}\boldsymbol{\mu}_0^* = \begin{pmatrix} 3-\sqrt{3} \\ (3-\sqrt{3})/2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4-\sqrt{3} \\ (3-\sqrt{3})/2 \end{pmatrix},$$

and

$$\begin{aligned}\sqrt{m}\, M \sum_{\mathbf{k}} (\boldsymbol{\mu}^{[1]*} + \mathbf{k}\boldsymbol{\mu}_0^*) h_{\mathbf{l}-M\mathbf{k}} &= \sqrt{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\left\{\begin{pmatrix} 3-\sqrt{3} \\ (3-\sqrt{3})/2 \end{pmatrix} h_0 + \left[\begin{pmatrix} 3-\sqrt{3} \\ (3-\sqrt{3})/2 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix}\right] h_2\right\} \\ &= \begin{pmatrix} 3-\sqrt{3} \\ (3-\sqrt{3})/2 \end{pmatrix},\end{aligned}$$

$$\begin{aligned}\sqrt{m}\, M \sum_{\mathbf{k}} (\boldsymbol{\mu}^{[1]*} + \mathbf{k}\boldsymbol{\mu}_0^*) h_{\mathbf{l}-M\mathbf{k}} &= \sqrt{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\left\{\begin{pmatrix} 3-\sqrt{3} \\ (3-\sqrt{3})/2 \end{pmatrix} h_1 + \left[\begin{pmatrix} 3-\sqrt{3} \\ (3-\sqrt{3})2 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix}\right] h_3\right\} \\ &= \begin{pmatrix} 4-\sqrt{3} \\ (3-\sqrt{3})/2 \end{pmatrix}.\end{aligned}$$

The approximation order condition for order two is also satisfied. It can be similarly checked that it does not have approximation order three.

5.3 Approximation Order in the Scalar Case

We now switch back to the full notation, so that $\phi^{(0)} = \phi$ denotes the scaling function and $\phi^{(1)}, \ldots, \phi^{(m-1)}$ denote the wavelets. We also restrict attention to the scalar case. Note that the word "scalar" refers to $r = 1$. These functions are still functions of $\mathbf{x} \in \mathbb{R}^d$.

Theorem 15

$$m_\alpha^{(0)} = 0 \text{ for all } 1 < |\alpha| \le p \text{ iff } \mu_\alpha^{(0)} = 0 \text{ for all } 1 \le |\alpha| \le p.$$

Similarly,

$$m_\alpha^{(j)} = 0 \text{ for all } 0 \le |\alpha| \le p \text{ iff } \mu_\alpha^{(j)} = 0 \text{ for all } 0 \le |\alpha| \le p,\ j = 1, 2, \ldots, m-1.$$

Proof. We prove by induction. We have

$$\widehat{\phi}^{(0)}(M^t\boldsymbol{\xi}) = H^{(0)}(\boldsymbol{\xi})\widehat{\phi}^{(0)}(\boldsymbol{\xi}),$$

$$\widehat{\phi}^{(j)}(M^t\boldsymbol{\xi}) = H^{(j)}(\boldsymbol{\xi})\widehat{\phi}(\boldsymbol{\xi}) \quad \text{for } j = 1, 2, \ldots, m-1.$$

Let $\boldsymbol{\xi} = 0$, then

$$H^{(0)}(\mathbf{0}) = 1, \quad H^{(j)}(\mathbf{0}) = 0 \text{ for all } j = 1, 2, \ldots, m-1$$

by the property of the wavelets. So for $|\boldsymbol{\alpha}| = 0$, we are done.

For large p, let $D^{[p]} = [D^{\boldsymbol{\alpha}}]_{|\boldsymbol{\alpha}|=p}$, with same order as in $X^{[p]}$ i.e,

$$D^{[p]} = \begin{pmatrix} D^{\boldsymbol{\alpha}_1} \\ D^{\boldsymbol{\alpha}_2} \\ \vdots \\ D^{\boldsymbol{\alpha}_{d_p}} \end{pmatrix}.$$

Now differentiating the above equation

$$D^{[p]}\widehat{\phi}^{(0)}(M^t\boldsymbol{\xi}) = D^{[p]}(H^{(0)}(\boldsymbol{\xi})\widehat{\phi}^{(0)}(\boldsymbol{\xi}))$$

we find that

$$M^{[p]}D^{[p]}\widehat{\phi}^{(0)}(M^t\boldsymbol{\xi}) = \left(D^{[p]}H^{(0)}(\boldsymbol{\xi})\right)\widehat{\phi}^{(0)}(\boldsymbol{\xi}) + H^0(\boldsymbol{\xi})\left(D^{[p]}\widehat{\phi}^{(0)}(\boldsymbol{\xi})\right) + \text{ middle order terms.}$$

Let $\boldsymbol{\xi} = 0$. Assume that all the partial derivatives of $H^{(0)}(\boldsymbol{\xi})$ for $0 < |\boldsymbol{\alpha}| \le p$, are zero for $\boldsymbol{\xi} = 0$ (i.e. $m_{\boldsymbol{\alpha}} = 0$).

Then the first term on the right and all the middle order terms vanish, and we get

$$M^{[p]}D^{[p]}\widehat{\phi}^{(0)}(0) = H^{(0)}D^{[p]}\widehat{\phi}^{(0)}(0).$$

Since $M^{[p]}$ is non-singular (because M is, and $H^{(0)}(0)$ is not an eigenvalue of $M^{[p]}$), this implies $D^{[p]}\widehat{\phi}^{(0)}(\mathbf{0}) = 0$, which implies that $\mu_{\boldsymbol{\alpha}}^{(0)} = 0$ for all $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}| = p$. Conversely assume $\mu_{\boldsymbol{\alpha}}^{(0)} = 0$ for $1 < |\boldsymbol{\alpha}| \le p$ and let, for $q < p$, all partial derivatives of $H^{(0)}$ of order q be zero. The left hand side is equal to zero and all the terms on the right except the first one are equal to zero by induction hypothesis. Therefore the $m_{\boldsymbol{\alpha}}^{(0)} = 0$. Similarly for the wavelets.

Notation: Let $\boldsymbol{\omega_i} = 2\pi M^{-t}\mathbf{d}_i$, where M is the dilation matrix.

Theorem 16 $\phi^{(0)}$ satisfies the approximation order condition (5.3) of order p

$$\iff D^{\boldsymbol{\alpha}} H^{(0)}(\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}=\boldsymbol{\omega_i}} = 0 \quad \text{for all } |\boldsymbol{\alpha}| \leq p-1,$$

except for $H^{(0)}(0) \neq 0$.

Proof: We prove by induction. We have the orthogonality conditions

$$H^{(0)}(\boldsymbol{\xi})H^{(0)}(\boldsymbol{\xi})^* + H^{(0)}(\boldsymbol{\xi}+\boldsymbol{\omega_1})H^{(0)}(\boldsymbol{\xi}+\boldsymbol{\omega_1})^* + \cdots + H^{(0)}(\boldsymbol{\xi}+\boldsymbol{\omega_{m-1}})H^{(0)}(\boldsymbol{\xi}+\boldsymbol{\omega_{m-1}})^* = 1,$$

$$H^{(0)}(\boldsymbol{\xi})H^{(i)}(\boldsymbol{\xi})^* + H^{(0)}(\boldsymbol{\xi}+\boldsymbol{\omega_1})H^{(i)}(\boldsymbol{\xi}+\boldsymbol{\omega_1})^* + \cdots + H^{0}(\boldsymbol{\xi}+\boldsymbol{\omega_{m-1}})H^{(i)}(\boldsymbol{\xi}+\boldsymbol{\omega_{m-1}})^* = 0$$

for $i = 1, 2, \ldots, m-1$.

Since the modulation matrix is unitary, its columns are orthogonal too.

$$H^{(0)}(\boldsymbol{\xi})H^{(0)}(\boldsymbol{\xi}+\boldsymbol{\omega_i})^* + H^{(1)}(\boldsymbol{\xi})H^{(1)}(\boldsymbol{\xi}+\boldsymbol{\omega_i})^* + \cdots + H^{(m-1)}(\boldsymbol{\xi})H^{(m-1)}(\boldsymbol{\xi}+\boldsymbol{\omega_i})^* = 0 \quad (5.4)$$

for all $i = 1, 2, \ldots, m-1$. Putting $\boldsymbol{\xi} = 0$, we get,

$$H^{(0)}(\mathbf{0})H^{(0)}(\boldsymbol{\omega_i})^* + H^{(1)}(\mathbf{0})H^{(1)}(\boldsymbol{\omega_i})^* + \cdots + H^{(m-1)}(\mathbf{0})H^{(m-1)}(\boldsymbol{\omega_i})^* = 0$$

$$\Rightarrow H^{(0)}(\boldsymbol{\omega_i})^* = 0 \ \text{ for all } j = 1, 2, \ldots, m-1.$$

Differentiating (5.4), we get

$$\sum_{k=0}^{m-1} \left\{ \sum_{\boldsymbol{\beta}=\mathbf{0}}^{\boldsymbol{\alpha}} \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} D^{\boldsymbol{\beta}} H^{(k)}(\boldsymbol{\xi}) D^{\boldsymbol{\alpha}-\boldsymbol{\beta}} H^{(k)}(\boldsymbol{\xi}+\boldsymbol{\omega_i})^* \right\} = 0.$$

Plugging in $\boldsymbol{\xi} = \mathbf{0}$, all the terms in the sum except the first one are equal to zero, because ϕ has approximation order p and thus all the derivatives up to order of p of $H^{(i)}$ vanish at 0.

We are left with

$$\sum_{\boldsymbol{\beta}=\mathbf{0}}^{\boldsymbol{\alpha}} \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} D^{\boldsymbol{\beta}} H^{(0)}(\mathbf{0}) D^{\boldsymbol{\alpha}-\boldsymbol{\beta}} H^{(0)}(\boldsymbol{\omega_i}) = 0.$$

In the last sum, because of the induction hypothesis, all terms disappear except for $\boldsymbol{\beta} = \mathbf{0}$. Thus we have

$$D^{\boldsymbol{\alpha}} H^{(0)}(\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}=\boldsymbol{\omega}_i} = 0$$

for all $i = 1, 2, \ldots, m-1$ and $|\boldsymbol{\alpha}| \leq p-1$. Conversely, if the derivatives in the statement of the theorem vanish, in a very similar way it can be shown that the all the moments up to order p of the wavelets vanish, which in turn implies the approximation order.

Theorem 17 Let $\phi^{(0)}$, $\widetilde{\phi}^{(0)}$ be a pair of dual scaling functions. Then $\phi^{(0)}$ satisfies the approximation order condition (5.3) of order p

$$\Longleftrightarrow D^{\boldsymbol{\alpha}} \widetilde{H}^{(i)}(\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}=\mathbf{0}} = 0, \text{ for all } |\boldsymbol{\alpha}| \leq p-1,\ i = 1, 2, \ldots, m-1.$$

Proof: Let $\phi^{(0)}$, $\widetilde{\phi}^{(0)}$ be a pair of dual scaling functions and $\phi^{(i)}$, $\widetilde{\phi^{(i)}}$, $i = 1, \ldots, m-1$ be the corresponding wavelets. Since $\phi^{(0)}$ has approximation order p, $\widetilde{\psi}^i$ has p vanishing moments. We prove by induction. We have biorthogonality condition

$$H^{(0)}(\boldsymbol{\xi})\widetilde{H^{(i)}}(\boldsymbol{\xi})^* + H^{(0)}(\boldsymbol{\xi}+\boldsymbol{\omega_1})\widetilde{H^{(i)}}(\boldsymbol{\xi}+\boldsymbol{\omega_1})^* + \cdots + H^{(0)}(\boldsymbol{\xi}+\boldsymbol{\omega_{m-1}})\widetilde{H^{(i)}}(\boldsymbol{\xi}+\boldsymbol{\omega_{m-1}})^* = 0$$

for $i = 1, 2, \ldots, m-1$.

Let $\boldsymbol{\alpha} = \mathbf{0}$ and let $\boldsymbol{\xi} = \mathbf{0}$. Since $\phi^{(0)}$ has approximation order p, we are left with

$$H^{(0)}(\mathbf{0})\widetilde{H^{(i)}}(\mathbf{0})^* = 0 \Rightarrow \widetilde{H^{(i)}}(\mathbf{0})^* = 0.$$

Now, differentiating the first term of the above sum and evaluating at $\boldsymbol{\xi} = \mathbf{0}$, we get

$$\sum_{\boldsymbol{\beta}=\mathbf{0}}^{\boldsymbol{\alpha}} \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} D^{\boldsymbol{\alpha}} H^{(0)}(\mathbf{0}) D^{\boldsymbol{\alpha}-\boldsymbol{\beta}} \widetilde{H^{(i)}}^*(\mathbf{0}) = 0.$$

All the other terms give

$$\sum_{\boldsymbol{\beta}=\mathbf{0}}^{\boldsymbol{\alpha}} \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} D^{\boldsymbol{\alpha}} H^{(0)}(\boldsymbol{\omega}_p) D^{\boldsymbol{\alpha}-\boldsymbol{\beta}} \widetilde{H^{(i)}}^*(\boldsymbol{\omega}_p) = 0$$

for all $p = 1, 2, \ldots, m-1$, which are all zero by the last theorem. Thus again, by the same argument, $\boldsymbol{\beta} = \mathbf{0}$ is the only term that contributes because of the induction hypothesis. Thus

$$D^{\boldsymbol{\alpha}} \widetilde{H}^{(i)} \boldsymbol{\xi})\Big|_{\boldsymbol{\xi}=\mathbf{0}} = 0, \text{ for all } |\boldsymbol{\alpha}| \leq p-1,\ i = 1, 2, \ldots, m-1.$$

CHAPTER 6. Lifting Multivariate Wavelets

6.1 Lifting

In this chapter, we only consider scalar multivariate wavelets with $m = 2$. For simplicity we write H, G instead of $H^{(0)}$, $H^{(1)}$.

Definition 26 A filter pair H, G is *complementary* if the corresponding polyphase matrix has determinant 1 .

Theorem 18 Let (H, G) be complementary , then any other finite filter G^{new} complementary to H is of the form

$$G^{new}(\mathbf{z}) = G(\mathbf{z}) + H(\mathbf{z})L(\mathbf{z}^M),$$

where $L(\mathbf{z})$ is a *Laurent polynomial.* Conversely, any filter of this form is complementary to G. In other words, the new polyphase matrix can be written as

$$P^{new}(\mathbf{z}) = \begin{bmatrix} 1 & 0 \\ L(\mathbf{z}) & 1 \end{bmatrix} P(\mathbf{z}).$$

This creates a new dual wavelet function given by the dual polyphase matrix

$$\tilde{P}^{new}(\mathbf{z}) = \begin{bmatrix} 1 & -L(\mathbf{z}^{-1}) \\ 0 & 1 \end{bmatrix} \tilde{P}(\mathbf{z}),$$

which implies that the dual scaling function is given by

$$\tilde{H}^{new}(\mathbf{z}) = \tilde{H}(\mathbf{z}) - \tilde{G}(\mathbf{z})L(\mathbf{z}^{-M}).$$

This can be used to raise the approximation order of $\widetilde{\phi}$. The approximation order of ϕ remains unchanged. This process can also be applied on the dual side.

Example: Let us consider H from section 4.4.

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

We know that it has approximation order one. Let us consider

$$L(\mathbf{z}) = a + bz_1 + cz_2 \;\;\Rightarrow\;\; L(\mathbf{z}^M) = a + bz_1z_2 + cz_1z_2^{-1},$$

or

$$L(\xi_1, \xi_2) = a + be^{-i\xi_1}e^{-i\xi_2} + ce^{-i\xi_1}e^{i\xi_2}.$$

Therefore

$$G^{new} = -\frac{1}{2} + \frac{1}{2}e^{-i\xi_1} + (a + be^{-i\xi_1}e^{-i\xi_2} + ce^{-i\xi_1}e^{i\xi_2})\left(\frac{1}{2} + \frac{1}{2}e^{-i\xi_1}\right).$$

Approximation order 1 :

$$G^{new}(0) = 0 \quad \Rightarrow a + b + c = 0$$

Approximation order two:

$$G^{new}(0) = 0 \;\text{ and }\; \frac{\partial}{\partial \xi_1}G^{new}(0) = \frac{\partial}{\partial \xi_2}G^{new}(0) = 0$$

$$\Rightarrow b + c = \frac{1}{2}, \quad \text{and} \quad b = c$$

$$\Rightarrow\; b = c = -\frac{1}{4}, \quad a = \frac{1}{2}.$$

Thus,

$$L(z) = \frac{1}{2} - \frac{1}{4}z_1 - \frac{1}{4}z_2.$$

Therefore

$$G(\mathbf{z}) = -\frac{1}{4} + \frac{3}{4}z_1 - \frac{1}{8}z_1z_2 - \frac{1}{8}z_1^2z_2 - \frac{1}{8}z_1z_2^{-1} - \frac{1}{8}z_1^2z_2^{-1}.$$

$\tilde{\phi}^{new}$, the new scaling function on the dual side, can be obtained from

$$\tilde{P}^{new} = \begin{pmatrix} 1 & -L(\mathbf{z}^{-1}) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} H_0 & H_1 \\ G_0 & G_1 \end{pmatrix}.$$

Thus

$$\tilde{H}^{new} = H - L(\mathbf{z}^{-M})G,$$

$$\begin{aligned}
\tilde{H}^{new} &= \frac{1}{2}+\frac{1}{2}e^{-i\xi_1}-\left(\frac{1}{2}-\frac{1}{4}e^{i(\xi_1+\xi_2)}-\frac{1}{4}e^{i(\xi_1-\xi_2)}\right)\left(\frac{-1}{2}+\frac{1}{2}e^{-i\xi_1}\right)\\
&= \frac{3}{4}+\frac{1}{4}e^{-i\xi_1}-\frac{1}{8}e^{i(\xi_1+\xi_2)}+\frac{1}{8}e^{i\xi_2}-\frac{1}{8}e^{i(\xi_1-\xi_2)}+\frac{1}{8}e^{-i\xi_2}\\
&= \frac{3}{4}+\frac{1}{4}z_1-\frac{1}{8}z_1^{-1}z_2^{-1}+\frac{1}{8}z_2^{-1}-\frac{1}{8}z_1^{-1}z_2+\frac{1}{8}z_2.
\end{aligned}$$

The new filter coefficients are located as follows

$$\begin{array}{ccc}
 & g_{1,1} & g_{2,1}\\
g_{0,0} & g_{1,0} & \\
 & g_{1,-1} & g_{2,-1}
\end{array}
\qquad
\begin{array}{ccc}
\tilde{h}_{-1,1} & \tilde{h}_{0,1} & \\
 & \tilde{h}_{0,0} & \tilde{h}_{1,0}\\
\tilde{h}_{-1,-1} & \tilde{h}_{0,-1} &
\end{array}.$$

We can verify that this new wavelet has approximation order 2. A direct computation yields

$$m_{1,0}=\frac{1}{2},\quad \text{and}\quad m_{0,1}=0,$$

$$\begin{aligned}
\boldsymbol{\mu}^{[1]} &= \left[\begin{pmatrix}1&1\\1&-1\end{pmatrix}-\begin{pmatrix}1&0\\0&1\end{pmatrix}\right]^{-1}\mathbf{m}^{[1]}\\
&= \begin{pmatrix}2&1\\1&0\end{pmatrix}\begin{pmatrix}1/2\\0\end{pmatrix}=\begin{pmatrix}1\\1/2\end{pmatrix}.
\end{aligned}$$

We need to have

$$\boldsymbol{\mu}^{[1]}=\sqrt{2}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\left\{\boldsymbol{\mu}^{[1]}\frac{1}{2\sqrt{2}}+\left[\boldsymbol{\mu}^{[1]}+\begin{pmatrix}-1\\0\end{pmatrix}\mu_0\right]\frac{1}{4\sqrt{2}}+\left[\boldsymbol{\mu}^{[1]}+\begin{pmatrix}0\\-1\end{pmatrix}\mu_0\right]\frac{1}{4\sqrt{2}}\right\},$$

and

$$\boldsymbol{\mu}^{[1]}+\begin{pmatrix}1\\0\end{pmatrix}\mu_0=\sqrt{2}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\left\{\boldsymbol{\mu}^{[1]}\frac{3}{2\sqrt{2}}-\left[\boldsymbol{\mu}^{[1]}+\begin{pmatrix}-1\\0\end{pmatrix}\mu_0\right]\frac{1}{4\sqrt{2}}-\left[\boldsymbol{\mu}^{[1]}+\begin{pmatrix}0\\-1\end{pmatrix}\mu_0\right]\frac{1}{4\sqrt{2}}\right\}.$$

Or,

$$\mu_1^{[1]}=\mu_1^{[1]}+\mu_2^{[1]}-\frac{1}{2}\tilde{\mu}_0,$$

and

$$\mu_2^{[1]}=\mu_1^{[1]}-\mu_2^{[1]},$$

These equations are identically satisfied if we let $\mu_2^{[1]}=\frac{1}{2}\mu_1^{[1]}$, $\mu_2^{[1]}=\frac{1}{2}\mu_0$, which were the actually computed values of moments. Thus it is verified that the new scaling function actually has approximation order two.

CHAPTER 7. Factorization

In the univariate case, the polyphase matrix of any orthogonal wavelet can be factored into linear terms.

$$P(z) = U \cdot F_1(z) \cdots F_k(z),$$

where U is unitary , $F_j = (I - A_j) + A_j z, \qquad A_j^2 = A_j = A_j^*.$

However, this does not work for $d \geq 2$, in general. This is a very hard unsolved problem. Conditions under which factorization is possible are not known. We will restrict ourselves to presenting some examples for the multivariate case.

Example:

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & z_1 \\ -z_2 & z_1 z_2 \end{pmatrix}$$

This is the orthogonal Haar wavelet that goes with the standard quincunx with $\Lambda = \{(0,0)^t, (1,1)^t\}$. This cannot be factored in the form

$$P = U\left[(I - A) + Az_1\right]\left[(I - B) + Bz_2\right], \quad \text{with} \qquad A^2 = A, B^2 = B.$$

However, it can be factored as

$$P = U\left[(I - B) + Bz_2\right]\left[(I - A) + Az_1\right]$$

with

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \; A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

On the other hand, this P is not the "natural" Haar wavelet. The wavelet has different support than the scaling function. If we complete

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & z_1 \\ * & * \end{pmatrix}$$

by

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & z_1 \\ -1 & z_1 \end{pmatrix},$$

then

$$P = U\,[(I-A) + Az_1]$$

where U and A are the same as before.

Example: DB2.

The one dimensional Daubechies wavelet with 2 vanishing moments factors as

$$P = \begin{pmatrix} h_0 & h_1 \\ -h_3 & h_2 \end{pmatrix} + \begin{pmatrix} h_2 & h_3 \\ -h_1 & h_0 \end{pmatrix} z = U\,[(I-A) + Az]\,,$$

where

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},\ A = \frac{1}{4} \begin{pmatrix} 3 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\ .$$

Now in the two-dimensional case, we have

$$P = \begin{pmatrix} h_0 & h_1 \\ * & * \end{pmatrix} + \begin{pmatrix} h_2 & h_3 \\ * & * \end{pmatrix} z_1 z_2$$

It seems natural to use the same completion with z replaced by z_1z_2 .This can be considered as

$$P = U\,[(I-A) + Az_1][(I-B) + Bz_2]\,,\ \text{with}\ B = I.$$

Note: The biorthogonal case is even more complicated than the orthogonal case, even for univariate wavelets, see [18].

CHAPTER 8. Completion

8.1 The Completion Problem

The completion problem is the problem of finding wavelets given the scaling function. This is equivalent to the problem of completing polyphase matrices whose first row is given.
If $m = 2$, it can be done as in the one-dimensional case, section 2.9. For the general completion problem, we have the following result [11].

Definition 27 Let q be a non-negative integer. A function f on $\mathbb{R}^d$ is *q-regular* if f is in class C^q and $|\frac{\partial^{\boldsymbol{\alpha}}}{\partial \mathbf{x}^{\boldsymbol{\alpha}}} f(\mathbf{x})| \leq \frac{C_k}{(1+|\mathbf{x}|)^k}$ for each $k = 0, 1, 2, \ldots$, and for each multi-index $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}| \leq q$. A q-regular MRA is an MRA where the scaling vector is q-regular.

Theorem 19 For each q-regular MRA of $\mathbb{R}^d$ with general dilation M such that $2(m-1)r \geq d$ there exists a wavelet set containing $2(m-1)r$, q-regular functions.

Proof: This is proved in [11].

Example 1: Haar, H

$$M = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

$\Lambda = \{(0,0)^t, (1,0)^t\}, \quad h_0 = h_1 = \frac{1}{\sqrt{2}}$.

The symbol is

$$H(\boldsymbol{\xi}) = \frac{1 + e^{-i\xi_1}}{2},$$

and the polyphase symbols are

$$H^{(0,0)}(\boldsymbol{\xi}) = H^{(0,1)}(\boldsymbol{\xi}\ ) = \frac{1}{\sqrt{2}}.$$

We note that P has

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

as the unitary completion. Thus the wavelet is given by

$$\begin{aligned} H^{(1)}(\boldsymbol{\xi}) &= \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}e^{-i\xi_1}\right) \\ &= -\frac{1}{2}+\frac{1}{2}e^{-i\xi_1}. \end{aligned}$$

When $m \geq 3$, there are many such examples.

Example: K-V scaling function. We take the determinant of the polyphase matrix as $\Delta = z_1 z_2$, so that the wavelet has the same support as scaling function. We find that

$$\begin{aligned} H^{(1,0)} &= -z_1 z_2(h_3 + h_7 z_1 + h_1 z_2 + h_5 z_1 z_2) \\ &= -h_5 - h_1 z_1 - h_7 z_2 - h_3 z_1 z_2 \\ \text{and}\ \ H^{(1,1)} &= h_4 + h_0 z_1 + h_6 z_2 + h_2 z_1 z_2 \end{aligned}$$

$$\begin{array}{ccccccccc} & h_6 & h_7 & & & -h_1 & h_0 & \\ h_2 & h_3 & h_4 & h_5 & -h_5 & h_4 & -h_3 & h_2 \\ & h_0 & h_1 & & & -h_7 & h_6 & \end{array}$$

scaling function wavelet

Recently, Judith and Marc in [17] have found a counterexample to the above theorem for the case $2(m-1) < d$, r = 1. They consider the dilation to be

$$M = \begin{bmatrix} 0 & 3 \\ I & 0 \end{bmatrix}$$

with dilation factor of 3, where I is a 3 ×3 identity. They showed that it is impossible to complete the matrix in such a way that the wavelet is a continuous function, even though the scaling function is smooth.

Even in the case where theorem 17 applies, it is usually difficult to complete the matrix. However, some completion methods are known for special cases. We present one such method in the following section.

8.2 A new Biorthogonal Completion Algorithm

We present here a method for constructing wavelets $\phi^{(1)}, \ldots, \phi^{(m-1)}$ such that $V_0^{(i)}$, $i = 1, 2, \ldots, m-1$ are orthogonal to $V_0^{(0)}$ and the spaces $V_0^{(i)}$, $i = 1, 2, \ldots, m-1$ are orthogonal among each other. This is a generalization of the method of Lai [12].

We have the following recursion relations

$$\phi^{(i)}(\mathbf{x}) = \sqrt{m} \sum_{\mathbf{k} \in \Lambda} h_{\mathbf{k}}^{(i)} \, \phi^{(0)} \left(M\mathbf{x} - \mathbf{k} \right), \;\; \mathbf{x} \in \mathbb{R}^d, \;\; i = 0, 1, \ldots m-1. \tag{8.1}$$

In the frequency domain, the above equation can be written as

$$\hat{\phi}^{(i)}(M^t \boldsymbol{\xi}) = H^{(i)}(\boldsymbol{\xi}) \hat{\phi}(\boldsymbol{\xi}), \;\; i = 0, 1, \ldots m-1,$$

where

$$H^{(j)}(\boldsymbol{\xi}) = \frac{1}{\sqrt{m}} \sum_{\mathbf{k}} h_{\mathbf{k}}^{(j)} e^{-i<\mathbf{k}, \boldsymbol{\xi}>}$$

are the symbols for the scaling function and the wavelets. Let

$$\Phi(\mathbf{z}) = \sum_{\mathbf{k} \in \mathbb{Z}^d} < \phi(\mathbf{x}), \phi(\mathbf{x} - \mathbf{k}) > \mathbf{z}^{\mathbf{k}}.$$

Lemma 4

$$\Phi(\mathbf{z}) = \frac{1}{(2\pi)^d} \sum_{\mathbf{p} \in \mathbb{Z}^d} |\hat{\phi}(\boldsymbol{\xi} + 2\pi \mathbf{p})|^2.$$

Proof:

$$\begin{aligned}
< \phi(\mathbf{x}), \phi(\mathbf{x} - \mathbf{k}) > &= \int_{\mathbb{R}^d} \phi(\mathbf{x}) \phi^*(\mathbf{x} - \mathbf{k}) d\mathbf{x} \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{\phi}(\boldsymbol{\xi}) \hat{\phi}^*(\boldsymbol{\xi}) e^{i<\mathbf{k}, \boldsymbol{\xi}>} d\boldsymbol{\xi} \\
&= \frac{1}{(2\pi)^d} \sum_{\mathbf{p} \in \mathbb{Z}^d} \int_{[0,2\pi]^d + 2\pi \mathbf{p}} |\hat{\phi}(\boldsymbol{\xi})|^2 e^{i<\mathbf{k}, \boldsymbol{\xi}>} d\boldsymbol{\xi} \\
&= \frac{1}{(2\pi)^d} \sum_{\mathbf{p} \in \mathbb{Z}^d} \int_{[0,2\pi]^d} |\hat{\phi}(\boldsymbol{\xi} + 2\pi \mathbf{p})|^2 e^{i<\mathbf{k}, \boldsymbol{\xi}>} d\boldsymbol{\xi} \\
&= \frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} \sum_{\mathbf{p} \in \mathbb{Z}^d} |\hat{\phi}(\boldsymbol{\xi} + 2\pi \mathbf{p})|^2 e^{i<\mathbf{k}, \boldsymbol{\xi}>} d\boldsymbol{\xi}.
\end{aligned}$$

We have used the fact that $[0, 2\pi]^d + 2\pi \mathbf{p}$ are disjoint for different $\mathbf{p}$. Thus we see that

$$(2\pi)^d < \phi(\mathbf{x}), \phi(\mathbf{x} - \mathbf{k}) >$$

is the $(-k)^{th}$ Fourier coefficient of the $2\pi\mathbb{Z}^d$-periodic function

$$\sum_{\mathbf{p}\in\mathbb{Z}} |\hat{\phi}(\ \boldsymbol{\xi} + 2\pi\mathbf{p})|^2.$$

Therefore,

$$\sum_{\mathbf{p}} |\hat{\phi}(\ \boldsymbol{\xi} + 2\pi\mathbf{p})|^2 = \sum_{\mathbf{k}} (2\pi)^d < \phi(\mathbf{x}), \phi(\mathbf{x}-\mathbf{k}) > e^{-i<\mathbf{k},\mathbf{x}>},$$

which implies

$$\Phi(\mathbf{z}) = \frac{1}{(2\pi)^d} \sum_{\mathbf{p}} |\hat{\phi}(\ \boldsymbol{\xi} + 2\pi\mathbf{p})|^2.$$

Definition 28 Let E stand for the operator that maps any Laurent polynomial f into the Laurent polynomial which contains all the terms of f indexed by the lattice points. In other words, E is a *downsampler.*

Example: Let us consider the standard quincunx lattice, and

$$f(z_1, z_2) = h_0 + h_1 z_1 + h_2 z_1 z_2 + h_3 z_1^2 z_2.$$

Then

$$E(f) = h_0 + h_2 z_1 z_2.$$

The operator E has the following properties

Lemma 5

$$\begin{aligned} E(A+B) &= E(A) + E(B), \\ E(E(A)B) &= E(A)E(B), \\ E(A(\mathbf{z}^M)B(\mathbf{z})) &= A(\mathbf{z}^M)E(B(\mathbf{z})). \end{aligned}$$

If the matrices M and M^t generate the same lattice, then also

$$E(A(\mathbf{z}^{M^t})B(\mathbf{z})) = A(\mathbf{z}^{M^t})E(B(\mathbf{z})).$$

Proof: Since the addition of two polynomial results in the addition of the coefficients of the terms of same index, the first equality follows. For the second one let B be a monomial, then the product $E(A)B$ will be indexed by lattice points iff B is indexed by the lattice points. The general case follows from the first equality. The third one follows from the second since $A(\mathbf{z}^M)$ are all indexed by the lattice points. Lastly, if M and M^t generate the same lattice, $A(\mathbf{z}^M)$ and $A(\mathbf{z}^{M^t})$ are indexed by the same lattice points. Thus it follows from the third.

Let $g_1, \ldots, g_k$ be any collection of functions defined by recursion relations of the form (8.1), and let

$$\mathcal{G}_k = \mathrm{clos}_{L^2(\mathbb{R}^d)}\{g_k(\mathbf{x}-\mathbf{m}),\ \mathbf{m} \in \mathbb{Z}^d\},\ k = 1, 2, \ldots m-1,$$

be the closure of the linear span of integer translates of g_k. Let G_i, $i = 1, 2, \ldots, m-1$ be their corresponding symbols. Then we have the following theorem.

Theorem 20 $\mathcal{G}_k$ is orthogonal to $\mathcal{G}_{k'}$ for $k \neq k'$, iff $E(G_k\overline{G_{k'}}\Phi) = 0$.

Proof: We have the following relations

$$\hat{g}_i(M^t\boldsymbol{\xi}) = G_i(\boldsymbol{\xi})\hat{\phi}(\boldsymbol{\xi}),\ i = 0, 1, \ldots m-1.$$

$$\begin{aligned}
< g_k(\mathbf{x}), g_{k'}(\mathbf{x}-\mathbf{j}) > &= \int_{\mathbb{R}^d} g_k(\mathbf{x}) g^*_{k'}(\mathbf{x}-\mathbf{j}) d\mathbf{x} \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{g}^*_k(\boldsymbol{\xi}) \hat{g}^*_{k'}(\boldsymbol{\xi}) e^{i<\mathbf{j},\boldsymbol{\xi}>} d\boldsymbol{\xi} \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} G_k(M^{-t}\boldsymbol{\xi}) \overline{G'_k(M^{-t}\boldsymbol{\xi})} |\hat{\phi}(M^{-t}\boldsymbol{\xi})|^2 e^{i<\mathbf{j},\ \boldsymbol{\xi}>} d\boldsymbol{\xi} \\
&= \frac{m}{(2\pi)^d} \sum_{\mathbf{p}\in\mathbb{Z}^d} \int_{[0,2\pi]^d+2\pi\mathbf{p}} G_k(\boldsymbol{\xi}) \overline{G'_k(\boldsymbol{\xi})} |\hat{\phi}(\boldsymbol{\xi})|^2 e^{i<\mathbf{j},M^t\boldsymbol{\xi}>} d\boldsymbol{\xi} \\
&= \frac{m}{(2\pi)^d} \sum_{\mathbf{p}\in\mathbb{Z}^d} \int_{[0,2\pi]^d} G_k(\boldsymbol{\xi}) \overline{G'_k(\boldsymbol{\xi})} |\hat{\phi}(\boldsymbol{\xi}+2\pi\mathbf{p})|^2 e^{i<\mathbf{j},M^t\boldsymbol{\xi}>} d\boldsymbol{\xi} \\
&= \frac{m}{(2\pi)^d} \int_{[0,2\pi]^d} \sum_{\mathbf{p}\in\mathbb{Z}^d} G_k(\boldsymbol{\xi}) \overline{G'_k(\boldsymbol{\xi})} |\hat{\phi}(\boldsymbol{\xi}+2\pi\mathbf{p})|^2 e^{i<\mathbf{j},M^t\boldsymbol{\xi}>} d\boldsymbol{\xi} \\
&= m \int_{[0,2\pi]^d} G_k(\boldsymbol{\xi}) \overline{G'_k(\boldsymbol{\xi})} \Phi(\boldsymbol{\xi}) e^{i<\mathbf{j},M^t\boldsymbol{\xi}>} d\boldsymbol{\xi}.
\end{aligned}$$

This implies that $\frac{1}{m} < g_k(\mathbf{x}), g'_k(\mathbf{x}-\mathbf{j}) >$ are the $(-M\mathbf{j})^{th}$ Fourier coefficients of the $2\pi\mathbb{Z}^d$ periodic function $G_k(\boldsymbol{\xi})\overline{G'_k(\ \boldsymbol{\xi})}\Phi(\boldsymbol{\xi})$. Therefore, in the Fourier expansion of $G_k(\boldsymbol{\xi})\overline{G'_k(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})$

the coefficients with index $M\mathbf{j}$ will not show up if $< g_k(\mathbf{x}), g'_k(\mathbf{x}-\mathbf{j}) >= 0$, so we have $E(G_k(\boldsymbol{\xi})\overline{G'_k(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})) = 0$. On the other hand, if $E(G_k(\boldsymbol{\xi})\overline{G'_k(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})) = 0$, then by the uniqueness of the Fourier coefficients $< g_k(\mathbf{x}), g'_k(\mathbf{x}-\mathbf{j}) >= 0$.

Let $D = \{\mathbf{d}_0, \mathbf{d}_1 \ldots, \mathbf{d}_{m-1}\}$ be the digit set for the corresponding dilation matrix. We will show that that there exists a linearly dependent set $g_0, g_1, \ldots, g_{m-1}$ defined by recursion relation (8.1) whose closure of integer translates spans the space W_0, the orthogonal complement of $V_0^{(0)}$ in $V_1^{(0)}$. We then find, under some extra condition, a linearly independent set which still spans the same space. Lastly, we orthogonalize the resulting set using something like the Gram-Schmidt orthogonalization process to get wavelets. For simplicity, we write ϕ, H instead of $\phi^{(0)}$, $H^{(0)}$ at this point.

Let us assume that $g_i(\mathbf{x}-\mathbf{k}) \perp V_0^{(0)}$ and

$$\phi(M\mathbf{x}-\mathbf{d}_k) = \sum_{\mathbf{m}} a_{k,\mathbf{m}}\phi(\mathbf{x}-\mathbf{m}) + b_{k,\mathbf{m}} g_k(\mathbf{x}-\mathbf{m}). \tag{8.2}$$

In other words, we are assuming that

$$\mathcal{G}_k \perp V_0^{(0)} \text{ and } V_1^{(0)} = V_0^{(0)} \bigoplus (\mathcal{G}_0 + \cdots + \mathcal{G}_{m-1}).$$

Taking the Fourier transform on each side of (8.2), we arrive at

$$H(M^{-t}\boldsymbol{\xi})A_k(\boldsymbol{\xi}) + G_k(M^{-t}\boldsymbol{\xi})B_k(\boldsymbol{\xi}) = \frac{1}{m} e^{-i<M^{-1}\mathbf{d}_k, \boldsymbol{\xi}>},$$

where

$$A_k = \sum_{\mathbf{m}} a_{k,\mathbf{m}} e^{-i<\mathbf{m},\boldsymbol{\xi}>} \text{and } B_k = \sum_{\mathbf{m}} b_{k,\mathbf{m}} e^{-i<\mathbf{m},\boldsymbol{\xi}>},$$

and G_i is the symbol for $g_i, i = 0, 1 \ldots, m-1$. Alternatively,

$$H(\boldsymbol{\xi})A_k(M^t\boldsymbol{\xi}) + G_k(\boldsymbol{\xi})B_k(M^t\boldsymbol{\xi}) = e^{-i<\mathbf{d}_k,\boldsymbol{\xi}>}. \tag{8.3}$$

Assuming both M and its transpose generate the same lattice, the above equations can be solved. Multiplying both sides of the above equation by $\overline{H}(\boldsymbol{\xi})\Phi(\boldsymbol{\xi})$ and applying the downsampling operator E on both sides, yields

$$A_k(M^t\boldsymbol{\xi}) = \frac{E[e^{-i<\mathbf{d}_k,\boldsymbol{\xi}>}\overline{H}(\boldsymbol{\xi})\Phi(\boldsymbol{\xi})]}{mE[H(\boldsymbol{\xi})\overline{H(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})]}.$$

Conversely, if we take this choice of A_k and also

$$B_k(M^t\boldsymbol{\xi}) = \frac{1}{mE[H(\boldsymbol{\xi})\overline{H(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})]},$$

$$G_k(\boldsymbol{\xi}) = E[H(\boldsymbol{\xi})\overline{H(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})]e^{-i<\mathbf{d}_k,\boldsymbol{\xi}>} - E[e^{-i<\mathbf{d}_k,\boldsymbol{\xi}>}\overline{H(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})]H(\boldsymbol{\xi}),$$

then (8.3) is satisfied. Furthermore,

$$V_0^{(0)} \perp \mathcal{G}_k, \;\; k = 0, 1, \ldots, m-1,$$

because

$$\begin{aligned}
& E[G_k(\boldsymbol{\xi})\overline{H(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})] \\
= \;& E\left[\left\{E[H(\boldsymbol{\xi})\overline{H(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})]e^{-i<\mathbf{d}_k,\boldsymbol{\xi}>} - E[e^{-i<\mathbf{d}_k,\boldsymbol{\xi}>}\overline{H(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})]H(\boldsymbol{\xi})\right\}\overline{H(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right] \\
= \;& E[H(\boldsymbol{\xi})\overline{H(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})]E[e^{-i<\mathbf{d}_k,\boldsymbol{\xi}>}\overline{H(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})] - E[e^{-i<\mathbf{d}_k,\boldsymbol{\xi}>}\overline{H(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})]E[H(\boldsymbol{\xi})\overline{H(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})] \\
= \;& 0.
\end{aligned}$$

We now switch back to the full notation. $H^{(0)}$ is the symbol of the scaling function $\phi^{(0)}$, $H^{(0,j)}, j = 0, 1, \ldots, m-1$ are its polyphase components.

Theorem 21 Suppose that the dilation matrix M and its transpose generate the same lattice and that there exists an integer k such that $H^{(0,k)}(\boldsymbol{\xi}) \neq 0$ for all $z = e^{-i\boldsymbol{\xi}}$ on the torus $\mathbb{T}^d$. Without loss of generality, let as assume that $H^{(0,0)}(\boldsymbol{\xi}) \neq 0$ for all $e^{-i\boldsymbol{\xi}} \in \mathbb{T}^d$. Then there exist coefficients $f_{\mathbf{m},k}$ such that

$$g_0(x) = \sum_{\mathbf{m}} \sum_{k=1}^{m-1} f_{\mathbf{m},k} g_k(\mathbf{x} - \mathbf{m}).$$

Equivalently,

$$\hat{\phi}^{(0)}(\boldsymbol{\xi})G_0(\boldsymbol{\xi}) = \sum_k G_k(\boldsymbol{\xi})\hat{\phi}^{(0)}(\boldsymbol{\xi})F_k(M^t\boldsymbol{\xi}),$$

where

$$F_k(\boldsymbol{\xi}) = \sum_{\mathbf{m}} f_{\mathbf{m},k} e^{-i<\mathbf{m},\boldsymbol{\xi}>}.$$

Proof. By actual calculations,

$$
\begin{aligned}
& H^{(0,1)}(M^t\boldsymbol{\xi})G_1(\boldsymbol{\xi}) + H^{(0,2)}(M^t\boldsymbol{\xi})G_2(\boldsymbol{\xi}) + \cdots + H^{(0,m-1)}(M^t\boldsymbol{\xi})G_{m-1}(\boldsymbol{\xi}) \\
= \; & E\left[H(\boldsymbol{\xi})\overline{H(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right]\left[H^{(0,1)}(M^t\boldsymbol{\xi})e^{-i<\mathbf{d}_1,\boldsymbol{\xi}>} + \cdots + H^{(0,m-1)}(M^t\boldsymbol{\xi})e^{-i<\mathbf{d}_{m-1},\boldsymbol{\xi}>}\right] \\
& -H(\boldsymbol{\xi})\left\{E\left[e^{-i<\mathbf{d}_1,\boldsymbol{\xi}>}\overline{H(\boldsymbol{\xi})}H^{(0,1)}(M^t\boldsymbol{\xi})\Phi(\boldsymbol{\xi})\right] + \cdots + E\left[e^{-i<\mathbf{d}_{m-1},\boldsymbol{\xi}>}\overline{H(\boldsymbol{\xi})}H^{(0,m-1)}(M^t\boldsymbol{\xi})\Phi(\boldsymbol{\xi})\right]\right\} \\
= \; & E\left[H(\boldsymbol{\xi})\overline{H(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right]\left[\sqrt{m}H(\boldsymbol{\xi}) - H^{(0,0)}(M^t\boldsymbol{\xi})\right] \\
& -H(\boldsymbol{\xi})\left\{E\left[(e^{-i<\mathbf{d}_1,\boldsymbol{\xi}>}H^{(0,1)}(M^t\boldsymbol{\xi}) + \cdots + e^{-i<\mathbf{d}_{m-1},\boldsymbol{\xi}>}H^{(0,m-1)}(M^t\boldsymbol{\xi}))\overline{H}(\boldsymbol{\xi})\Phi(\boldsymbol{\xi})\right]\right\} \\
= \; & E\left[H(\boldsymbol{\xi})\overline{H(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right]\left[\sqrt{m}H(\boldsymbol{\xi}) - H^{(0,0)}(M^t\boldsymbol{\xi})\right] - H(\boldsymbol{\xi})\left\{E\left[(\sqrt{m}H(\boldsymbol{\xi}) - H^{(0,0)}(M^t\boldsymbol{\xi}))\overline{H(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right]\right\} \\
= \; & E\left[H(\boldsymbol{\xi})\overline{H(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right]\left[\sqrt{m}H(\boldsymbol{\xi}) - H^{(0,0)}(M^t\boldsymbol{\xi})\right] \\
& -H(\boldsymbol{\xi})E\left[\sqrt{m}H(\boldsymbol{\xi})\overline{H(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right] + H(\boldsymbol{\xi})E\left[H^{(0,0)}(M^t\boldsymbol{\xi})\overline{H(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right] \\
= \; & \sqrt{m}H(\boldsymbol{\xi})E\left[H(\boldsymbol{\xi})\overline{H(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right] - H^{(0,0)}(M^t\boldsymbol{\xi})E\left[H(\boldsymbol{\xi})\overline{H(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right] \\
& -\sqrt{m}H(\boldsymbol{\xi})E\left[H(\boldsymbol{\xi})\overline{H(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right] + H(\boldsymbol{\xi})E\left[H^{(0,0)}(M^t\boldsymbol{\xi})\overline{H(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right] \\
= \; & -E\left[H(\boldsymbol{\xi})\overline{H(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right]H^{(0,0)}(M^t\boldsymbol{\xi}) + H^(0,0)(M^t\boldsymbol{\xi})H(\boldsymbol{\xi})E\left[\overline{H(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right] \\
= \; & -H^{(0,0)}(M^t\boldsymbol{\xi})\left\{E\left[H(\boldsymbol{\xi})\overline{H(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right] - H(\boldsymbol{\xi})E\left[\overline{H(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right]\right\} \\
= \; & -H^{(0,0)}(M^t\boldsymbol{\xi})G_0(\boldsymbol{\xi}).
\end{aligned}
$$

Thus

$$
G_0(\boldsymbol{\xi}) = -\frac{H^{(0,1)}(M^t\boldsymbol{\xi})G_1(\boldsymbol{\xi}) + H^{(0,2)}(M^t\boldsymbol{\xi})G_2(\boldsymbol{\xi}) + \cdots + H^{(0,m-1)}(M^t\boldsymbol{\xi})G_{m-1}(\boldsymbol{\xi})}{H^{(0,0)}(M^t\boldsymbol{\xi})}.
$$

Note: By a dimension count, the remaining basis functions $g_1, \ldots, g_{m-1}$ so obtained are linearly independent, which guarantees that the integer translates of g_i, $i = 1, 2, \ldots, m-1$ form a basis for W_0.

The next step is to find compactly supported wavelets $\phi^{(i)}$, $i = 1, 2, \ldots, m-1$. We use an orthogonalization technique to construct $\phi^{(i)}$ from g_i [12]. We first let, $\phi^{(1)} = g_1$. Let

$$
\phi^{(2)}(\mathbf{x}) = \sum_{\mathbf{m}\in\mathbb{Z}^d} c_{1,m}\phi^{(1)}(\mathbf{x}-\mathbf{m}) + c_{2,m}g_2(\mathbf{x}-\mathbf{m})
$$

for some coefficients $c_{1,m}$ and $c_{2,m}$. In the frequency domain,

$$\begin{aligned}
\hat{\phi}^{(2)}(\boldsymbol{\xi}) &= C_1(\boldsymbol{\xi})\hat{\phi}^{(1)}(\boldsymbol{\xi}) + C_2(\boldsymbol{\xi})\hat{g}_2(\boldsymbol{\xi}) \\
&= \left(C_1(\boldsymbol{\xi})G_1(M^{-t}\boldsymbol{\xi}) + C_2(\boldsymbol{\xi})G_2(M^{-t}\boldsymbol{\xi})\right)\phi(M^{-t}\boldsymbol{\xi}), \\
\text{or} \qquad \hat{\phi}^{(2)}(M^t\boldsymbol{\xi}) &= \left(C_1(M^t\boldsymbol{\xi})G_1(\boldsymbol{\xi}) + C_2(M^t\boldsymbol{\xi})G_2(\boldsymbol{\xi})\right)\phi(\boldsymbol{\xi}),
\end{aligned}$$

where C_1 and C_2 are discrete Fourier transforms of the sequences $c_{1,m}$ and $c_{2,m}$.
In order to have $V_0^{(2)} \perp V_0^{(1)}$, the orthogonality condition of theorem 20 implies that

$$\begin{aligned}
&E\left[\left(C_1(M^t\boldsymbol{\xi})G_1(\boldsymbol{\xi}) + C_2(M^t\boldsymbol{\xi})G_2(\boldsymbol{\xi})\right)\overline{G_1(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right] = 0, \\
\text{or} \qquad &C_1(M^t\boldsymbol{\xi})E\left[G_1(\boldsymbol{\xi})\overline{G_1(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right] + C_2(M^t\boldsymbol{\xi})E\left[G_2(\boldsymbol{\xi})\overline{G_1(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right] = 0. \qquad (8.4)
\end{aligned}$$

By choosing

$$\begin{aligned}
C_1(M^t\boldsymbol{\xi}) &= E\left[G_2(\boldsymbol{\xi})\overline{G_1(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right], \\
C_2(M^t\boldsymbol{\xi}) &= -E\left[G_1(\boldsymbol{\xi})\overline{G_1(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right],
\end{aligned}$$

we see that (8.4) is satisfied, thus we have $V_0^{(2)} \perp V_0^{(1)}$. Likewise, we construct $\phi^{(3)}$ by letting

$$\phi^{(3)}(\mathbf{x}) = \sum_{\mathbf{m}\in\mathbb{Z}^d} d_{1,m}\phi^{(1)}(M^j\mathbf{x}-\mathbf{m}) + d_{2,m}\phi^{(2)}(M^j\mathbf{x}-\mathbf{m}) + d_{3,m}g_3(M^j\mathbf{x}-\mathbf{m})$$

for some sequences $d_{1,m}, d_{2,m}, d_{3,m}$. In the frequency domain it can be written as

$$\begin{aligned}
\hat{\phi}^{(3)}(\boldsymbol{\xi}) &= D_1(\boldsymbol{\xi})\hat{\phi}^{(1)}(\boldsymbol{\xi}) + D_2(\boldsymbol{\xi})\hat{\phi}^{(2)}(\boldsymbol{\xi}) + D_3(\boldsymbol{\xi})\hat{g}_3(\boldsymbol{\xi}) \\
&= \left(D_1(\boldsymbol{\xi})A(M^{-t}\boldsymbol{\xi}) + D_2(\boldsymbol{\xi})B(M^{-t}\boldsymbol{\xi}) + D_3(\boldsymbol{\xi})G_3(M^{-t}\boldsymbol{\xi})\right)\hat{\phi}(\boldsymbol{\xi}),
\end{aligned}$$

or

$$\hat{\phi}^{(3)}(M^t\boldsymbol{\xi}) = \left(D_1(M^t\boldsymbol{\xi})A(\boldsymbol{\xi}) + D_2(M^t\boldsymbol{\xi})B(\boldsymbol{\xi}) + D_3(M^t\boldsymbol{\xi})G_3(\boldsymbol{\xi})\right)\hat{\phi}(\boldsymbol{\xi}).$$

Again, $D_1(M^t\boldsymbol{\xi})A(\boldsymbol{\xi}) + D_2(M^t\boldsymbol{\xi})B(\boldsymbol{\xi}) + D_3(M^t\boldsymbol{\xi})G_3(\boldsymbol{\xi})$ is the symbol for the corresponding wavelet $\phi^{(3)}$. The orthogonality conditions, $V_0^{(3)} \perp V_0^{(1)}$ and $V_0^{(3)} \perp V_0^{(2)}$ give

$$D_1(M^t\boldsymbol{\xi})E\left[A(\boldsymbol{\xi})\overline{A(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right] + D_2(M^t\boldsymbol{\xi})E\left[B(\boldsymbol{\xi})\overline{A(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right] + D_3(M^t\boldsymbol{\xi})E\left[G_3(\boldsymbol{\xi})\overline{A(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right] = 0$$

and

$$D_1(M^t\boldsymbol{\xi})E\left[A(\boldsymbol{\xi})\overline{B(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right]+D_2(M^t\boldsymbol{\xi})E\left[B(\boldsymbol{\xi})\overline{B(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right]+D_3(M^t\boldsymbol{\xi})E\left[G_3(\boldsymbol{\xi})\overline{B(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right]=0.$$

Since we already have $V_0^{(1)} \perp V_0^{(2)}$, this reduces to

$$D_1(M^t\boldsymbol{\xi})E\left[A(\boldsymbol{\xi})\overline{A(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right]+D_3(M^t\boldsymbol{\xi})E\left[G_3(\boldsymbol{\xi})\overline{A(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right]=0, \tag{8.5}$$

$$D_2(M^t\boldsymbol{\xi})E\left[B(\boldsymbol{\xi})\overline{B(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right]+D_3(M^t\boldsymbol{\xi})E\left[G_3(\boldsymbol{\xi})\overline{B(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right]=0. \tag{8.6}$$

It is not hard to solve the above system of equations. A solution may be found as

$$\begin{aligned}
D_1(M^t\boldsymbol{\xi}) &= E\left[B(\boldsymbol{\xi})\overline{B(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right]E\left[G_3(\boldsymbol{\xi})\overline{A(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right],\\
D_2(M^t\boldsymbol{\xi}) &= E\left[A(\boldsymbol{\xi})\overline{A(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right]E\left[G_3(\boldsymbol{\xi})\overline{B(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right],\\
D_3(M^t\boldsymbol{\xi}) &= -E\left[A(\boldsymbol{\xi})\overline{A(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right]E\left[B(\boldsymbol{\xi})\overline{B(\boldsymbol{\xi})}\Phi(\boldsymbol{\xi})\right].
\end{aligned}$$

Thus, with these choice of D_1, D_2, D_3 the equations (8.5) and (8.6) are satisfied, giving the wavelet $\phi^{(3)}$. The above process can be repeated to orthogonalize all of them.

Example: Haar wavelet H.

Consider the Haar scaling function with standard quincunx dilation matrix. The symbol is

$$H^{(0)}(\boldsymbol{\xi})=\frac{1+e^{-i\xi_1}}{2}.$$

We calculate

$$\begin{aligned}
\Phi(\boldsymbol{\xi}) &= 1,\\
H^{(0)}\overline{H}^{(0)}\Phi(\boldsymbol{\xi}) &= \frac{e^{-i\xi_1}+2+e^{i\xi_1}}{4},\\
E[H^{(0)}\overline{H}^{(0)}\Phi(\boldsymbol{\xi})] &= \frac{1}{2}.
\end{aligned}$$

Since $H^{(0,0)}=\frac{1}{2}\neq 0$, the algorithm applies. We get

$$\begin{aligned}
G_0(\boldsymbol{\xi}) &= E[H^{(0)}\overline{H}^{(0)}\Phi]-E[\overline{H}^{(0)}\Phi]\\
&= \frac{1}{2}-\frac{1}{2}\left(\frac{1+e^{-i\xi_1}}{2}\right)=\frac{1-e^{-i\xi_1}}{4}\\
G_1(\boldsymbol{\xi}) &= E[H^{(0)}\overline{H}^{(0)}\Phi]e^{-i\xi_1}-E[e^{-i\xi_1}\overline{H}^{(0)}\Phi]H^{(0)}\\
&= \frac{1}{2}e^{-i\xi_1}-\frac{1}{2}\left(\frac{1+e^{-i\xi_1}}{2}\right)=\frac{-1+e^{-i\xi_1}}{4}\\
&= -G_0(\boldsymbol{\xi}).
\end{aligned}$$

The resulting wavelet, after normalization, yields

$$h^{(1)}(\boldsymbol{\xi}) = \frac{1 - e^{-i\xi_1}}{2},$$

which is the standard Haar wavelet. It can be checked that the symbols of the wavelet and the scaling function satisfy the orthogonality condition, since

$$\begin{aligned} E[H^{(1)}\overline{H}^{(0)}\Phi(\boldsymbol{\xi})] &= E\left[\left(\frac{1 - e^{-i\xi_1}}{2}\right)\left(\frac{1 + e^{i\xi_1}}{2}\right)\right] \\ &= E\left[\frac{e^{i\xi_1} - e^{-i\xi_1}}{4}\right] \\ &= 0. \end{aligned}$$

Example: Four-point orthogonal example. Let M be the dilation

$$M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

with the digit set

$$D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

Let us consider the symbol

$$H(\boldsymbol{\xi}) = \frac{1}{2}[h_0 + h_1 e^{-i\xi_1} + h_2 e^{-i(\xi_1+\xi_2)} + h_3 e^{-i(2\xi_1+\xi_2)}].$$

The orthogonality conditions are

$$h_0^2 + h_1^2 + h_2^2 + h_3^2 = 1.$$

Any such choice of h_j will result in an orthogonal scaling function. The polyphase symbols are

$$\begin{aligned} H^{(0,0)}(\boldsymbol{\xi}) &= h_0, \\ H^{(0,1)}(\boldsymbol{\xi}) &= h_1, \\ H^{(0,2)}(\boldsymbol{\xi}) &= h_3 e^{-i\xi_1}, \\ H^{(0,3)}(\boldsymbol{\xi}) &= h_2, \end{aligned}$$

and

$$\begin{aligned}
\Phi(\boldsymbol{\xi}) &= 1,\\
E[H^{(0)}\bar{H}^{(0)}\Phi] &= \frac{1}{4},\\
E[\overline{H}] &= \frac{h_0}{2},\\
E[e^{-i\xi_1}\overline{H}\Phi] &= \frac{h_1}{2},\\
E[e^{-i\xi_2}\overline{H}\Phi] &= \frac{1}{2}h_3e^{2i\xi_1},\\
E[e^{-i(\xi_1+\xi_2)}\overline{H}\Phi] &= \frac{h_2}{2}.
\end{aligned}$$

At least one of the h_j is nonzero. Without loss of generality we can assume that $h_0 \neq 0$, otherwise we can use one of the other polyphase components. The wavelet symbols come out to be

$$\begin{aligned}
G_0(\boldsymbol{\xi}) &= E[H^{(0)}\overline{H}^{(0)}\Phi] - E[\overline{H}^{(0)}\Phi]H^{(0)}(\boldsymbol{\xi})\\
&= \frac{1}{4} - \frac{h_0}{2}H^{(0)}(\boldsymbol{\xi})\\
&= \frac{1}{4}[1 - h_0^2 - h_0h_1e^{-i\xi_1} - h_0h_2e^{-i(\xi_1+\xi_2)} - h_0h_3e^{-i(2\xi_1+\xi_2)}],\\
G_1(\boldsymbol{\xi}) &= E[H^{(0)}\overline{H}^{(0)}\Phi]e^{-i\xi_1} - E[e^{-i\xi_1}\overline{H}^{(0)}\Phi]H^{(0)}(\boldsymbol{\xi})\\
&= \frac{1}{4}e^{-i\xi_1} - \frac{h_1}{2}H^{(0)}(\boldsymbol{\xi})\\
&= \frac{1}{4}[-h_0h_1 + (1-h_1^2)e^{-i\xi_1} - h_1h_2e^{-i(\xi_1+\xi_2)} - h_1h_3e^{-i(2\xi_1+\xi_2)}],\\
G_2(\boldsymbol{\xi}) &= E[H^{(0)}\overline{H}^{(0)}\Phi]e^{-i\xi_2} - E[e^{-i\xi_2}\overline{H}^{(0)}\Phi]H^{(0)}(\boldsymbol{\xi})\\
&= \frac{1}{4}e^{-i\xi_2} - \frac{h_3}{2}e^{2i\xi_1}H^{(0)}(\boldsymbol{\xi})\\
&= \frac{1}{4}[(1-h_3^2)e^{-i\xi_2} - h_0h_3e^{2i\xi_1} - h_1h_3e^{i\xi_1} - h_2h_3e^{i(2\xi_1-\xi_2)}],\\
G_3(\boldsymbol{\xi}) &= E[H^{(0)}\overline{H}^{(0)}\Phi]e^{-i(\xi_1+\xi_2)} - E[e^{-i(\xi_1+\xi_2)}\overline{H}^{(0)}\Phi]H^{(0)}(\boldsymbol{\xi})\\
&= \frac{1}{4}e^{-i(\xi_1+\xi_2)} - \frac{1}{2}h_2H^{(0)}(\boldsymbol{\xi})\\
&= \frac{1}{4}[-h_0h_2 - h_1h_2e^{-i\xi_1} + (1-h_2^2)e^{-i(\xi_1+\xi_2)} - h_2h_3e^{-i(2\xi_1+\xi_2)}].
\end{aligned}$$

A further calculation shows that ($\sum h_i^2 = 1$ has been used)

$$G_0 = \frac{-H^{(0,1)}G_1 - H^{(0,2)}G_2 - H^{(0,3)}G_3}{H^{(0,0)}}.$$

A simple calculation shows that $C_1(M^t\boldsymbol{\xi}) = E[G_2\overline{G_1}]$ is a constant multiple of $e^{-2i\xi_1}$ and $C_2(M^t\boldsymbol{\xi}) = E[G_1\overline{G_1}]$ is a constant. Let $C_1(M^t\boldsymbol{\xi}) = p_1e^{-2i\xi_1}$, and $C_2(M^t\boldsymbol{\xi}) = p_2$, where p and q are constants. Thus

$$H^{(1)} = G_1 = \frac{1}{4}[h_0^{(1)} + h_1^{(1)}e^{-i\xi_1} + h_1^{(1)}e^{-i(\xi_1+\xi_2)} + h_1^{(1)}e^{-i(2\xi_1+\xi_2)}],$$

$$H^{(2)} = C_1G_1 + C_2G_2 = h_0^{(2)} + h_1^{(2)}e^{-i\xi_1} + h_2^{(2)}e^{-i(\xi_1+\xi_2)} + h_3^{(2)}e^{-i(2\xi_1+\xi_2)},$$

where $h_j^{(1)}$, $h_j^{(2)}$ can be expressed in terms of h_j, p_1,p_2. Similarly, since $E[H^{(2)}\overline{H^{(2)}}\Phi]$, $E[H^{(1)}\overline{H^{(1)}}\Phi]$, $E[G_{(3)}\overline{H^{(2)}}\Phi]$, and $E[G_{(3)}\overline{H^{(1)}}\Phi]$, are all constants, we find that

$$\begin{aligned}
D_1(M^t\boldsymbol{\xi}) &= E[H^{(2)}\overline{H^{(2)}}\Phi]E[G_{(3)}\overline{H^{(1)}}\Phi], \\
D_2(M^t\boldsymbol{\xi}) &= E[H^{(1)}\overline{H^{(1)}}\Phi]E[G_{(3)}\overline{H^{(2)}}\Phi],
\end{aligned}$$

and

$$D_3(M^t\boldsymbol{\xi}) = -E[H^{(1)}\overline{H^{(1)}}\Phi]E[H^{(2)}\overline{H^{(2)}}\Phi]$$

are all constants. Let these be q_1, q_2, and q_3 respectively. Therefore

$$\begin{aligned}
H^{(3)} &= p_1H^{(1)} + p_2H^{(2)} + p_3G_3 \\
&= h_0^{(3)} + h_1^{(3)}e^{-i\xi_1} + h_2^{(3)}e^{-i(\xi_1+\xi_2)} + h_3^3e^{-i(2\xi_1+\xi_2)}.
\end{aligned}$$

Example: Six point orthogonal example.

Let the dilation matrix be

$$M = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$

with the digit set

$$D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

Let us consider the symbol,

$$H(\boldsymbol{\xi}) = \frac{1}{\sqrt{3}}\left(h_0 + h_1e^{-i\xi_1} + h_2e^{-2i\xi_1} + h_3e^{-i(2\xi_1+\xi_2)} + h_4e^{-i(3\xi_1+\xi_2)} + h_5e^{-i(4\xi_1+\xi_2)}\right).$$

The orthogonality conditions are given by

$$h_0^2 + h_1^2 + h_2^2 + h_3^2 + h_4^2 + h_5^2 = 1,$$

$$h_0h_4 + h_1h_5 + h_2h_6 = 0.$$

The polyphase symbols are

$$\begin{aligned} H^{(0,0)}(\boldsymbol{\xi}) &= h_0 + h_3e^{-i\xi_1}, \\ H^{(0,1)}(\boldsymbol{\xi}) &= h_1e^{i\xi_2} + h_4e^{-i\xi_1+i\xi_2}, \\ H^{(0,2)}((\boldsymbol{\xi}) &= h_2e^{i\xi_2} + h_5e^{-i\xi_1+i\xi_2}, \end{aligned}$$

and

$$\begin{aligned} E[H\overline{H}] &= \frac{1}{3}, \\ E[\overline{H}] &= \frac{1}{\sqrt{3}}\left(h_0 + h_3e^{i(2\xi_1+\xi_2)}\right), \\ E[e^{-i\xi_2}\overline{H}] &= \frac{1}{\sqrt{3}}\left(h_1e^{i(\xi_1-\xi_2)} + h_4e^{3i\xi_1}\right), \\ E[e^{-i(\xi_1+\xi_2)}\overline{H}] &= \frac{1}{\sqrt{3}}\left(h_2e^{i(\xi_1-\xi_2)} + h_5e^{3i\xi_1}\right). \end{aligned}$$

Therefore

$$\begin{aligned} G_0(\boldsymbol{\xi}) &= \frac{1}{3} - \frac{1}{\sqrt{3}}\left(h_0 + h_3e^{i(2\xi_1+\xi_2)}\right)H(\boldsymbol{\xi}) \\ &= \frac{1}{3}[1 - h_0^2 - h_3^2 - (h_0h_1 + h_3h_4)e^{-i\xi_1} - (h_0h_2 + h_3h_5)e^{-2i\xi_1} - h_0h_3e^{-i(2\xi_1+\xi_2)} \\ &\quad -h_0h_4e^{-i(3\xi_1+\xi_2)} - h_0h_5e^{-i(4\xi_1+\xi_2)} - h_0h_3e^{i(2\xi_1+\xi_2)} - h_1h_3e^{i(\xi_1+\xi_2)} - h_2h_3e^{i\xi_2}], \\ G_1(\boldsymbol{\xi}) &= \frac{1}{3}e^{-i\xi_2} - \frac{1}{\sqrt{3}}\left(h_1e^{i(\xi_1-\xi_2)} + h_4e^{3i\xi_1}\right)H(\boldsymbol{\xi}) \\ &= \frac{1}{3}[1 - h_1^2 - h_4^2e^{-i\xi_2} - (h_1h_2 + h_4h_5)e^{-i(\xi_1+\xi_2)} - (h_0h_1 + h_3h_4)e^{i(\xi_1-\xi_2)} \\ &\quad -h_1h_3e^{-i(\xi_1+2\xi_2)} - h_1h_4e^{-i(2\xi_1+2\xi_2)} - h_1h_5e^{-i(3\xi_1+2\xi_2)} - h_0h_4e^{i3\xi_1} - h_1h_4e^{2i\xi_1} - h_2h_3e^{i\xi_2}], \\ G_2(\boldsymbol{\xi}) &= \frac{1}{3}e^{-i(\xi_1+\xi_2)} - \frac{1}{\sqrt{3}}\left(h_2e^{i(\xi_1-\xi_2)} + h_5e^{3i\xi_1}\right)H(\boldsymbol{\xi}) \\ &= \frac{1}{3}[1 - h_2^2 - h_5^2e^{-i(\xi_2+\xi_2)} - (h_1h_2 + h_4h_5)e^{-i\xi_2} - (h_0h_2 + h_3h_5)e^{i(\xi_1-\xi_2)} \\ &\quad -h_2h_3e^{-i(\xi_1+2\xi_2)} - h_2h_4e^{-i(2\xi_1+2\xi_2)} - h_2h_5e^{-i(3\xi_1+2\xi_2)} - h_0h_5e^{i3\xi_1} - h_1h_5e^{2i\xi_1} - h_2h_5e^{i\xi_1}]. \end{aligned}$$

It is clear that the supports of the G_is are bigger than that of the scaling function we started with. To orthogonalize them, we note that

$$C_1(M^t\boldsymbol{\xi}) = E[G_2\overline{G_1}] = c_1 + c_2 z_1^{-2} z_2^{-1} + c_3 z_1^2 z_2 + c_4 z_1^4 z_2^2 + c_5 z_1^{-4} z_2^{-2},$$

and

$$C_2(M^t\boldsymbol{\xi}) = -E[G_2\overline{G_1}] = e_1 + e_2 z_1^{-2} z_2^{-1} + e_3 z_1^2 z_2 + e_4 z_1^4 z_2^2 + e_5 z_1^{-4} z_2^{-2},$$

where c_i and e_i are some constants. Thus the symbols for the wavelets are given by

$$\begin{aligned} H^{(1)}(\boldsymbol{\xi}) &= G_1(\boldsymbol{\xi}), \\ H^{(2)}(\boldsymbol{\xi}) &= C_1(M^t\boldsymbol{\xi})H^{(1)}(\boldsymbol{\xi}) + C_2(M^t\boldsymbol{\xi})G_2(\boldsymbol{\xi}). \end{aligned}$$

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