Analysis and approximations of terminal-state tracking optimal control problems and controllability problems constrained by linear and semilinear parabolic partial differential equations

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ABSTRACT

Terminal-state tracking optimal control problems for linear and semilinear parabolic equations are studied. The control objective is to track a desired terminal state and the control is of the distributed type. A distinctive feature of this work is that the controlled state and the target state are allowed to have nonmatching boundary conditions.

In the linear case, analytic solution formulae for the optimal control problems are derived in the form of eigen series. Pointwise-in-time L^2 norm estimates for the optimal solutions are obtained and approximate controllability results are established. Exact controllability is shown when the target state and the controlled state have matching boundary conditions. One-dimensional computational results are presented which illustrate the terminal-state tracking properties for the solutions expressed by the series formulae.

In the semilinear case, the existence of an optimal control solution is shown. The dynamics of the optimal control solution is analyzed. Error estimates are obtained for semidiscrete (spatially discrete) approximations of the optimal control problem in two and three space dimensions. A gradient algorithm is discussed and numerical results are presented.

1 INTRODUCTION

In this thesis we study terminal-state tracking optimal control problems for linear and semilinear second order parabolic partial differential equations (PDE) defined over the time interval $[0,T] \subset [0,\infty)$ and on a bounded, C^2 (or convex) spatial domain $\Omega \subset I\!R^d$, $d = 1$ or 2 or 3. Let a target function $W \in L^2(\Omega)$ and an initial condition $w \in L^2(\Omega)$ be given and let $f \in L^2((0,T)\times\Omega)$ denote the distributed control. We wish to find a control f that drives the state to W at time T. We will use the optimal control approach. Obviously, the topics we study are closely related the to exact and approximate controllability problem.

In Chapter 2 the linear optimal control problems we study are to minimize the terminal-state tracking functional

$$\mathcal{J}(u,f) = \frac{T}{2}\int_\Omega |u(T,\mathbf{x}) - W(\mathbf{x})|^2\,d\mathbf{x} + \frac{\gamma}{2}\int_0^T\!\!\int_\Omega |f(t,\mathbf{x})|^2\,d\mathbf{x}\,dt \tag{1.1}$$

or

$$\mathcal{K}(u,f) = \frac{T}{2}\int_\Omega |u(T,\mathbf{x}) - W(\mathbf{x})|^2\,d\mathbf{x} + \frac{\gamma}{2}\int_0^T\!\!\int_\Omega |f(t,\mathbf{x}) - F(t,\mathbf{x})|^2\,d\mathbf{x}\,dt \tag{1.2}$$

(where γ is a positive constant and F is a given reference function) subject to the parabolic PDE

$$u_t - \operatorname{div}[A(\mathbf{x})\nabla u] = f, \qquad (t,\mathbf{x}) \in (0,T)\times\Omega \tag{1.3}$$

with the homogeneous boundary condition

$$u = 0, \qquad (t,\mathbf{x}) \in (0,T)\times\partial\Omega \tag{1.4}$$

and the initial condition

$$u(0,\mathbf{x}) = w(\mathbf{x}), \qquad \mathbf{x} \in \Omega\,. \tag{1.5}$$

In (1.3), $A(\mathbf{x})$ is a symmetric matrix-valued, $C^1(\overline{\Omega})$ function that is uniformly positive definite.

Similar optimal control problems have been studied in the literature from different aspects or in different settings. For instance, in [19] the existence and regularity of an optimal solution was studied; in [1] the connection between optimal solutions and controllability was examined, and in [26] eigenseries solutions were studied wherein the control f was assumed to belong to a bounded set in $L^2((0,T)\times\Omega)$ (due to the boundedness constraint the tracking functional of [26] did not contain the term involving f.) Both optimal control problems and controllability problems are studied in this paper. Our main achievements concerning optimal control problems include: the introduction of an F in (1.2) that results in an optimal solution that approaches the target more effectively (even for $t << T$ and moderate parameter γ;) the derivation and justification of explicit eigenseries solution formulae for optimal solutions; pointwise-in-time estimates for optimal solutions and the approximate controllability properties for the optimal solutions. A distinctive feature of this work is that the desired terminal-state W and the admissible state u are allowed to have nonmatching boundary conditions, though the reference function F needs be suitably chosen in the formulation of cost functional (1.2) (the details about the choice of F will be revealed in Section 2.1.)

Terminal-state tracking problems are optimal control problems in their own right. They are also closely related to approximate and exact controllability problems which were studied in, among others, [1, 3, 4, 5, 8, 9, 12, 13, 14, 15, 16, 17, 18, 21, 22, 23, 24]. As mentioned in the foregoing the boundary value for the target state W may be nonzero so that the parabolic problem (1.3)–(1.5) in general is not exactly controllable when the solution for (1.3)–(1.5) is defined in the standard weak sense (see [6]). Contributions of this work on controllability consist of the proof of approximate controllability when the target state has an inhomogeneous boundary value and the derivation of explicit series solution formulae for the exact controllability problem when the target state vanishes on the boundary.

The plan of Chapter 2 is described as follows. In Section 2.1 we formulate the optimal control problems and controllability problems in an appropriate mathematical framework. In Section 2.2 we review and establish certain results concerning eigenfunction expansions for both spatial and temporal-spatial functions. In Section 2.3 we derive explicit eigenseries solution formulae for the optimal control problems. In Section 2.4 we derive pointwise-in-time estimates for the optimal solutions and show that as the parameter $\gamma \to 0$, the optimal solutions at the terminal time T approach the target state W. In Section 2.5 we justify eigenseries solution formulae for the exact controllability problem by assuming homogeneous boundary values for the target state. In Section 2.6 we present some one-dimensional computational results that illustrate the terminal-state tracking properties for the solutions expressed by the series formulae of Section 2.3.

In Chapter 3 we study terminal-state tracking optimal control problems for a semilinear second order parabolic partial differential equation: minimize the terminal-state tracking functional

$$\mathcal{J}(u,f) = \frac{T}{2}\int_\Omega |u(T,\mathbf{x}) - W(\mathbf{x})|^2\,d\mathbf{x} + \frac{\gamma}{2}\int_0^T\int_\Omega |f(t,\mathbf{x}) - F(t,\mathbf{x})|^2\,d\mathbf{x}\,dt \qquad (1.6)$$

(where γ is a positive constant and F is a given reference function) subject to the parabolic PDE

$$u_t - \operatorname{div}\left[A(\mathbf{x})\nabla u\right] + \Phi(u) + a(\mathbf{x})u = f, \qquad (t,\mathbf{x}) \in (0,T)\times\Omega \qquad (1.7)$$

with the homogeneous boundary condition

$$u = 0, \qquad (t,\mathbf{x}) \in (0,T)\times\partial\Omega \qquad (1.8)$$

and the initial condition

$$u(0, \mathbf{x}) = w(\mathbf{x}), \qquad \mathbf{x} \in \Omega\,. \tag{1.9}$$

In (1.7), $A(\mathbf{x})$ is a symmetric matrix-valued, $C^1(\overline{\Omega})$ function that is uniformly positive definite and

$$a(\mathbf{x}) \in L^\infty(\Omega), \quad a(\mathbf{x}) \geq -C_1 \tag{1.10}$$

where C_1 is a positive constant. We assume that the function $\Phi(u) \in C^1(I\!\!R)$ satisfies the conditions

$$\Phi'(u) \geq 0 \quad \text{and} \quad M_1|u|^{p_0} + M_2|u| \geq \Phi(u) \cdot u \geq \mu_0|u|^{p_0} \tag{1.11}$$

where M_1, M_2, $\mu_0 > 0$ and $p_0 > 2$.

Null and approximate controllability of the semilinear heat equation were studied in [7, 10, 11] (In a null controllability problem one seeks a control f such that the corresponding initial-boundary problem possesses a solution u with $u(T) = 0$). In those papers they assumed that the control acts on any open and nonempty subset of Ω or on a part of the boundary. Approximate controllability was also studied in [3] in which the distributed control acts on the whole domain.

Chapter 3 is organized as follows. In Section 3.1 we formulate the semilinear optimal control problems in an appropriate mathematical framework. In Section 3.2 we prove the existence of an optimal solution. In Section 3.3 we show that the optimal solution at the terminal time T approaches the target state W as the parameter $\gamma \to 0$, i.e. the optimal solution is a solution of approximate controllability problem. In Section 3.4 we discretize the spatial variables by finite element methods and consider dynamics of the semidiscrete optimal solution. In Section 3.5 we introduce a two-dimensional algorithm based on the gradient method to compute the optimal solution; we also will present some computational results.

2 LINEAR OPTIMAL CONTROL PROBLEMS

Following the plan outlined in chapter 1 we study in this chapter the linear optimal control problems of minimizing the terminal-state tracking functional (1.1) or (1.2) subject to the linear parabolic equations (1.3)–(1.5).

2.1 Formulation of optimal control and controllability problems

Throughout we freely make use of standard Sobolev space notations $H^m(\Omega)$ and $H^1_0(\Omega)$. We denote the norm for Sobolev space $H^m(\Omega)$ by $\|\cdot\|_m$. Note that $H^0(\Omega) = L^2(\Omega)$ so that $\|\cdot\|_0$ is the $L^2(\Omega)$ norm. We will need the temporal-spatial function space

$$H^{2,1}((0,T)\times\Omega) = \{v \in L^2(0,T;H^2(\Omega)) : v_t \in L^2(0,T;L^2(\Omega))\}\,.$$

A temporal-spatial function $v(t,\mathbf{x})$ often will be simply written as $v(t)$.

Functional (1.1) can be written as

$$\mathcal{J}(u,f) = \frac{T}{2}\|u(T)-W\|_0^2 + \frac{\gamma}{2}\int_0^T \|f(t)\|_0^2\,dt\,. \tag{2.1.1}$$

Regarding functional (1.2) the idea for constructing the reference function F is that we first choose a reference function $U(t,\mathbf{x})$ satisfying $U(T,\mathbf{x}) = W$ (i.e., U is a given path that reaches W at time T) and then set

$$F = U_t - \operatorname{div}[A(\mathbf{x})\nabla U] \qquad \text{in } [0,T]\times\Omega\,.$$

However, W (and thus U) in general does not vanish on the boundary. The series method to be studied in this paper will involve eigenseries expressions for reference functions F and U. The validity of these expressions requires U to vanish on the boundary. To resolve this difficulty we choose the reference function $F = F^{(\gamma)}$ (which is dependent on γ) as follows. We first choose a one-parameter set of functions $\{W^{(\gamma)} : \gamma > 0\} \subset H^2(\Omega) \cap H_0^1(\Omega)$ such that

$$\|W^{(\gamma)} - W\|_0 \to 0 \quad \text{as } \gamma \to 0\,. \tag{2.1.2}$$

(If $W \in H^2(\Omega) \cap H_0^1(\Omega)$, then we may simply choose $W^{(\gamma)} = W$ that is independent of γ. In general, W has an inhomogeneous boundary condition and $W^{(\gamma)}$ approximates W in the $L^2(\Omega)$ sense.) Next, for each given $\gamma > 0$, we choose a function $V^{(\gamma)}(t, \mathbf{x})$ that satisfies

$$\begin{gathered} V^{(\gamma)} \in L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega))\,, \;\; V_t^{(\gamma)} \in L^2(0,T; L^2(\Omega))\,, \\ V^{(\gamma)}(T) = W^{(\gamma)} \qquad \text{in } \Omega\,; \end{gathered} \tag{2.1.3}$$

in other words, $V^{(\gamma)}$ is an arbitrarily chosen smooth path that reaches $W^{(\gamma)}$ at time T. By virtue of (2.1.2)–(2.1.3) we have

$$\|V^{(\gamma)}(T) - W\|_0 \to 0 \quad \text{as } \gamma \to 0\,. \tag{2.1.4}$$

We also assume that

$$\|V^{(\gamma)}(0)\|_0 \le C \quad \text{where } C > 0 \text{ is a constant independent of } \gamma. \tag{2.1.5}$$

The choices of a $V^{(\gamma)}$ that satisfies (2.1.3), (2.1.4) and (2.1.5) are certainly nonvacuous, e.g., the steady-state function $V^{(\gamma)}(t, \cdot) = W^{(\gamma)}$ is a particular and convenient choice. Here we allow more general choices of such a path $V^{(\gamma)}(t, \cdot)$ than the steady-state one. The reference function F is now defined by

$$F = F^{(\gamma)} \equiv V_t^{(\gamma)} - \operatorname{div}\,[A(\mathbf{x})\nabla V^{(\gamma)}] \qquad \text{in } (0,T) \times \Omega\,. \tag{2.1.6}$$

Functional (1.2) may be written

$$\begin{aligned}\mathcal{K}(u,f) &= \frac{T}{2}\|u(T)-W\|_0^2 + \frac{\gamma}{2}\int_0^T \|f(t)-F(t)\|_0^2\,dt \\ &= \frac{T}{2}\|u(T)-W\|_0^2 + \frac{\gamma}{2}\int_0^T \left\|f(t) - \frac{d}{dt}V^{(\gamma)}(t) - \operatorname{div}\left[A(\mathbf{x})\nabla V^{(\gamma)}(t)\right]\right\|_0^2 dt\,.\end{aligned} \tag{2.1.7}$$

The solution to the constraint equations (1.3)–(1.5) is understood in the following weak sense:

Definition 2.1.1 *Let $f \in L^2((0,T);L^2(\Omega))$ and $w \in L^2(\Omega)$ be given. u is said to be a solution of (1.3)–(1.5) if $u \in L^2((0,T);H_0^1(\Omega))$, $u_t \in L^2((0,T);H^{-1}(\Omega))$, and u satisfies*

$$\begin{cases} \langle u_t(t),\phi\rangle + \displaystyle\int_\Omega [A(\mathbf{x})\nabla u(t)]\cdot\nabla\phi\,d\mathbf{x} \\ \qquad = \langle f(t),\phi\rangle \qquad \forall\phi\in H_0^1(\Omega),\ a.e.\ t\in(0,T) \\ u(0) = w \quad in\ \Omega \end{cases} \tag{2.1.8}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Remark 2.1.2 *A weak solution in the sense of Definition 2.1.1 belongs to $C([0,T];L^2(\Omega))$; see* [6].

An admissible element for the optimal control problem is a pair (u,f) that satisfies the initial boundary value problem (2.1.8). The precise definition is given as follow.

Definition 2.1.3 *Let $w \in L^2(\Omega)$ be given. A pair (u,f) is said to be an admissible element if $u \in L^2((0,T);H_0^1(\Omega))$, $u_t \in L^2((0,T);H^{-1}(\Omega))$, $f \in L^2((0,T);L^2(\Omega))$, and (u,f) satisfies equation (2.1.8). The set of all admissible elements is denoted by $\mathcal{V}_{ad}((0,T),w)$ or simply $\mathcal{V}_{ad}$.*

The optimal control problems we study can be concisely stated as

(OP1) seek a pair $(\widehat{u},\widehat{f}) \in \mathcal{V}_{ad}$ such that $\mathcal{J}(\widehat{u},\widehat{f}) = \inf_{(u,f)\in\mathcal{V}_{ad}} \mathcal{J}(u,f)$

where the functional $\mathcal{J}$ is defined by (2.1.1).

and

(OP2)
$$\text{seek a pair } (\widehat{u}, \widehat{f}) \in \mathcal{V}_{ad} \text{ such that } \mathcal{K}(\widehat{u}, \widehat{f}) = \inf_{(u,f)\in\mathcal{V}_{ad}} \mathcal{K}(u, f)$$
where the functional $\mathcal{K}$ is defined by (2.1.7).

The existence and uniqueness of optimal solutions for (OP1) and (OP2) follow from classical optimal control theories (see, e.g., [19]:)

Theorem 2.1.4 *Assume that $w \in L^2(\Omega)$ and $W \in L^2(\Omega)$. Then there exists a unique solution $(\widehat{u}, \widehat{f}) \in \mathcal{V}_{ad}$ to* (OP1) *and to* (OP2)*. If, in addition, $w \in H_0^1(\Omega)$, then $\widehat{u} \in H^{2,1}((0,T) \times \Omega)$.*

The approximate and exact controllability problems are formulated as follows:

(AP-CON)
$$\text{seek a one-parameter set } \{(u_\epsilon, f_\epsilon) : \epsilon > 0\} \subset \mathcal{V}_{ad}$$
$$\text{such that } \lim_{\epsilon\to 0} \|u_\epsilon(T) - W\|_0 = 0$$

and

(EX-CON)
$$\text{seek a pair } (u, f) \in \mathcal{V}_{ad} \text{ such that } u(T) = W \text{ in } \Omega.$$

Of course, exact controllability, whenever it holds, implies approximate controllability. In particular, if w and W belong to $H_0^1(\Omega)$, then the exact controllability holds.

Theorem 2.1.5 *Assume that $w \in H_0^1(\Omega)$. Then* (EX-CON) *has a solution if and only if $W \in H_0^1(\Omega)$.*

proof: If (EX-CON) has a solution (u, f), then regularity for parabolic PDEs ([6, p.360, Theorem 5; p.288, Theorem 4]) implies $u \in H^{2,1}(Q)$ and $u \in C([0,T]; H^1(\Omega))$ so that $W = u(T) \in H^1(\Omega)$. Since $u = 0$ on $(0,T) \times \partial\Omega$, we have that

$$\|W\|_{1/2,\partial\Omega} = \lim_{t\to T-} \|u(T) - u(t)\|_{1/2,\partial\Omega} \le C \lim_{t\to T-} \|u(T) - u(t)\|_1 = 0$$

where $\|\cdot\|_{1/2,\partial\Omega}$ denotes the norm for the Sobolev space $H^{1/2}(\partial\Omega)$. Thus, $W \in H_0^1(\Omega)$.

Conversely, assume that $W \in H_0^1(\Omega)$. Let $\widetilde{u}$ be a function satisfying

$$\widetilde{u} \in H^{2,1}((0,T)\times\Omega)\,, \qquad \widetilde{u} = 0 \text{ on } (0,T)\times\partial\Omega\,, \qquad \widetilde{u}|_{t=0} = w \in H_0^1(\Omega).$$

The existence of such a $\widetilde{u}$ is guaranteed by the trace theorem [20, Vol. II, p.18, Theorem 2.3] or by the existence and regularity results (see [6]) for the parabolic problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in } (0,T)\times\Omega\,, \\ u = 0 & \text{on } (0,T)\times\partial\Omega\,, \\ u|_{t=0} = w\,. \end{cases}$$

Likewise, there exists a $\widetilde{\widetilde{u}}$ satisfying

$$\widetilde{\widetilde{u}} \in H^{2,1}((0,T)\times\Omega)\,, \qquad \widetilde{\widetilde{u}} = 0 \text{ on } (0,T)\times\partial\Omega\,, \qquad \widetilde{\widetilde{u}}|_{t=T} = W \in H_0^1(\Omega).$$

We choose a function $\theta = \theta(t) \in C^\infty[0,T]$ such that

$$\begin{cases} \theta(t) = 1 \quad \forall\, t \in [0, T/3], \\ 0 \le \theta(t) \le 1 \quad \forall\, t \in [T/3, 2T/3], \\ \theta(t) = 0 \quad \forall\, t \in [2T/3, T] \end{cases}$$

and set

$$u = \theta(t)\widetilde{u} + [1-\theta(t)]\widetilde{\widetilde{u}} \quad \text{in } (0,T)\times\Omega.$$

Clearly,

$$u \in H^{2,1}((0,T)\times\Omega)\,, \qquad u = 0 \text{ on } (0,T)\times\partial\Omega\,, \qquad u|_{t=0} = w\,, \qquad u|_{t=T} = W\,.$$

By defining

$$f \equiv u_t - \operatorname{div}[A(\mathbf{x})\nabla u] \in L^2((0,T)\times\Omega)$$

we see that (u, f) solves the exact controllability problem (EX-CON).

Remark 2.1.6 *In the statement of the exact controllability result of* [1, Theorem 3.7] $H^2(\Omega)$ *should be in* $H^2(\Omega)\cap H_0^1(\Omega)$*. The proof of that theorem indeed required the target state to have the homogeneous boundary condition.*

2.2 Results concerning eigenfunction expansions

The main objective of this chapter is to find explicit solution formulae, expressed in terms of eigenfunction expansions, for optimal control problems (OP1) and (OP2) and for controllability problem (EX-CON). In this section we will review some properties for the eigenpairs and eigenfunction expansions. We recall the following lemma (see [6, p.335, Theorem 1]):

Lemma 2.2.1 *The set* Λ *of all eigenvalues for the elliptic operator* $-\mathrm{div}\,(A(\mathbf{x})\nabla)$ *where* $A(\mathbf{x})$ *is defined by (1.3) may be written* $\Lambda = \{\lambda_i\}_{i=1}^{\infty} \subset I\!R$ *where*

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \cdots \qquad \text{and} \qquad \lambda_i \to \infty \text{ as } i \to \infty\,.$$

Furthermore, there exists a set of corresponding eigen functions $\{e_i\}_{i=1}^{\infty} \subset H^2(\Omega)\cap H_0^1(\Omega)$ *which form an orthonormal basis of* $L^2(\Omega)$ *(with respect to the* $L^2(\Omega)$ *inner product.)*

In the sequel we let $\{(\lambda_i, e_i)\}_{i=1}^{\infty}$ denote a set of eigenpairs as stated in Lemma 2.2.1.

Lemma 2.2.2 *The set* $\{e_i/\sqrt{\lambda_i}\}_{i=1}^{\infty}$ *forms an orthonormal basis of* $H_0^1(\Omega)$ *with respect to the inner product*

$$(u,v) \mapsto B[u,v] \equiv \int_\Omega A(\mathbf{x})\nabla u\cdot\nabla v\,d\mathbf{x}, \qquad \forall\, u,v \in H_0^1(\Omega)\,. \tag{2.2.1}$$

The set $\{e_i/\lambda_i\}_{i=1}^{\infty}$ *forms an orthonormal basis of* $H^2(\Omega)\cap H_0^1(\Omega)$ *with respect to the inner product*

$$\begin{aligned}(u,v) \mapsto \widetilde{B}[u,v] \equiv \int_\Omega \mathrm{div}\,[A(\mathbf{x})\nabla u]\,\mathrm{div}\,[A(\mathbf{x})\nabla v]\,d\mathbf{x},\\ \forall\, u,v \in H^2(\Omega)\cap H_0^1(\Omega)\,.\end{aligned} \tag{2.2.2}$$

proof: The first statement of this lemma is proved in [6, p.335, Theorem 1; p.337, step 3]). The proof for the second statement is a verbatim repetition of [6, p.335, Theorem 1; p.337, step 3]) with the inner product $B[\cdot,\cdot]$ replaced by $\widetilde{B}[\cdot,\cdot]$ (defined in (2.2.2)).

Based on Lemmas 2.2.1 and 2.2.2 we may establish the following characterizations of $H_0^1(\Omega)$.

Lemma 2.2.3 *Assume that* $y \in L^2(\Omega)$ *and* $y = \sum_{i=1}^{\infty} y_i e_i$ *in* $L^2(\Omega)$. *Then the following statements are equivalent:*

$$\text{i)} \quad y \in H_0^1(\Omega);$$
$$\text{ii)} \quad y = \sum_{i=1}^{\infty} y_i e_i \quad \textit{in } H_0^1(\Omega);$$
$$\text{iii)} \quad \sum_{i=1}^{\infty} \lambda_i |y_i|^2 < \infty.$$

proof: We first prove i) implies ii). But this follows from [6, p.335, Theorem 1; p.337, step 2 & 3].

We next prove ii) implies iii). Assume that $y = \sum_{i=1}^{\infty} y_i e_i$ in $H_0^1(\Omega)$. By Lemma 2.2.2 we may write

$$y = \sum_{i=1}^{\infty} \frac{\bar{y}_i}{\sqrt{\lambda_i}} e_i \quad \text{in } H_0^1(\Omega) \quad \text{and} \quad \sum_{i=1}^{\infty} |\bar{y}_i|^2 = B[y, y] < \infty.$$

Comparing

$$y = \sum_{i=1}^{\infty} \frac{\bar{y}_i}{\sqrt{\lambda_i}} e_i \quad \text{and} \quad y = \sum_{i=1}^{\infty} y_i e_i \quad \text{in } L^2(\Omega),$$

we obtain $\bar{y}_i = \sqrt{\lambda_i} y_i$ so that

$$\sum_{i=1}^{\infty} \lambda_i |y_i|^2 = \sum_{i=1}^{\infty} |\bar{y}_i|^2 < \infty\,.$$

Finally, we prove iii) implies i). Assume that $\sum_{i=1}^{\infty} \lambda_i |y_i|^2 < \infty$. We note that the definition of the eigenpairs implies

$$B[e_i, v] = \lambda_i \int_\Omega e_i v \, d\mathbf{x} \qquad \forall\, v \in H_0^1(\Omega)$$

so that $B[e_i, e_j] = 0$ if $j \neq i$ and $B[e_i, e_i] = \lambda_i$. Thus,

$$B\Big[\sum_{i=n}^{n+p} y_i e_i, \sum_{j=n}^{n+p} y_j e_j\Big] = \sum_{i=n}^{n+p} \lambda_i |y_i|^2$$

so that $\{\sum_{i=1}^{n} y_i e_i\}_{n=1}^{\infty} \subset H_0^1(\Omega)$ is a Cauchy sequence with respect to the $H_0^1(\Omega)$ norm induced by the $B[\cdot, \cdot]$ inner product. Hence $\sum_{i=1}^{\infty} y_i e_i = \bar{y}$ in $H_0^1(\Omega)$ for some $\bar{y} \in H_0^1(\Omega)$. But $y = \sum_{i=1}^{\infty} y_i e_i$ in $L^2(\Omega)$ and we conclude $y = \bar{y} \in H_0^1(\Omega)$.

Similar arguments yield the following characterizations of $H^2(\Omega) \cap H_0^1(\Omega)$.

Lemma 2.2.4 *Assume that* $y \in L^2(\Omega)$ *and* $y = \sum_{i=1}^{\infty} y_i e_i$ *in* $L^2(\Omega)$. *Then the following statements are equivalent:*

$$\begin{aligned} &\text{i)} \quad y \in H^2(\Omega) \cap H_0^1(\Omega); \\ &\text{ii)} \quad y = \sum_{i=1}^{\infty} y_i e_i \quad \text{in } H^2(\Omega) \cap H_0^1(\Omega); \\ &\text{iii)} \quad \sum_{i=1}^{\infty} |\lambda_i|^2 |y_i|^2 < \infty. \end{aligned}$$

The main results of this section are the two theorems below concerning term-by-term differentiations of eigenseries for functions in $H^{2,1}((0,T) \times \Omega) \cap C([0,T]; H_0^1(\Omega))$. We first quote a lemma (see [25, p.169, Lemma 1.1] and [6, p.286, Theorem 2])

Lemma 2.2.5 *Assume that* $u \in L^2(0,T; L^2(\Omega))$ *and* $u_t \in L^2(0,T; L^2(\Omega))$. *Then*

$$-\int_0^T \phi'(t) \int_\Omega u(t) v \, d\mathbf{x} \, dt = \int_0^T \phi(t) \int_\Omega u_t(t) v \, d\mathbf{x} \, dt \qquad \forall \, \phi \in C_0^\infty(0,T), \; \forall \, v \in L^2(\Omega) \, .$$

Theorem 2.2.6 *Assume that* $u \in H^{2,1}((0,T) \times \Omega)$, $u = 0$ *on* $(0,T) \times \partial\Omega$ *and*

$$u(t) = \sum_{i=1}^{\infty} u_i(t) e_i \quad \text{in } L^2(\Omega) \, , \text{a.e. } t \in (0,T) \, .$$

Then

$$\sum_{i=1}^{\infty} \int_0^T \Big(|u_i'(t)|^2 + |\lambda_i|^2 |u_i(t)|^2 \Big) dt = \|u_t\|^2_{L^2(0,T;L^2(\Omega))} + \int_0^T \widetilde{B}[u,u] \, dt < \infty \, , \tag{2.2.3}$$

$$\sum_{i=1}^{\infty} |\lambda_i| |u_i(0)|^2 \, dt < \infty \, , \tag{2.2.4}$$

$$u_t(t) = \sum_{i=1}^{\infty} u_i'(t) e_i \quad \text{in } L^2(\Omega) \, , \text{a.e. } t \in (0,T) \tag{2.2.5}$$

and

$$-\text{div} \, [A(\mathbf{x}) \nabla u(t)] = \sum_{i=1}^{\infty} \lambda_i u_i(t) e_i \quad \text{in } L^2(\Omega) \, , \; \text{a.e. } t \, . \tag{2.2.6}$$

proof: We first note the continuous embedding $H^{2,1}((0,T) \times \Omega) \hookrightarrow C([0,T]; H^1(\Omega))$ and the boundary condition $u = 0$ on $(0,T) \times \partial\Omega$ imply that $u(t) \in H_0^1(\Omega)$ for every $t \in [0,T]$. By Lemma 2.2.3 we have

$$u(t) = \sum_{i=1}^{\infty} u_i(t) e_i \quad \text{in } H_0^1(\Omega) \, , \forall \, t \in [0,T] \, .$$

In particular, since $u(0) \in H_0^1(\Omega)$, Lemma 2.2.3 yields (2.2.4).

Using the $L^2(\Omega)$ orthonormality of $\{e_i\}$ we have

$$\|u\|^2_{L^2(0,T;L^2(\Omega))} = \int_0^T \|u(t)\|_0^2\, dt = \int_0^T \sum_{i=1}^{\infty} |u_i(t)|^2\, dt \geq \int_0^T |u_j(t)|^2\, dt, \qquad \forall\, j$$

so that each $u_j \in L^2(0,T)$. Since $u_t \in L^2(0,T;L^2(\Omega))$, we may write

$$u_t(t) = \sum_{i=1}^{\infty} v_i(t) e_i \quad \text{in } L^2(\Omega)\,, \text{ a.e } t$$

and

$$\|u_t\|^2_{L^2(0,T;L^2(\Omega))} = \int_0^T \|u_t(t)\|_0^2\, dt = \int_0^T \sum_{i=1}^{\infty} |v_i(t)|^2\, dt \geq \int_0^T |v_j(t)|^2\, dt, \qquad \forall\, j \qquad (2.2.7)$$

so that each $v_j \in L^2(0,T)$. Using Lemma 2.2.5 we have that

$$-\int_0^T \phi'(t) \int_\Omega u(t) e_j\, d\mathbf{x}\, dt = \int_0^T \phi(t) \int_\Omega u_t(t) e_j\, d\mathbf{x}\, dt, \qquad \forall\, \phi \in C_0^\infty(0,T),\ j = 1, 2, \cdots .$$

Substituting series expressions for u and u_t into the last equation and using the $L^2(\Omega)$ orthonormality of $\{e_i\}$ we obtain

$$-\int_0^T \phi'(t) u_j(t)\, dt = \int_0^T \phi(t) v_j(t)\, dt, \qquad \forall\, \phi \in C_0^\infty(0,T),\ j = 1, 2, \cdots$$

so that $v_j = u_j'$ for $j = 1, 2, \cdots$. This proves (2.2.5).

Since $u(t) \in H^2(\Omega) \cap H_0^1(\Omega)$ for almost every t, Lemma 2.2.4 implies that

$$u(t) = \sum_{i=1}^{\infty} u_i(t) e_i \quad \text{in } H^2(\Omega) \cap H_0^1(\Omega)\,, \text{ a.e. } t$$

so that

$$-\text{div}\,[A(\mathbf{x})\nabla u(t)] = \sum_{i=1}^{\infty} -\text{div}\,[A(\mathbf{x})\nabla u_i(t) e_i] = \sum_{i=1}^{\infty} \lambda_i u_i(t) e_i \quad \text{in } L^2(\Omega)\,, \text{ a.e. } t\,,$$

i.e., (2.2.6) holds.

From (2.2.6) we obtain

$$\int_0^T \widetilde{B}[u,u]\, dt = \int_0^T \|\text{div}\,[A(\mathbf{x})\nabla u(t)]\|_0^2\, dt = \int_0^T \sum_{i=1}^{\infty} |\lambda_i|^2 |u_i(t)|^2\, dt\,. \qquad (2.2.8)$$

Adding up (2.2.7) and (2.2.8) and applying the Monotone Convergence Theorem we arrive at (2.2.3).

Theorem 2.2.7 *Assume that the set of functions* $\{u_i(t)\}_{i=1}^{\infty} \subset H^1(0,T)$ *satisfies*

$$\sum_{i=1}^{\infty} \int_0^T \Big(|u_i'(t)|^2 + |\lambda_i|^2 |u_i(t)|^2 \Big) dt < \infty \tag{2.2.9}$$

and

$$\sum_{i=1}^{\infty} |\lambda_i| |u_i(0)|^2 \, dt < \infty \,. \tag{2.2.10}$$

Then the function u *formally defined by* $u(t) = \sum_{i=1}^{\infty} u_i(t) e_i$ *satisfies* $u \in H^{2,1}((0,T) \times \Omega)$, $u = 0$ *on* $(0,T) \times \partial\Omega$,

$$u_t(t) = \sum_{i=1}^{\infty} u_i'(t) e_i \quad in \ L^2(\Omega) \,, \ a.e. \ t \tag{2.2.11}$$

and

$$-\text{div}\,[A(\mathbf{x}) \nabla u(t)] = \sum_{i=1}^{\infty} \lambda_i u_i(t) e_i \quad in \ L^2(\Omega) \,, \ a.e. \ t \,. \tag{2.2.12}$$

proof: We note that

$$\sum_{i=1}^{\infty} \int_0^T |u_i(t)|^2 \, dt \le \frac{1}{|\lambda_1|^2} \sum_{i=1}^{\infty} \int_0^T |\lambda_i|^2 |u_i(t)|^2 \, d < \infty$$

so that $u(t) = \sum_{i=1}^{\infty} u_i(t) e_i$ in $L^2(\Omega)$ for almost every $t \in (0,T)$.

By assumption (2.2.9) we are justified to define $f \in L^2((0,T); L^2(\Omega))$ as the series function

$$f = \sum_{i=1}^{\infty} f_i(t) e_i \equiv \sum_{i=1}^{\infty} [u_i'(t) + \lambda_i u_i(t)] e_i \quad \text{in } L^2(\Omega) \,, \ \text{a.e. } t \in (0,T) \,.$$

It is well known that $H^1(0,T)$ is continuously embedded into $C[0,T]$ so that $u_i(0)$ is well defined for each i. Assumption (2.2.10) and Lemma 2.2.3 imply that $u|_{t=0} \in H_0^1(\Omega)$ where $u|_{t=0} = \sum_{i=1}^{\infty} u_i(0) e_i$.

Let $\tilde{u}$ be the solution for the parabolic problem

$$\begin{cases} \tilde{u}_t - \text{div}\,[A(\mathbf{x}) \nabla \tilde{u}] = f \quad \text{in } (0,T) \times \Omega \,, \\ \\ \tilde{u} = 0 \quad \text{on } (0,T) \times \partial\Omega \,, \\ \\ \tilde{u}|_{t=0} = u|_{t=0} \end{cases} \tag{2.2.13}$$

in the sense of Definition 2.1.1. The regularity for parabolic PDEs implies $\widetilde{u} \in H^{2,1}((0,T)\times\Omega)$. We write $\widetilde{u} = \sum_{i=1}^{\infty} \widetilde{u}_i(t)e_i$ in $L^2(\Omega)$ for almost every $t \in (0,T)$. Employing Theorem 2.2.6 we have

$$\widetilde{u}_t(t) = \sum_{i=1}^{\infty} \widetilde{u}_i'(t)e_i \quad \text{in } L^2(\Omega)\,, \text{ a.e. } t \tag{2.2.14}$$

and

$$-\text{div}\,[A(\mathbf{x})\nabla\widetilde{u}(t)] = \sum_{i=1}^{\infty} \lambda_i\widetilde{u}_i(t)e_i \quad \text{in } L^2(\Omega)\,, \text{ a.e. } t\,. \tag{2.2.15}$$

Thus, we may write (2.2.13) in the series form

$$\begin{cases} \sum_{i=1}^{\infty}[\widetilde{u}_i'(t) + \lambda_i\widetilde{u}_i(t)]e_i = \sum_{i=1}^{\infty} f_i(t)e_i \quad \text{in } L^2(\Omega)\,, \text{ a e. } t \\ \sum_{i=1}^{\infty} \widetilde{u}_i(0) = \sum_{i=1}^{\infty} u_i(0)e_i \quad \text{in } L^2(\Omega) \end{cases}$$

so that for each i,

$$\begin{cases} \widetilde{u}_i(t) + \lambda_i\widetilde{u}_i(t) = f_i(t) \quad \text{in } (0,T)\,, \\ \widetilde{u}_i(0) = u_i(0). \end{cases} \tag{2.2.16}$$

From the definition of f_i we see that each u_i satisfies the same equations as $\widetilde{u}_i$. The uniqueness of the solution for the initial value problem (2.2.16) implies $u_i \equiv \widetilde{u}_i$ in $(0,T)$ for each i so that $u(t) = \widetilde{u}(t)$ in $L^2(\Omega)$ for every t. Hence, $u = \widetilde{u} \in H^{2,1}((0,T)\times\Omega)$ and $u = \widetilde{u} = 0$ on $(0,T)\times\partial\Omega$. Also, equations (2.2.14) and (2.2.15) yield (2.2.11) and (2.2.12).

2.3 Solutions of the optimal control problems

We express all functions involved as $L^2(\Omega)$-convergent series of $\{e_i\}$:

$$u(t,\mathbf{x}) = \sum_{i=1}^{\infty} u_i(t)e_i(\mathbf{x})\,, \qquad f(t,\mathbf{x}) = \sum_{i=1}^{\infty} f_i(t)e_i(\mathbf{x})\,, \qquad w(\mathbf{x}) = \sum_{i=1}^{\infty} w_ie_i(\mathbf{x})\,,$$

$$W(\mathbf{x}) = \sum_{i=1}^{\infty} W_ie_i(\mathbf{x})\,, \qquad V^{(\gamma)}(t,\mathbf{x}) = \sum_{i=1}^{\infty} V_i^{(\gamma)}(t)e_i(\mathbf{x})\,.$$

We work out below an explicit formula for the optimal solution of (OP1) expressed as a series of eigenfunctions $\{e_i\}$. (For the existence of optimal solutions, see Theorem 2.1.4.)

Theorem 2.3.1 *Assume that* $w \in H_0^1(\Omega)$, $W \in L^2(\Omega)$, *and* $(\widehat{u}, \widehat{f}) \in H^{2,1}((0,T)\times\Omega)\times L^2((0,T)\times\Omega)$ *is the solution of* (OP1). *Then*

$$\widehat{u}(t,\mathbf{x}) = \sum_{i=1}^{\infty} \widehat{u}_i(t) e_i(\mathbf{x}) \tag{2.3.1}$$

where

$$\widehat{u}_i(t) = w_i\Big(e^{-\lambda_i t} - \frac{Te^{-\lambda_i T}(e^{\lambda_i t} - e^{-\lambda_i t})}{2\lambda_i\gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T})}\Big) + W_i \frac{T(e^{\lambda_i t} - e^{-\lambda_i t})}{2\lambda_i\gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T})}\,. \tag{2.3.2}$$

proof: Let (u,f) be an arbitrary admissible element, then $u \in H^{2,1}((0,T)\times\Omega)\cap C([0,T];H_0^1(\Omega))$. We may write $u = \sum_{i=1}^{\infty} u_i(t)e_i$ and $f = \sum_{i=1}^{\infty} f_i(t)e_i$ in $L^2(\Omega)$ for almost every t. Moreover, Theorem 2.2.6 implies

$$u_t = \sum_{i=1}^{\infty} u_i'(t)e_i \qquad \text{in } L^2(\Omega)\,, \text{ a.e. } t$$

and

$$-\mathrm{div}\,[A(\mathbf{x})\nabla u] = \sum_{i=1}^{\infty} \lambda_i u_i(t) e_i \qquad \text{in } L^2(\Omega)\,, \text{ a.e. } t\,.$$

Thus we may rewrite the constraint equations (2.1.8) as

$$\begin{cases} \int_\Omega \Big(\sum_{j=1}^{\infty}[u_j'(t) + \lambda_j u_j(t)]e_j\Big)e_i\,d\mathbf{x} = \int_\Omega \Big(\sum_{j=1}^{\infty} f_j(t)e_j\Big)e_i\,d\mathbf{x} \quad i = 1,2,\cdots \\ \int_\Omega \Big(\sum_{j=1}^{\infty} u_j(0)e_j\Big)e_i\,d\mathbf{x} = \int_\Omega \Big(\sum_{j=1}^{\infty} w_j e_j\Big)e_i\,d\mathbf{x} \quad i = 1,2,\cdots \end{cases}$$

so that for each i,

$$\begin{cases} u_i'(t) + \lambda_i u_i(t) = f_i(t) \quad \text{in } (0,T)\,, \\ u_i(0) = w_i\,. \end{cases} \tag{2.3.3}$$

The functional $\mathcal{J}$ also can be written in the series form

$$\mathcal{J}(u,f) = \frac{T}{2}\sum_{i=1}^{\infty}|u_i(T) - W_i|^2 + \frac{\gamma}{2}\sum_{i=1}^{\infty}\int_0^T |f_i(t)|^2\,dt\,. \tag{2.3.4}$$

The optimal control problem (OP1) is recast into:

$(\widetilde{\mathrm{OP1}})$ minimize functional (2.3.4) subject to the constraints (2.3.3) for all $i = 1, 2, \cdots$.

Since the constraint equations are fully uncoupled for each i, the optimal control problem $(\widetilde{\mathrm{OP1}})$ is equivalent to

$(\widetilde{\mathrm{OP1}}_i)$ for each $i = 1, 2, \cdots$, minimize $\mathcal{J}_i(u_i, f_i)$ subject to the constraints (2.3.3).

where the functional $\mathcal{J}_i(u_i, f_i)$ is defined by

$$\mathcal{J}_i(u_i, f_i) = \frac{T}{2}|u_i(T) - W_i|^2 + \frac{\gamma}{2}\int_0^T |f_i(t)|^2\, dt\,.$$

The pair $(\widehat{u}, \widehat{f}) = (\sum_{i=1}^{\infty} \widehat{u}_i(t)e_i(\mathbf{x}), \sum_{i=1}^{\infty} \widehat{f}_i(t)e_i(\mathbf{x}))$ is the solution for (OP1) if and only if $(\widehat{u}_i, \widehat{f}_i)$ is the solution for $(\widetilde{\mathrm{OP1}}_i)$ for every i.

To solve the constrained minimization problem $(\widetilde{\mathrm{OP1}}_i)$ we introduce a Lagrange multiplier ξ_i and form the Lagrangian

$$\mathcal{L}_i(u_i, f_i, \xi_i) = \frac{T}{2}|u_i(T) - W_i|^2 - u_i(T)\xi_i(T) + w_i\xi_i(0) + \int_0^T \Big(\frac{\gamma}{2}|f_i(t)|^2 + u_i(t)\xi_i'(t) - \lambda_i u_i(t)\xi_i(t) + f_i(t)\xi_i(t)\Big)dt\,.$$

By taking variations of the Lagrangian with respect to ξ_i, u_i and f_i, respectively, we obtain an optimality system which consists of (2.3.3),

$$\begin{cases} \xi_i'(t) - \lambda_i\xi_i(t) = 0 \quad \text{in } (0, T)\,, \\ \xi_i(T) = T(u_i(T) - W_i), \end{cases} \tag{2.3.5}$$

and

$$\xi_i(t) = -\gamma f_i(t)\,. \tag{2.3.6}$$

We proceed to solve for $(\widehat{u}_i, \widehat{f}_i)$ from the optimality system formed by (2.3.3), (2.3.5) and (2.3.6).

By eliminating ξ_i from (2.3.5)–(2.3.6) we have

$$\begin{cases} f_i'(t) - \lambda_i f_i(t) = 0 \quad \text{in } (0,T)\,, \\[2ex] f_i(T) = -\frac{T}{\gamma}(u_i(T) - W_i)\,. \end{cases} \tag{2.3.7}$$

Combining (2.3.7) and (2.3.3) we arrive at a second order ordinary differential equation with initial and terminal conditions:

$$\begin{cases} u_i''(t) - \lambda_i^2 u_i(t) = 0 \quad \text{in } (0,T)\,, \\[2ex] u_i(0) = w_i\,, \\[2ex] u_i'(T) + \lambda_i u_i(T) = -\frac{T}{\gamma}(u_i(T) - W_i)\,. \end{cases} \tag{2.3.8}$$

The general solution to this differential equation is

$$u_i(t) = C_1 e^{-\lambda_i t} + C_2 e^{\lambda_i t}.$$

The initial and terminal conditions yield:

$$\begin{cases} C_1 + C_2 = w_i \\[2ex] \frac{T}{\gamma} e^{-\lambda_i T} C_1 + (2\lambda_i e^{\lambda_i T} + \frac{T}{\gamma} e^{\lambda_i T}) C_2 = \frac{T}{\gamma} W_i. \end{cases} \tag{2.3.9}$$

Solving for C_1 and C_2 and then plugging them into the general solution we find the formula for the solution $\widehat{u}_i$ to (2.3.8) and that formula is precisely (2.3.2). Hence, the solution to (OP1) is expressed by (2.3.1)–(2.3.2).

Similarly, we may derive an explicit formula for the optimal solution of (OP2).

Theorem 2.3.2 *Assume that* $w \in H_0^1(\Omega)$, $W \in L^2(\Omega)$, $W^{(\gamma)} \in H^2(\Omega) \cap H_0^1(\Omega)$, $V^{(\gamma)}$ *satisfies (2.1.3) and* F *is defined by (2.1.6). Let* $(\widehat{u}, \widehat{f}) \in H^{2,1}((0,T)\times\Omega) \times L^2((0,T)\times\Omega)$ *be the solution of* (OP2). *Then*

$$\widehat{u}(t,\mathbf{x}) = \sum_{i=1}^{\infty} \widehat{u}_i(t) e_i(\mathbf{x}) \tag{2.3.10}$$

where

$$\hat{u}_i(t) = V_i^{(\gamma)}(t) + [w_i - V_i^{(\gamma)}(0)]\Big(e^{-\lambda_i t} - \frac{Te^{-\lambda_i T}(e^{\lambda_i t} - e^{-\lambda_i t})}{2\lambda_i \gamma e^{\lambda_i T} + Te^{\lambda_i T} - Te^{-\lambda_i T}}\Big)$$
$$+[W_i - V_i^{(\gamma)}(T)]\frac{T(e^{\lambda_i t} - e^{-\lambda_i t})}{2\lambda_i \gamma e^{\lambda_i T} + Te^{\lambda_i T} - Te^{-\lambda_i T}}\,. \tag{2.3.11}$$

proof: As in the proof of Theorem 2.3.1 we may write the constraint equations as

$$\begin{cases} u_i'(t) + \lambda_i u_i(t) = f_i(t) \quad \text{in } (0,T)\,, \\ \\ u_i(0) = w_i \end{cases} \tag{2.3.12}$$

for $i = 1, 2, \cdots$.

To simplify the notation we drop the superscript $(\cdot)^{(\gamma)}$ and write V in place of $V^{(\gamma)}$. Since $V \in H^{2,1}((0,T)\times\Omega)$, we are justified by Theorem 2.2.6 to express (2.1.6) in the series form

$$\begin{aligned} \sum_{i=1}^{\infty} F_i(t)e_i &= F(t,\mathbf{x}) = V_t - \operatorname{div}[A(\mathbf{x})\nabla V] \\ &= \sum_{i=1}^{\infty}\Big[V_i'(t) + \lambda_i V_i(t)\Big]e_i \quad \text{in } L^2(\Omega)\,, \text{ a.e. } t \end{aligned} \tag{2.3.13}$$

so that

$$F_i(t) = V_i'(t) + \lambda_i V_i(t) \quad \text{a.e. } t\,.$$

The functional $\mathcal{K}$ also can be written in the series form

$$\begin{aligned} \mathcal{K}(u,f) &= \frac{T}{2}\sum_{i=1}^{\infty}|u_i(T) - W_i|^2 + \frac{\gamma}{2}\sum_{i=1}^{\infty}\int_0^T |f_i(t) - F_i(t)|^2\,dt \\ &= \frac{T}{2}\sum_{i=1}^{\infty}|u_i(T) - W_i|^2 + \frac{\gamma}{2}\sum_{i=1}^{\infty}\int_0^T |f_i(t) - V_i'(t) - \lambda_i V_i(t)|^2\,dt\,. \end{aligned} \tag{2.3.14}$$

The optimal control problem (OP2) is recast into:

$(\widetilde{\text{OP2}})$ minimize functional (2.3.14) subject to the constraints (2.3.12) for all $i = 1, 2, \cdots$.

Since the constraint equations are fully uncoupled for each i, the optimal control problem $(\widetilde{\text{OP2}})$ is equivalent to

$(\widetilde{\text{OP2}}_i)$ for each $i = 1, 2, \cdots$, minimize $\mathcal{K}_i(u_i, f_i)$ subject to the constraints (2.3.12).

where the functional $\mathcal{K}_i(u_i, f_i)$ is defined by

$$\mathcal{K}_i(u_i, f_i) = \frac{T}{2}|u_i(T) - W_i|^2 + \frac{\gamma}{2}\int_0^T |f_i(t) - V_i'(t) - \lambda_i V_i(t)|^2\, dt\,.$$

The pair $(\widehat{u}, \widehat{f}) = (\sum_{i=1}^{\infty} \widehat{u}_i(t) e_i(\mathbf{x}), \sum_{i=1}^{\infty} \widehat{f}_i(t) e_i(\mathbf{x}))$ is the solution for (OP2) if and only if $(\widehat{u}_i, \widehat{f}_i)$ is the solution for $(\widetilde{\text{OP2}_i})$ for every i.

To solve the constrained minimization problem $(\widetilde{\text{OP2}_i})$ we introduce a Lagrange multiplier ξ_i and form the Lagrangian

$$\begin{aligned}\mathcal{L}_i(u_i, f_i, \xi_i) = &\frac{T}{2}|u_i(T) - W_i|^2 - u_i(T)\xi_i(T) + w_i\xi_i(0) \\ &+ \int_0^T \Big(\frac{\gamma}{2}|f_i(t) - V_i'(t) - \lambda_i V_i(t)|^2 + u_i(t)\xi_i'(t) - \lambda_i u_i(t)\xi_i(t) + f_i(t)\xi_i(t)\Big) dt\,.\end{aligned}$$

By taking variations of the Lagrangian with respect to ξ_i, u_i and f_i, respectively, we obtain an optimality system which consists of (2.3.12),

$$\begin{cases} \xi_i'(t) - \lambda_i \xi_i(t) = 0 \quad \text{in } (0, T)\,, \\ \xi_i(T) = T(u_i(T) - W_i) \end{cases} \tag{2.3.15}$$

and

$$\xi_i(t) = -\gamma[f_i(t) - V_i'(t) - \lambda_i V_i(t)] \quad \text{in } (0, T)\,. \tag{2.3.16}$$

We proceed to solve for $(\widehat{u}_i, \widehat{f}_i)$ from the optimality system formed by (2.3.12), (2.3.15) and (2.3.16).

By eliminating ξ_i from (2.3.15)–(2.3.16) we have

$$\begin{cases} f_i'(t) - \lambda_i f_i(t) = V_i''(t) - \lambda_i^2 V_i(t) \quad \text{in } (0, T)\,, \\ f_i(T) = V_i'(T) + \lambda_i V_i(T) - \frac{T}{\gamma}(u_i(T) - W_i)\,. \end{cases} \tag{2.3.17}$$

Combining (2.3.17) and (2.3.12) we arrive at a second order ordinary differential equation with initial and terminal conditions:

$$\begin{cases} u_i''(t) - \lambda_i^2 u_i(t) = V_i''(t) - \lambda_i^2 V_i(t) \quad \text{in } (0, T)\,, \\ u_i(0) = w_i\,, \\ u_i'(T) + \lambda_i u_i(T) = V_i'(T) + \lambda_i V_i(T) - \frac{T}{\gamma}(u_i(T) - W_i). \end{cases} \tag{2.3.18}$$

Evidently, $V_i(t)$ is a particular solution of this differential equation so that the general solution is

$$u_i(t) = V_i(t) + C_1 e^{-\lambda_i t} + C_2 e^{\lambda_i t}.$$

The initial and terminal conditions yield:

$$\begin{cases} C_1 + C_2 = w_i - V_i(0) \\ \frac{T}{\gamma} e^{-\lambda_i T} C_1 + (2\lambda_i e^{\lambda_i T} + \frac{T}{\gamma} e^{\lambda_i T}) C_2 = \frac{T}{\gamma}[W_i - V_i(T)]\,. \end{cases} \tag{2.3.19}$$

Solving for C_1 and C_2 and then plugging them into the general solution we find formula (2.3.11) for the solution $\hat{u}_i$ to (2.3.18). Hence, the solution to (OP2) is expressed by (2.3.10).

Remark 2.3.3 *In order for series expressions (2.3.13) to be valid,* $V(t, \mathbf{x}) = \sum_{i=1}^{\infty} V_i(t) e_i(\mathbf{x})$ *must satisfy* $\sum_{i=1}^{\infty} |\lambda_i|^2 |V_i(t)|^2 < \infty$ *for almost every* t. *But then by Lemma 2.2.4,* $V(t) = V(t, \cdot)$ *must belong to* $H^2(\Omega) \cap H_0^1(\Omega)$. *This is precisely the reason for choosing* $W^{(\gamma)} \in H^2(\Omega) \cap H_0^1(\Omega)$ *that approximates* W *so as to define* V *and* F.

Remark 2.3.4 *As in the proof of Theorem (2.5.1) we may verify that the optimal solution* $\hat{u}$ *given by (2.3.1)–(2.3.2) or (2.3.10)–(2.3.11) indeed belongs to* $H^{2,1}((0,T) \times \Omega)$ *and satisfies* $\hat{u} = 0$ *on* $(0,T) \times \partial\Omega$.

2.4 Dynamics of the optimal control solutions

In this section we will derive pointwise-in-time estimates for $\|\hat{u}(t) - W\|_0$ (in the case of (OP1)) or $\|\hat{u}(t) - V^{(\gamma)}(t)\|_0$ (in the case of (OP2)) where $\hat{u}$ is the optimal solution for (OP1) or (OP2). The derivation will be based on the explicit solution formulae that were expressed as series of eigenfunctions $\{e_i\}$. We recall that $\{e_i\}$ is orthonormal in $L^2(\Omega)$ so that for any function $\phi(\mathbf{x}) = \sum_{i=1}^{\infty} \phi_i e_i(\mathbf{x})$ in $L^2(\Omega)$ we have $\|\phi\|_0^2 = \sum_{i=1}^{\infty} |\phi_i|^2$.

Lemma 2.4.1 *Let* $\lambda > 0$ *be given. Then* $2\lambda t \le e^{\lambda t} - e^{-\lambda t} \le e^{\lambda T} - e^{-\lambda T}$ *for all* $t \in [0, T]$.

proof: The right inequality follows from the fact that the function $g(t) \equiv e^{\lambda t} - e^{-\lambda t}$ is increasing on $[0,T]$ (as $g'(t) \geq 0$.) The left inequality can be proved by the power series expression for exponential functions:

$$e^{\lambda t} - e^{-\lambda t} = \sum_{m=0}^{\infty} \frac{\lambda^m t^m}{m!} - \sum_{m=0}^{\infty} \frac{(-1)^m \lambda^m t^m}{m!} = 2\sum_{m=1}^{\infty} \frac{\lambda^{2m-1} t^{2m-1}}{(2m-1)!} \geq 2\lambda t\,.$$

This completes the proof.

Theorem 2.4.2 *Assume that* $w \in H_0^1(\Omega)$ *and* $W \in L^2(\Omega)$. *Let* $(\widehat{u}, \widehat{f}) \in H^{2,1}((0,T)\times\Omega) \times L^2((0,T)\times\Omega)$ *be the solution of* (OP1). *Then*

$$\|\widehat{u}(t) - W\|_0^2 \leq 6e^{-2\lambda_1 t}\|w\|_0^2 + 3\|W\|_0^2 \qquad \forall\, t \in [0,T] \tag{2.4.1}$$

and for every integer $n \geq 1$,

$$\|\widehat{u}(T) - W\|_0^2 \leq \frac{2\gamma^2\|w\|_0^2}{T^4} + \frac{8\gamma^2}{T^2}\sup_{1\leq i\leq n}\frac{|\lambda_i|^2}{(1-e^{-2\lambda_i T})^2}\sum_{i=1}^{n}|W_i|^2 + 2\sum_{i=n+1}^{\infty}|W_i|^2\,. \tag{2.4.2}$$

Furthermore, the optimal solution $\widehat{u}$ *as a function of the parameter* γ *satisfies the approximate controllability property*

$$\lim_{\gamma\to 0}\|\widehat{u}(T) - W\|_0 = 0\,.$$

proof: Let $t \in [0,T]$ be given. Using solution formulae (2.3.1)–(2.3.2) and adding and subtracting terms we have:

$$\begin{aligned}
&\|\widehat{u}(t) - W\|_0^2 = \sum_{i=1}^{\infty}|u_i(t) - W_i|^2 \\
&= \sum_{i=1}^{\infty}\left|w_i\Big(e^{-\lambda_i t} - \frac{Te^{-\lambda_i T}(e^{\lambda_i t} - e^{-\lambda_i t})}{2\lambda_i\gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T})}\Big) + W_i\frac{T(e^{\lambda_i t} - e^{-\lambda_i t})}{2\lambda_i\gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T})} - W_i\right|^2 \\
&= \sum_{i=1}^{\infty}\Bigg\{w_i(e^{-\lambda_i t} - e^{-\lambda_i T}) + \frac{2\lambda_i\gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T}) - T(e^{\lambda_i t} - e^{-\lambda_i t})}{2\lambda_i\gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T})}e^{-\lambda_i T}w_i \\
&\qquad - \frac{2\lambda_i\gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T}) - T(e^{\lambda_i t} - e^{-\lambda_i t})}{2\lambda_i\gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T})}W_i\Bigg\}^2\,.
\end{aligned} \tag{2.4.3}$$

Applying the inequality $|\sum_{i=1}^{3} a_i|^2 \leq 3\sum_{i=1}^{3}|a_i|^2$ to (2.4.3) and using the relation

$$0 \leq \frac{2\lambda_i\gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T}) - T(e^{\lambda_i t} - e^{-\lambda_i t})}{2\lambda_i\gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T})} \leq 1 \qquad \text{(see Lemma 2.4.1)}$$

we obtain

$$\|\hat{u}(t) - W\|_0^2 \leq 3\|w\|_0^2 \sup_{1\leq i<\infty} |e^{-\lambda_i t} - e^{-\lambda_i T}|^2 + 3e^{-2\lambda_1 T}\|w\|_0^2 + 3\|W\|_0^2 \tag{2.4.4}$$

so that (2.4.1) holds.

Using formulae (2.3.1)–(2.3.2) with $t = T$ we have, for each integer $n \geq 1$,

$$\begin{aligned}
&\|\hat{u}(T) - W\|_0^2 = \sum_{i=1}^{\infty} |u_i(T) - W_i|^2 \\
&= \sum_{i=1}^{\infty} \left\{ \frac{2\lambda_i\gamma}{2\lambda_i\gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T})} w_i - \frac{2\lambda_i\gamma e^{\lambda_i T}}{2\lambda_i\gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T})} W_i \right\}^2 \\
&\leq 2\sum_{i=1}^{\infty} \left| \frac{2\lambda_i\gamma w_i}{2\lambda_i\gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T})} \right|^2 + 2\sum_{i=1}^{\infty} \left| \frac{2\lambda_i\gamma W_i}{2\lambda_i\gamma + T(1 - e^{-2\lambda_i T})} \right|^2 \qquad (2.4.5) \\
&\leq \frac{8\gamma^2}{T^2} \sup_{1\leq i<\infty} \frac{|\lambda_i|^2}{(e^{\lambda_i T} - e^{-\lambda_i T})^2} \sum_{i=1}^{\infty} |w_i|^2 \\
&\qquad + \frac{8\gamma^2}{T^2} \sup_{1\leq i\leq n} \frac{|\lambda_i|^2}{(1 - e^{-2\lambda_i T})^2} \sum_{i=1}^{n} |W_i|^2 + 2\sum_{i=n+1}^{\infty} |W_i|^2.
\end{aligned}$$

Using Lemma 2.4.1 we have

$$\frac{|\lambda_i|^2}{(e^{\lambda_i T} - e^{-\lambda_i T})^2} \leq \frac{|\lambda_i|^2}{(2\lambda_i T)^2} = \frac{1}{4T^2} \qquad \forall\, i\,. \tag{2.4.6}$$

Combining (2.4.5) and (2.4.6) we arrive at (2.4.2).

It remains to prove

$$\lim_{\gamma\to 0} \|\hat{u}(T) - W\|_0 = 0\,.$$

Let $\epsilon > 0$ be given. There exists an n such that

$$\sum_{i=n+1}^{\infty} |W_i|^2 < \frac{\epsilon^2}{6}\,.$$

Holding this n fixed, we may choose a γ_0 such that

$$\frac{8|\gamma_0|^2}{T^2}\sup_{1\le i\le n}\frac{|\lambda_i|^2}{(1-e^{-2\lambda_i T})^2}\sum_{i=1}^{n}|W_i|^2<\frac{\epsilon^2}{3}\quad\text{and}\quad\frac{2|\gamma_0|^2\|w\|_0^2}{T^4}<\frac{\epsilon^2}{3}\,.$$

Thus, we obtain from (2.4.2) that $\|\widehat{u}(T)-W\|_0<\epsilon$ for each $\gamma\in[0,\gamma_0]$.

We may similarly derive a pointwise-in-time, $L^2(\Omega)$ estimate for the solution of (OP2).

Theorem 2.4.3 *Assume that* $w\in H_0^1(\Omega)$, $W\in L^2(\Omega)$, $W^{(\gamma)}\in H^2(\Omega)\cap H_0^1(\Omega)$, $V^{(\gamma)}$ *satisfies (2.1.3) and* F *is defined by (2.1.6). Let* $(\widehat{u},\widehat{f})\in H^{2,1}((0,T)\times\Omega)\times L^2((0,T)\times\Omega)$ *be the solution of* (OP2)*. Then*

$$\begin{aligned}\|\widehat{u}(t)-V^{(\gamma)}(t)\|_0^2&\le 3e^{-2\lambda_1 t}\|w-V^{(\gamma)}(0)\|_0^2\\&+3e^{-2\lambda_1 T}\|w-V^{(\gamma)}(0)\|_0^2+3\|W-V^{(\gamma)}(T)\|_0^2\qquad\forall\,t\in[0,T]\end{aligned}\tag{2.4.7}$$

and

$$\|\widehat{u}(T)-V^{(\gamma)}(T)\|_0^2\le\frac{2\gamma^2}{T^4}\|w-V^{(\gamma)}(0)\|_0^2+2\|W-V^{(\gamma)}(T)\|_0^2\,.\tag{2.4.8}$$

If, in addition, hypotheses (2.1.2), (2.1.4) and (2.1.5) hold, then the optimal solution $\widehat{u}$ *as a function of the parameter* γ *satisfies*

$$\lim_{\gamma\to 0}\|\widehat{u}(T)-W\|_0=0\,.$$

proof: Using solution formulae (2.3.10)–(2.3.11) and writing V in place of $V^{(\gamma)}$ (for notation abbrevity) we obtain:

$$\begin{aligned}\|\widehat{u}(t)-V(t)\|_0^2&=\sum_{i=1}^{\infty}|u_i(t)-V_i(t)|^2\\&=\sum_{i=1}^{\infty}\Big\{[w_i-V_i(0)]e^{-\lambda_i t}+\frac{[W_i-V_i(T)]T(e^{\lambda_i t}-e^{-\lambda_i t})}{2\lambda_i\gamma e^{\lambda_i T}+T(e^{\lambda_i T}-e^{-\lambda_i T})}\\&\qquad+\frac{[V_i(0)-w_i]Te^{-\lambda_i T}}{2\lambda_i\gamma e^{\lambda_i T}+T(e^{\lambda_i T}-e^{-\lambda_i T})}(e^{\lambda_i t}-e^{-\lambda_i t})\Big\}^2\\&=\sum_{i=1}^{\infty}\Big\{[w_i-V_i(0)](e^{-\lambda_i t}-e^{-\lambda_i T})+\frac{[W_i-V_i(T)]T(e^{\lambda_i t}-e^{-\lambda_i t})}{2\lambda_i\gamma e^{\lambda_i T}+T(e^{\lambda_i T}-e^{-\lambda_i T})}\\&\quad+\frac{2\lambda_i\gamma e^{\lambda_i T}+T(e^{\lambda_i T}-e^{-\lambda_i T})-T(e^{\lambda_i t}-e^{-\lambda_i t})}{2\lambda_i\gamma e^{\lambda_i T}+T(e^{\lambda_i T}-e^{-\lambda_i T})}[w_i-V_i(0)]e^{-\lambda_i T}\Big\}^2\end{aligned}\tag{2.4.9}$$

so that

$$\|\widehat{u}(t) - V(t)\|_0^2 \leq 3\sum_{i=1}^{\infty}[w_i - V_i(0)]^2|e^{-\lambda_i t} - e^{-\lambda_i T}|^2 + 3\sum_{i=1}^{\infty}|W_i - V_i(T)|^2$$
$$+ 3\sum_{i=1}^{\infty}\Big|\frac{2\lambda_i\gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T}) - T(e^{\lambda_i t} - e^{-\lambda_i t})}{2\lambda_i\gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T})}\Big|^2[w_i - V_i(0)]^2 e^{-2\lambda_i T}$$
$$\leq 3e^{-2\lambda_1 t}\|w - V(0)\|_0^2 + 3\|W - V(T)\|_0^2 + 3e^{-2\lambda_1 T}\|w - V(0)\|_0^2 \quad \forall\, t \in [0,T],$$

i.e., (2.4.7) holds.

Setting $t = T$ in (2.4.9) and using (2.4.6) we have

$$\|\widehat{u}(T) - V(T)\|_0^2$$
$$\leq 2\sum_{i=1}^{\infty}\Big|\frac{2\lambda_i\gamma}{2\lambda_i\gamma e^{\lambda_i T} + T(e^{\lambda_i T} - e^{-\lambda_i T})}\Big|^2[w_i - V_i(0)]^2 + 2\sum_{i=1}^{\infty}|W_i - V_i(T)|^2$$
$$\leq 8\gamma^2\|w - V(0)\|_0^2 \sup_{1\leq i<\infty}\frac{|\lambda_i|^2}{T^2(e^{\lambda_i T} - e^{-\lambda_i T})^2} + 2\|W - V(T)\|_0^2$$
$$\leq \frac{2\gamma^2}{T^4}\|w - V(0)\|_0^2 + 2\|W - V(T)\|_0^2 .$$

This proves (2.4.8).

The relation

$$\lim_{\gamma\to 0}\|\widehat{u}(T) - W\|_0 = 0$$

follows easily from the triangle inequality $\|\widehat{u}(T)-W\|_0 \leq \|\widehat{u}(T)-V(T)\|_0+\|V(T)-W\|_0$, estimate (2.4.8) and assumption (2.1.2).

The particular choice of $V^{(\gamma)}(t) \equiv W^{(\gamma)}$ satisfies (2.1.3), (2.1.4) and (2.1.5). Thus Theorem 2.4.3 yields the following result.

Corollary 2.4.4 *Assume that $w \in H_0^1(\Omega)$, $W \in L^2(\Omega)$, $W^{(\gamma)} \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfying (2.1.2), $V^{(\gamma)}(t,\cdot) \equiv W^{(\gamma)}$ and F is defined by*

$$F = F^{(\gamma)} \equiv -\mathrm{div}\,[A(\mathbf{x})\nabla W^{(\gamma)}] \qquad \textit{in } (0,T)\times\Omega\,.$$

Let $(\widehat{u},\widehat{f}) = (\sum_{i=1}^\infty \widehat{u}_i(t)e_i(\mathbf{x}), \sum_{i=1}^\infty \widehat{f}_i(t)e_i(\mathbf{x})) \in H^{2,1}((0,T)\times\Omega)\times L^2((0,T)\times\Omega)$ *be the solution of* (OP2) *given by*

$$\begin{aligned}\widehat{u}_i(t) = W_i^{(\gamma)} + [w_i - W_i^{(\gamma)}]\Big(e^{-\lambda_i t} - \frac{Te^{-\lambda_i T}(e^{\lambda_i t} - e^{-\lambda_i t})}{2\lambda_i\gamma e^{\lambda_i T} + Te^{\lambda_i T} - Te^{-\lambda_i T}}\Big)\\ +[W_i - W_i^{(\gamma)}]\frac{T(e^{\lambda_i t} - e^{-\lambda_i t})}{2\lambda_i\gamma e^{\lambda_i T} + Te^{\lambda_i T} - Te^{-\lambda_i T}}\,.\end{aligned} \tag{2.4.10}$$

Then

$$\|\widehat{u}(t) - W^{(\gamma)}\|_0^2 \le 3e^{-2\lambda_1 t}\|w - W^{(\gamma)}\|_0^2 + 3e^{-2\lambda_1 T}\|w - W^{(\gamma)}\|_0^2 + 3\|W - W^{(\gamma)}\|_0^2 \qquad \forall\, t\in[0,T]$$

and

$$\|\widehat{u}(T) - W^{(\gamma)}\|_0^2 \le \frac{2\gamma^2}{T^4}\|w - W^{(\gamma)}\|_0^2 + 2\|W - W^{(\gamma)}\|_0^2\,.$$

Moreover, the optimal solution $\widehat{u}$ *as a function of* γ *satisfies*

$$\lim_{\gamma\to 0}\|\widehat{u}(T) - W\|_0 = 0\,.$$

When $W \in H^2(\Omega)\cap H_0^1(\Omega)$ we may simply choose $W^{(\gamma)} = W$ and $V^{(\gamma)} \equiv W$. Then Corollary 2.4.4 reduces to:

Corollary 2.4.5 *Assume that* $w \in H_0^1(\Omega)$, $W \in H^2(\Omega)\cap H_0^1(\Omega)$, $V^{(\gamma)} \equiv W$ *and* F *is defined by*

$$F \equiv -\mathrm{div}\,[A(\mathbf{x})\nabla W] \qquad \text{in } (0,T)\times\Omega\,.$$

Let $(\widehat{u},\widehat{f}) = (\sum\limits_{i=1}^\infty \widehat{u}_i(t)e_i(\mathbf{x}), \sum\limits_{i=1}^\infty \widehat{f}_i(t)e_i(\mathbf{x})) \in H^{2,1}((0,T)\times\Omega)\times L^2((0,T)\times\Omega)$ *be the solution of* (OP2) *given by*

$$\widehat{u}_i(t) = W_i + [w_i - W_i]\Big(e^{-\lambda_i t} - \frac{Te^{-\lambda_i T}(e^{\lambda_i t} - e^{-\lambda_i t})}{2\lambda_i\gamma e^{\lambda_i T} + Te^{\lambda_i T} - Te^{-\lambda_i T}}\Big)\,. \tag{2.4.11}$$

Then

$$\|\widehat{u}(t) - W\|_0^2 \le 2e^{-2\lambda_1 t}\|w - W\|_0^2 + 2e^{-2\lambda_1 T}\|w - W\|_0^2 \qquad \forall\, t\in[0,T]$$

and

$$\|\widehat{u}(T) - W\|_0^2 \le \frac{2\gamma^2}{T^4}\|w - W\|_0^2\,.$$

Moreover, the optimal solution $\hat{u}$ *as a function of* γ *satisfies*

$$\lim_{\gamma \to 0} \|\hat{u}(T) - W\|_0 = 0\,.$$

2.5 Solutions to the exact controllability problem

Recall that the exact controllability problem (EX-CON) is solvable if $w \in H_0^1(\Omega)$ and $W \in H_0^1(\Omega)$. Formally setting $\gamma = 0$ in (2.3.2) and (2.3.11) we expect to obtain solution formulae for the exact controllability problem (EX-CON). But these formulae needs justification as infinite series functions are involved. We first examine the solution obtained by setting $\gamma = 0$ in (2.3.2).

Theorem 2.5.1 *Assume that* $w \in H_0^1(\Omega)$ *and* $W \in H_0^1(\Omega)$. *Then the functions*

$$u(t, \mathbf{x}) = \sum_{i=1}^{\infty} u_i(t) e_i(\mathbf{x}) \qquad \textit{and} \qquad f(t, \mathbf{x}) = \sum_{i=1}^{\infty} f_i(t) e_i(\mathbf{x})\,,$$

where

$$u_i(t) = w_i e^{-\lambda_i t} - w_i \frac{e^{-\lambda_i T}(e^{\lambda_i t} - e^{-\lambda_i t})}{e^{\lambda_i T} - e^{-\lambda_i T}} + W_i \frac{e^{\lambda_i t} - e^{-\lambda_i t}}{e^{\lambda_i T} - e^{-\lambda_i T}} \tag{2.5.1}$$

and

$$f_i(t) = -2\lambda_i w_i \frac{e^{-\lambda_i T} e^{\lambda_i t}}{e^{\lambda_i T} - e^{-\lambda_i T}} + 2\lambda_i W_i \frac{e^{\lambda_i t}}{e^{\lambda_i T} - e^{-\lambda_i T}}\,, \tag{2.5.2}$$

form a solution pair to the exact controllability problem (EX-CON).

proof: Since $u_i(0) = w_i$ and $u_i(T) = W_i$, we have that $u(0) = w$ and $u(T) = W$. To show that the pair (u, f) is a solution to (EX-CON) we need to show that

$$\begin{cases} u_t - \operatorname{div}[A(\mathbf{x})\nabla u] = f \quad \text{in } (0,T) \times \Omega \\ \\ u = 0 \quad \text{on } (0,T) \times \partial\Omega \end{cases}$$

and we will do so by employing Theorem 2.2.7.

We proceed to verify the assumptions of Theorem 2.2.7.

Lemma 2.2.3 and the assumptions $w, W \in H_0^1(\Omega)$ imply

$$\sum_{i=1}^{\infty} |\lambda_i||w_i|^2 < \infty \qquad \text{and} \qquad \sum_{i=1}^{\infty} |\lambda_i||W_i|^2 < \infty \,. \tag{2.5.3}$$

Since $u_i(0) = w_i$, we obviously have

$$\sum_{i=1}^{\infty} \lambda_i |u_i(0)|^2 = \sum_{i=1}^{\infty} \lambda_i |w_i|^2 < \infty \,.$$

Let $C_T = 1 - e^{-2\lambda_1 T} \in (0,1)$. Then we have

$$2\lambda_i T \geq 2\lambda_1 T = -\ln(1 - C_T)$$

so that

$$e^{2\lambda_i T} \geq 1/(1 - C_T),$$

or equivalently,

$$e^{\lambda_i T} - e^{-\lambda_i T} \geq C_T e^{\lambda_i T} \qquad \forall\, i \,.$$

From (2.5.1) and the last inequality we have

$$\begin{aligned}
&\int_0^T |\lambda_i|^2 |u_i(t)|^2 \, dt \\
&\leq 3|\lambda_i|^2|w_i|^2 \int_0^T e^{-2\lambda_i t}\, dt + \frac{3|\lambda_i|^2|w_i|^2}{(C_T)^2 e^{4\lambda_i T}} \int_0^T e^{2\lambda_i t}\, dt + \frac{3|\lambda_i|^2|W_i|^2}{(C_T)^2 e^{2\lambda_i T}} \int_0^T e^{2\lambda_i t}\, dt \\
&\leq 3|\lambda_i|^2|w_i|^2 \frac{1}{2\lambda_i} + \frac{3|\lambda_i|^2|w_i|^2}{(C_T)^2 e^{4\lambda_i T}} \cdot \frac{e^{2\lambda_i T}}{2\lambda_i} + \frac{3|\lambda_i|^2|W_i|^2}{(C_T)^2 e^{2\lambda_i T}} \cdot \frac{e^{2\lambda_i T}}{2\lambda_i} \\
&\leq |\lambda_i||w_i|^2 \Big(\frac{3}{2} + \frac{3}{2(C_T)^2 e^{2\lambda_1 T}} \Big) + |\lambda_i||W_i|^2 \frac{3}{2(C_T)^2}
\end{aligned} \tag{2.5.4}$$

Combining (2.5.4) and (2.5.3) we arrive at

$$\sum_{i=1}^{\infty} \int_0^T |\lambda_i|^2 |u_i(t)|^2 \, dt < \infty \,.$$

Differentiation of (2.5.1) yields

$$u_i'(t) = -\lambda_i w_i e^{-\lambda_i t} - \lambda_i w_i \frac{e^{-\lambda_i T}(e^{\lambda_i t} + e^{-\lambda_i t})}{e^{\lambda_i T} - e^{-\lambda_i T}} + \lambda_i W_i \frac{e^{\lambda_i t} + e^{-\lambda_i t}}{e^{\lambda_i T} - e^{-\lambda_i T}} \,.$$

Note that $e^{\lambda_i t} + e^{-\lambda_i t} \le 2e^{\lambda_i t}$ so that estimations similar to those of (2.5.4) lead us to

$$\int_0^T |u_i'(t)|^2\,dt \le |\lambda_i||w_i|^2\Big(\frac{3}{2} + \frac{6}{(C_T)^2 e^{2\lambda_i T}}\Big) + |\lambda_i||W_i|^2 \frac{6}{(C_T)^2} < \infty\,.$$

Thus we have verified all assumptions of Theorem 2.2.7. Using that theorem we conclude that $u \in H^{2,1}((0,T)\times\Omega)$, $u = 0$ on $(0,T)\times\partial\Omega$, and

$$\begin{aligned} u_t(t) - \operatorname{div}[A(\mathbf{x})\nabla u(t)] &= \sum_{i=1}^{\infty} u_i'(t)e_i + \sum_{i=1}^{\infty}\lambda_i u_i(t)e_i \\ &= \sum_{i=1}^{\infty}\Big(-2\lambda_i w_i \frac{e^{-\lambda_i T}e^{\lambda_i t}}{e^{\lambda_i T} - e^{-\lambda_i T}} + 2\lambda_i W_i \frac{e^{\lambda_i t}}{e^{\lambda_i T} - e^{-\lambda_i T}}\Big)e_i \quad \text{in } L^2(\Omega)\,, \text{ a.e. } t\,. \end{aligned}$$

By a comparison of the last relation with (2.5.2) we deduce

$$f(t) = u_t(t) - \operatorname{div}[A(\mathbf{x})\nabla u(t)] \quad \text{in } L^2(\Omega)$$

for almost every t so that

$$f = u_t - \operatorname{div}[A(\mathbf{x})\nabla u] \in L^2(0,T;L^2(\Omega)).$$

Hence, the pair (u,f) is indeed a solution to (EX-CON).

If $W \in H^2(\Omega)\cap H_0^1(\Omega)$, then by choosing $V^{(\gamma)} \equiv W$ and setting $\gamma = 0$ in formula (2.3.11) we obtain another solution for the exact controllability problem (EX-CON). The proof of the following theorem is similar to that of Theorem 2.5.1 and is omitted.

Theorem 2.5.2 *Assume that* $w \in H_0^1(\Omega)$ *and* $W \in H^2(\Omega)\cap H_0^1(\Omega)$. *Then the functions*

$$u(t,\mathbf{x}) = \sum_{i=1}^{\infty} u_i(t)e_i(\mathbf{x}) \qquad \text{and} \qquad f(t,\mathbf{x}) = \sum_{i=1}^{\infty} f_i(t)e_i(\mathbf{x})\,,$$

where

$$u_i(t) = w_i\Big(e^{-\lambda_i t} - \frac{e^{-\lambda_i T}(e^{\lambda_i t} - e^{-\lambda_i t})}{e^{\lambda_i T} - e^{-\lambda_i T}}\Big) + W_i\Big(1 - e^{-\lambda_i t} + \frac{e^{-\lambda_i T}(e^{\lambda_i t} - e^{-\lambda_i t})}{e^{\lambda_i T} - e^{-\lambda_i T}}\Big)$$

and

$$f_i(t) = -2\lambda_i w_i \frac{e^{-\lambda_i T}e^{\lambda_i t}}{e^{\lambda_i T} - e^{-\lambda_i T}} + \lambda_i W_i\Big(1 + \frac{2e^{-\lambda_i T}e^{\lambda_i t}}{e^{\lambda_i T} - e^{-\lambda_i T}}\Big)\,,$$

form a solution pair to the exact controllability problem (EX-CON).

2.6 One-dimensional numerical simulations

In one space dimension the eigenpairs $\{e_i\}$ are well known so that optimal solutions for (OP1) and (OP2) can be computed from series solution formulae (2.3.1)–(2.3.2) or (2.3.10)–(2.3.11), respectively.

The one-dimensional constraint equations are defined on the spatial interval $\Omega = (0,1)$:

$$\begin{cases} u_t - u_{xx} = f \quad \text{in } (0,T)\times(0,1), \\ u(t,0) = u(t,1) = 0 \\ u(0,x) = w(x)\,. \end{cases}$$

The eigenpairs $\{(\lambda_i, e_i)\}_{i=1}^{\infty}$ are determined from

$$\begin{cases} -e''(x) = \lambda e(x) \quad 0 \le x \le 1\,, \\ e(0) = e(1) = 0. \end{cases}$$

It is well known that

$$\lambda_i = (i\pi)^2 \quad \text{and} \quad e_i(x) = \sqrt{2}\,\sin(i\pi x) \qquad i = 1, 2, \cdots\,.$$

Given a target function $W(x)$, the solution to optimal control problem (OP1) is explicitly given by (2.3.1)–(2.3.2). To find the solution to (OP2) we first need to construct $W^{(\gamma)}$ and $V^{(\gamma)}$ satisfying (2.1.2), (2.1.3), (2.1.4) and (2.1.5); we then use solution formulae (2.3.10)–(2.3.11). Note that w_i, W_i and $V_i^{(\gamma)}(t)$ in (2.3.2) and (2.3.11) are calculated by

$$w_i = \int_0^1 w(x)e_i(x)\,dx\,, \quad W_i = \int_0^1 W(x)e_i(x)\,dx \quad \text{and} \quad V_i^{(\gamma)}(t) = \int_0^1 V^{(\gamma)}(t,x)e_i(x)\,dx\,.$$

We consider two sets of data (the initial condition w, the target function W and the terminal time T):

DATA I.
$$\begin{cases} T = 2\,, \\ w(x) = \sum_{i=1}^{5} i\, e_i(x)/\sqrt{2}\,, \\ W(x) = T \sum_{i=1}^{5} i\, e_i(x)/\sqrt{2}\,. \end{cases}$$

DATA II.
$$\begin{cases} T = 1\,, \\ w(x) = \sum_{i=1}^{1000} i\, e_i(x)/\sqrt{2}\,, \\ W(x) = 1 = \sum_{i=1}^{\infty} W_i e_i(x) \text{ in } L^2(\Omega) \text{ where } W_i = \int_0^1 e_i = \sqrt{2}\,\frac{1-(-1)^i}{i\pi}\, d\mathbf{x}. \end{cases}$$

For each data set we solve (OP1) by series solution formulae (2.3.1)–(2.3.2). In each case we vary the parameter γ and plot the optimal solution $\widehat{u}$ at the terminal time T (the "$*$" curve) versus the target function W (the "$-$" curve.) See Figures 2.1 and 2.3.

For each data set we solve (OP2) by series solution formulae (2.3.10)–(2.3.11). In the case of DATA I, we choose

$$W^{(\gamma)}(x) = W(x) = T \sum_{i=1}^{5} i\, e_i(x)/\sqrt{2}$$

and

$$V^{(\gamma)}(t,x) = W(x) = T \sum_{i=1}^{5} i\, e_i(x)/\sqrt{2}$$

which evidently satisfy assumptions (2.1.2), (2.1.3), (2.1.4) and (2.1.5); in addition, formula (2.3.11) takes on the simpler form (2.4.11), i.e.,

$$\widehat{u} = \sum_{i=1}^{5} \widehat{u}_i(t)\sqrt{2}\sin(i\pi x)$$

where

$$\widehat{u}_i = \frac{2i}{\sqrt{2}} - \frac{i}{\sqrt{2}}\Big(e^{-i^2\pi^2 t} - \frac{e^{-2i^2\pi^2}(e^{i^2\pi^2 t} - e^{-i^2\pi^2 t})}{i^2\pi^2\gamma e^{2i^2\pi^2} + e^{2i^2\pi^2} - e^{-2i^2\pi^2}}\Big)\,.$$

In the case of DATA II, we choose

$$W^{(\gamma)}(x) = \frac{\sqrt{2}}{\pi} \sum_{i=1}^{N_\gamma} \frac{1-(-1)^i}{i} e_i(x)$$

and

$$V^{(\gamma)}(t,x) = W^{(\gamma)}(x) = \frac{\sqrt{2}}{\pi} \sum_{i=1}^{N_\gamma} \frac{1-(-1)^i}{i} e_i(x)$$

where $N_\gamma \to \infty$ as $\gamma \to 0$ (e.g., N_γ is the integer part of the decimal number $[1000 + \ln(1/\gamma)]$.) It can be verified that $W^{(\gamma)}$ and $V^{(\gamma)}$ satisfy assumptions (2.1.2), (2.1.3), (2.1.4) and (2.1.5); in addition, formula (2.3.11) takes on the simpler form (2.4.10), i.e.,

$$\hat{u} = \sum_{i=1}^{N_\gamma} \hat{u}_i(t)\sqrt{2}\sin(i\pi x)$$

where

$$\begin{aligned} \hat{u}_i &= \frac{\sqrt{2}}{i\pi}[1-(-1)^i] \\ &+ \Big(\frac{i}{\sqrt{2}} - \frac{\sqrt{2}}{i\pi}[1-(-1)^i]\Big)\Big(e^{-i^2\pi^2 t} - \frac{e^{-i^2\pi^2}(e^{i^2\pi^2 t} - e^{-i^2\pi^2 t})}{2i^2\pi^2\gamma e^{i^2\pi^2} + e^{i^2\pi^2} - e^{-i^2\pi^2}}\Big) \quad i = 1, 2, \cdots, 1000 \end{aligned}$$

and

$$\hat{u}_i = \frac{\sqrt{2}}{i\pi}[1-(-1)^i]\Big(1 - e^{-i^2\pi^2 t} + \frac{e^{-i^2\pi^2}(e^{i^2\pi^2 t} - e^{-i^2\pi^2 t})}{2i^2\pi^2\gamma e^{i^2\pi^2} + e^{i^2\pi^2} - e^{-i^2\pi^2}}\Big) \quad i = 1001, 1002, \cdots, N_\gamma\,.$$

As in the case of (OP1), for each data set we vary the parameter γ and plot the optimal solution $\hat{u}$ for (OP2) at the terminal time T (the "$*$" curve) versus the target function W (the "$-$" curve;) see the first column of plots in Figures 2.2 and 2.4. Note that unlike in the case of (OP1), the optimal solution $\hat{u}(T)$ for (OP2) matches W very well even for $\gamma = 1$. This phenomenon is expected from Corollary 2.4.4 and Corollary 2.4.5.

Moreover, in the case of (OP2), we again from Corollary 2.4.4 and Corollary 2.4.5 anticipate optimal solution $\hat{u}(t)$ to yield good matching to W even for moderate γ and $t << T$. When $\gamma = 1$, we plot some snapshots of the optimal solution $\hat{u}$ (the "$*$" curve) versus the target function W (the "$-$" curve.) See the second column of plots in Figures 2.2 and 2.4.

For DATA I the admissible state and the target state have matching boundary conditions (both have homogeneous boundary conditions.) For DATA II the admissible state and the target function have nonmatching boundary conditions. For both data sets and for sufficiently small γ, the optimal solutions expressed by the series formulae did a good job of tracking the target functions in the interior at the terminal time T, as predicted by Theorems 2.4.2 and 2.4.3. The optimal solutions of (OP2) furnish good matchings to the target state even for moderate γ and $t << T$, as predicted by Corollaries 2.4.4 and 2.4.5.

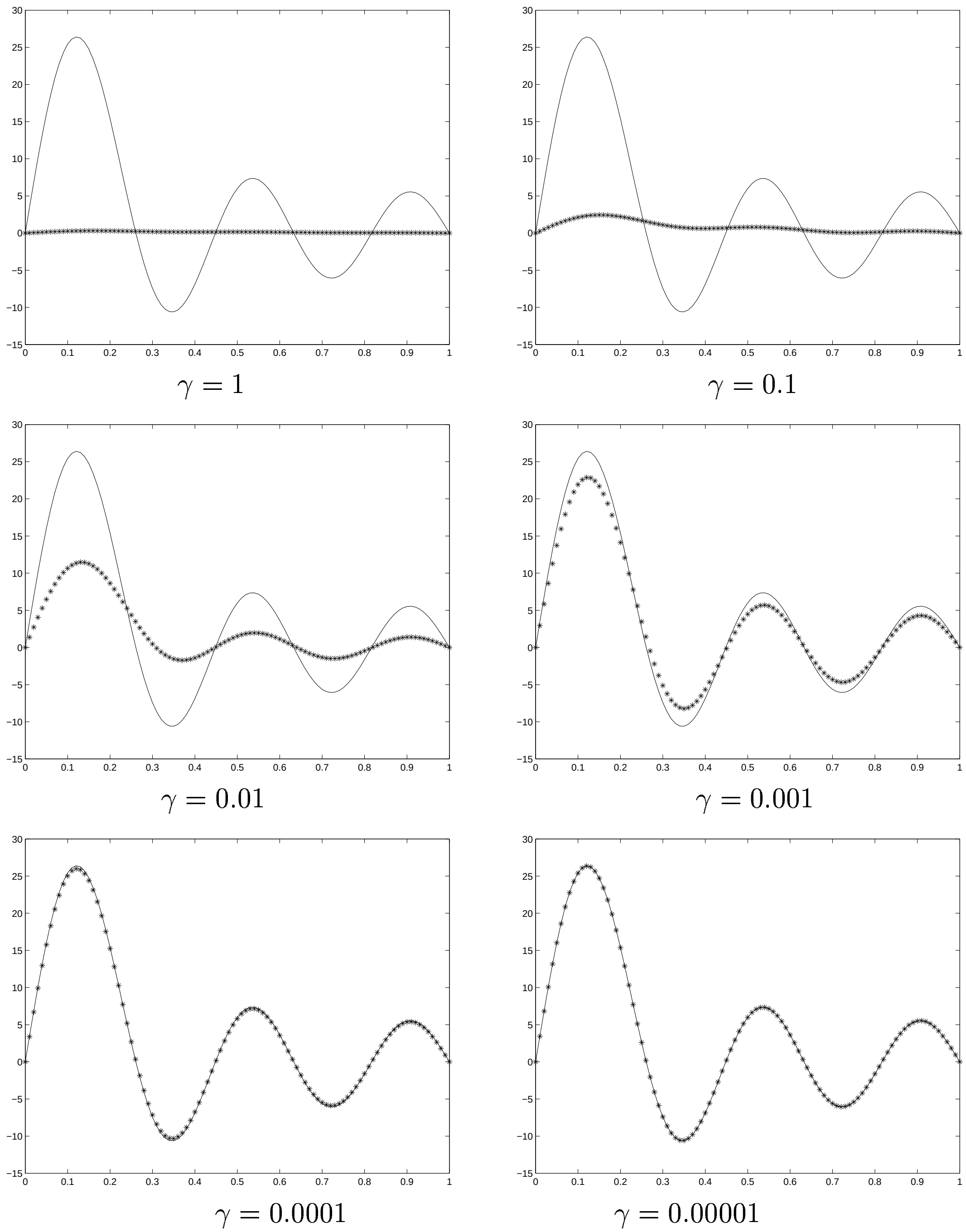


Figure 2.1 Optimal solution $\hat{u}(T)$ and target W for (OP1) with DATA I $(T = 2)$
$*$: optimal solution $\hat{u}(T)$ $-$: target function W

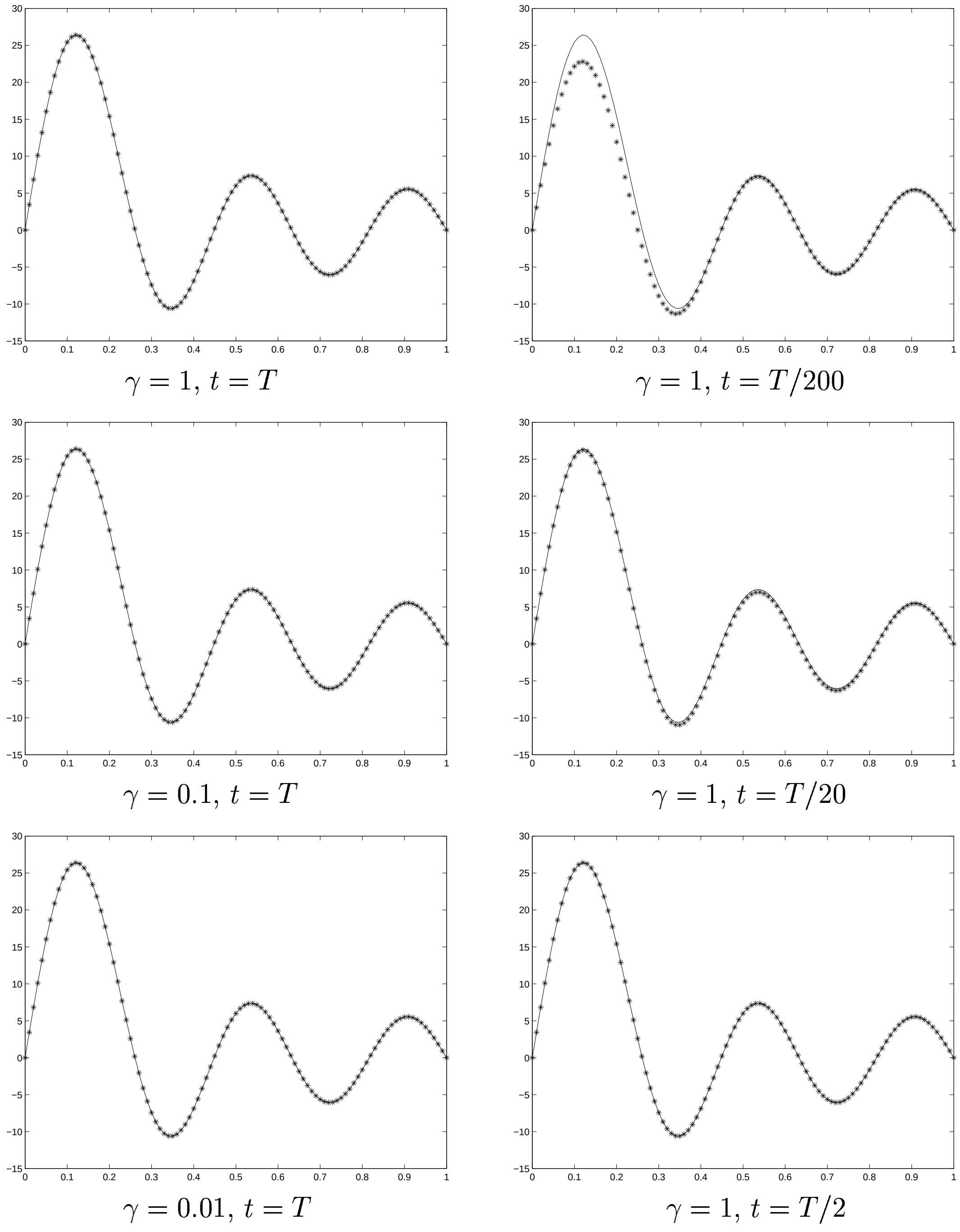
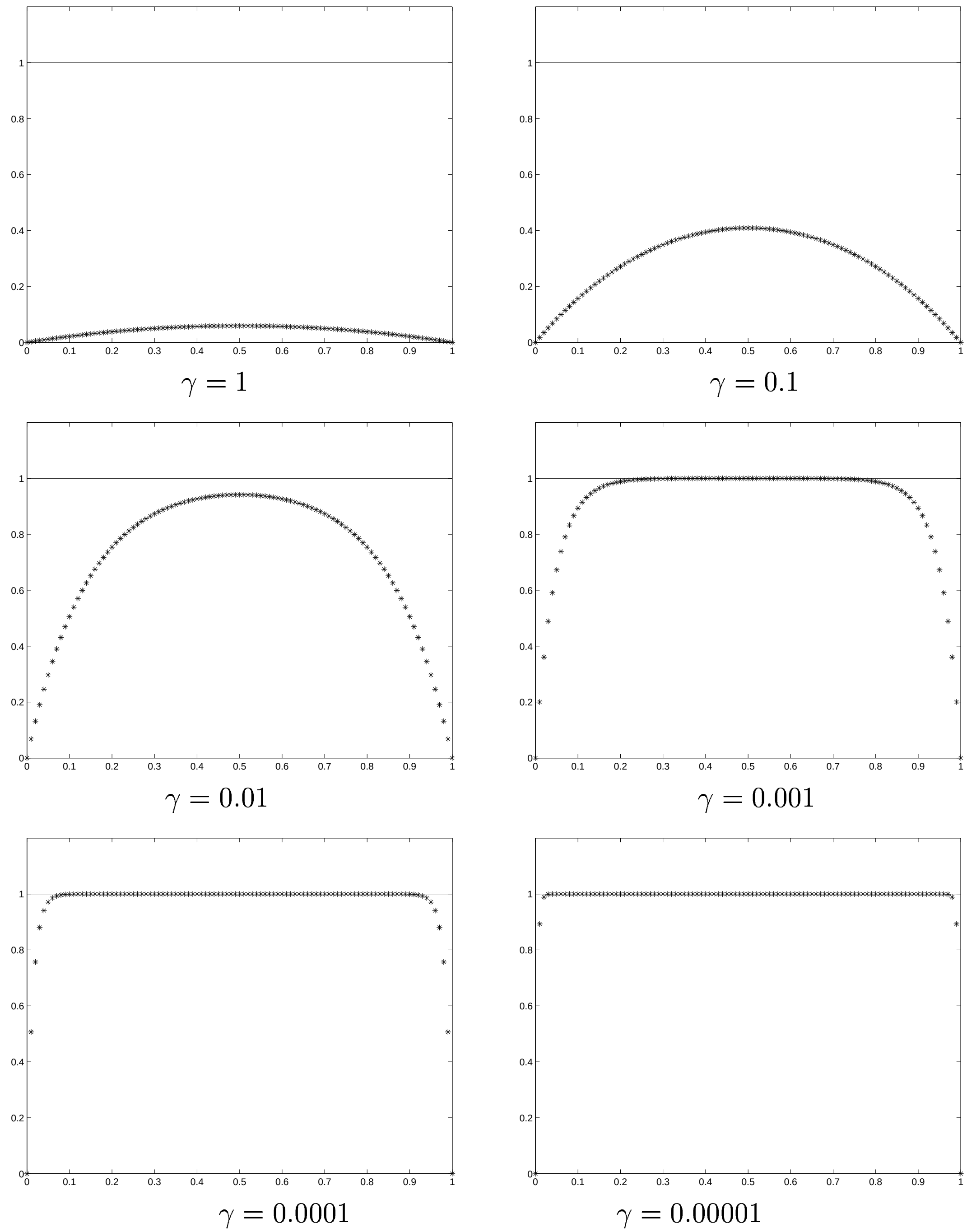


Figure 2.2 Optimal solution $\widehat{u}(t)$ and target W for (OP2) with DATA I ($T = 2$)
$*$: optimal solution $\widehat{u}(t)$ $-$: target function W



$\gamma = 1$ $\quad$ $\gamma = 0.1$

$\gamma = 0.01$ $\quad$ $\gamma = 0.001$

$\gamma = 0.0001$ $\quad$ $\gamma = 0.00001$

Figure 2.3 Optimal solution $\widehat{u}(t)$ and target W for (OP1) with DATA II ($T = 1$)
$*$: optimal solution $\widehat{u}(t)$ $\qquad$ $-$: target function W

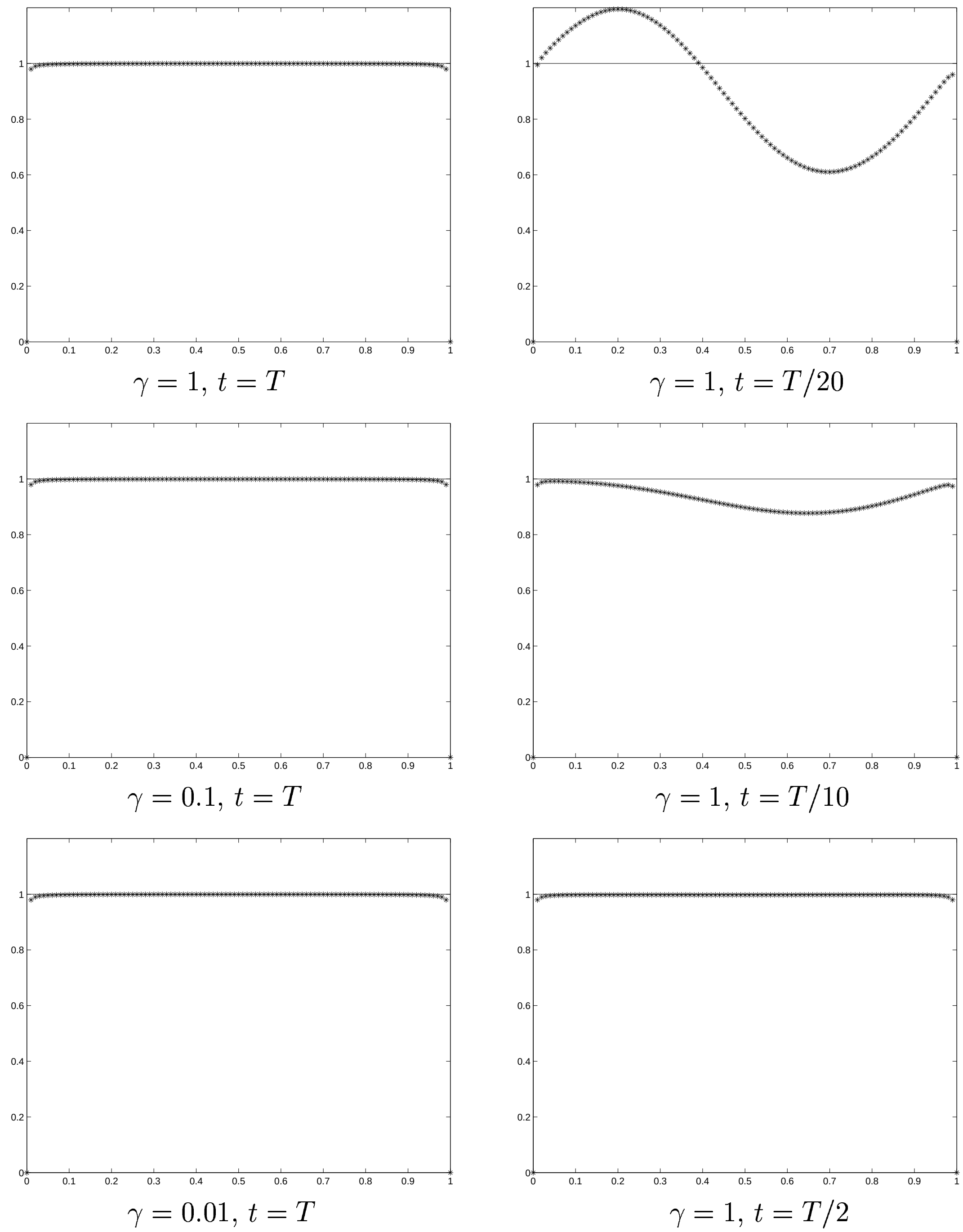


Figure 2.4 Optimal solution $\hat{u}(t)$ and target W for (OP2) with DATA II ($T = 1$)
$*$: optimal solution $\hat{u}(t)$ $-$: target function W

3 SEMILINEAR OPTIMAL CONTROL PROBLEMS

As outlined in chapter 1, we study in this chapter the semilinear optimal control problems: minimize the terminal-state tracking functional (1.6) subject to the semilinear parabolic equations (1.7)–(1.9).

3.1 Formulation of optimal control problems

Similar to the linear case, we construct the reference function F in (1.6) as follows. We first choose $\{W^{(\gamma)} : \gamma > 0\} \subset H^2(\Omega) \cap H_0^1(\Omega) \cap L^{p_0}(\Omega)$ such that

$$\|W^{(\gamma)} - W\|_0 \to 0 \quad \text{as } \gamma \to 0\,. \tag{3.1.1}$$

where p_0 is defined by (1.11). Next, for each given $\gamma > 0$, we choose a function $V^{(\gamma)}(t, \mathbf{x})$ that satisfies

$$\begin{aligned} &V^{(\gamma)} \in L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega)) \cap L^{p_0}(0,T; L^{p_0}(\Omega))\,, \\ &V_t^{(\gamma)} \in L^2(0,T; L^2(\Omega))\,, \\ &V^{(\gamma)}(T) = W^{(\gamma)} \qquad \text{in } \Omega\,; \end{aligned} \tag{3.1.2}$$

By virtue of (3.1.1)–(3.1.2) we have

$$\|V^{(\gamma)}(T) - W\|_0 \to 0 \quad \text{as } \gamma \to 0\,. \tag{3.1.3}$$

We also assume

$$\|V^{(\gamma)}(0)\|_0 \le C \quad \text{where } C > 0 \text{ is a constant independent of } \gamma. \tag{3.1.4}$$

The reference function F is now defined by

$$F = F^{(\gamma)} \equiv V_t^{(\gamma)} - \operatorname{div}[A(\mathbf{x})\nabla V^{(\gamma)}] + \Phi(V^{(\gamma)}) + a(\mathbf{x})V^{(\gamma)} \qquad \text{in } (0,T)\times\Omega\,. \tag{3.1.5}$$

Functional (1.6) may be written

$$\mathcal{J}(u,f) = \frac{T}{2}\|u(T)-W\|_0^2 + \frac{\gamma}{2}\int_0^T \|f(t)-F(t)\|_0^2\,dt\,. \tag{3.1.6}$$

Denote

$$B = \{u(t,\mathbf{x})|u \in L^2(0,T;H_0^1(\Omega)),\ u \in L^{p_0}(Q)\}. \tag{3.1.7}$$

Then B is a Banach space with

$$\|u\|_B = \|u\|_{L^2(0,T;H_0^1(\Omega))} + \|u\|_{L^{p_0}(Q)}.$$

The solution to the constraint equations (1.7)–(1.9) is understood in the following weak sense:

Definition 3.1.1 *Let $f \in L^2((0,T);L^2(\Omega))$ and $w \in L^2(\Omega)$ be given. u is said to be a solution of (1.7)–(1.9) if $u \in L^2((0,T);H_0^1(\Omega)) \cap L^{p_0}((0,T);L^{p_0}(\Omega))$, $u_t \in B^*$ which is a dual space of B and u satisfies*

$$\begin{cases} \langle u_t(t),\phi\rangle + \int_\Omega [A(\mathbf{x})\nabla u(t)]\cdot\nabla\phi\,d\mathbf{x} + \int_\Omega \Phi(u)\,\phi\,d\mathbf{x} + \int_\Omega a(\mathbf{x})u\,\phi\,d\mathbf{x} \\ \qquad = \int_\Omega f(t)\,\phi\,d\mathbf{x} \qquad \forall\,\phi \in H_0^1(\Omega)\cap L^{p_0}(\Omega),\ \ a.e.\ \ t\in(0,T) \\ u(0) = w \quad in\ \Omega \end{cases} \tag{3.1.8}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H_0^1(\Omega)\cap L^{p_0}(\Omega)$ and $(H_0^1(\Omega)\cap L^{p_0}(\Omega))^*$.

An admissible element for the optimal control problem is a pair (u,f) that satisfies the initial boundary value problem (3.1.8). The precise definition is given as follow.

Definition 3.1.2 *Let $w \in L^2(\Omega)$ be given. A pair (u,f) is said to be an admissible element if $u \in L^2((0,T);H_0^1(\Omega)) \cap L^{p_0}((0,T);L^{p_0}(\Omega))$, $f \in L^2((0,T);L^2(\Omega))$, and (u,f) satisfies equation (3.1.8). The set of all admissible elements is denoted by $\mathcal{V}_{ad}((0,T),w)$ or simply $\mathcal{V}_{ad}$.*

The optimal control problems we study can be concisely stated as

$$\text{(OP)} \qquad \begin{array}{l} \text{seek a pair } (\hat{u}, \hat{f}) \in \mathcal{V}_{ad} \text{ such that } \mathcal{J}(\hat{u}, \hat{f}) = \inf_{(u,f)\in\mathcal{V}_{ad}} \mathcal{J}(u, f) \\ \\ \text{where the functional } \mathcal{J} \text{ is defined by (3.1.6).} \end{array}$$

3.2 Existence of solution for optimal control problems

Now we shall show the existence of optimal solutions for (OP). Denote

$$Q = (0, T) \times \Omega.$$

Lemma 3.2.1 *For any $\delta > 0$ and $p > 2$, there exists $C = C(\delta) > 0$ such that*

$$|u|^2 \leq \delta |u|^p + C$$

proof: For given $\delta > 0$ and sufficient large $R \in I\!R$, if $|u| \geq R$, then $\delta|u|^p - |u|^2 \geq 0$ since $p > 2$. And if $|u| \leq R$, then there exists $C(\delta) > 0$ such that $\delta|u|^p - |u|^2 \geq -C(\delta)$. So the lemma is proved.

Theorem 3.2.2 *Assume that $w \in L^2(\Omega)$ and $W \in L^{p_0}(\Omega)$. Then there exists an optimal solution for* (OP).

proof: By the existence of solution for a parabolic equation with a monotone principle part [2, p.38, Theorem 3.1], the admissible set $\mathcal{V}_{ad}$ is non-empty. Thus we may choose a minimizing sequence $\{(u_m, f_m)\}$ in $\mathcal{V}_{ad}$ such that

$$\lim_{m\to\infty} \mathcal{J}(u_m, f_m) = \inf_{(u,f)\in\mathcal{V}_{ad}} \mathcal{J}(u, f)$$

and

$$\begin{cases} \langle (u_m)_t, \phi\rangle + \int_\Omega [A(\mathbf{x})\nabla u_m(t)] \cdot \nabla\phi \, d\mathbf{x} + \int_\Omega \Phi(u_m)\,\phi\, d\mathbf{x} + \int_\Omega a(\mathbf{x}) u_m\, \phi\, d\mathbf{x} \\ \qquad = \int_\Omega f_m(t)\,\phi\, d\mathbf{x} \qquad \forall\, \phi \in H_0^1(\Omega) \cap L^{p_0}(\Omega), \text{ a.e. } t \in (0, T) \\ u_m(0) = w \quad \text{in } \Omega \end{cases} \tag{3.2.1}$$

Thus we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_m^2(t)\,d\mathbf{x}+\int_\Omega A(\mathbf{x})\nabla u_m\cdot\nabla u_m\,d\mathbf{x}+\int_\Omega \Phi(u_m)u_m\,d\mathbf{x} \\ +\int_\Omega a(\mathbf{x})u_m^2\,d\mathbf{x}=\int_\Omega f_m u_m\,d\mathbf{x}. \tag{3.2.2}$$

By (1.10), (1.11) and integrating with respect to s over $[0,t]$ we obtain

$$\frac{1}{2}\int_\Omega u_m^2(t)\,d\mathbf{x}+\int_0^t\int_\Omega A(\mathbf{x})\nabla u_m\cdot\nabla u_m\,d\mathbf{x}ds+\mu_0\int_0^t\int_\Omega |u_m|^{p_0}\,d\mathbf{x}ds \\ -(C_1+\frac{1}{2})\int_0^t\int_\Omega u_m^2\,d\mathbf{x}ds\le \frac{1}{2}\int_0^t\int_\Omega f_m^2\,d\mathbf{x}ds+\frac{1}{2}\int_\Omega w^2\,d\mathbf{x}. \tag{3.2.3}$$

By lemma 3.2.1, we get

$$\frac{1}{2}\int_\Omega u_m^2(t)\,d\mathbf{x}+\int_0^t\int_\Omega A(\mathbf{x})\nabla u_m\cdot\nabla u_m\,d\mathbf{x}ds+\mu_0\int_0^t\int_\Omega |u_m|^{p_0}\,d\mathbf{x}ds \\ -(C_1+\frac{1}{2})\int_0^t\int_\Omega(\delta|u_m|^{p_0}+C(\delta))\,d\mathbf{x}ds \\ \le\frac{1}{2}\|f_m\|^2_{L^2(Q)}+\frac{1}{2}\|w\|^2_{L^2(\Omega)}; \tag{3.2.4}$$

i.e.

$$\frac{1}{2}\int_\Omega u_m^2(t)\,d\mathbf{x}+\int_0^t\int_\Omega A(\mathbf{x})\nabla u_m\cdot\nabla u_m\,d\mathbf{x}ds+\Big(\mu_0-\delta(C_1+\frac{1}{2})\Big)\int_0^t\int_\Omega |u_m|^{p_0}\,d\mathbf{x}ds \\ \le C(\delta)T(C_1+\frac{1}{2})meas(\Omega)+\frac{1}{2}\|f_m\|^2_{L^2(Q)}+\frac{1}{2}\|w\|^2_{L^2(\Omega)} \tag{3.2.5}$$

where $meas(\Omega)$ is a measure of Ω. Since we may choose δ such that $\mu_0-\delta(C_1+\frac{1}{2})\ge 0$ and $\{(u_m,f_m)\}$ is a minimizing sequence, $\{u_m\}$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))\cap L^{p_0}(Q)$. Now we can extract a subsequence, still denoted by $\{(u_m,f_m)\}$, such that

$$u_m\rightharpoonup\hat{u}\quad\text{weakly in } L^2(0,T;H_0^1(\Omega))\cap L^{p_0}(Q)$$

$$u_m\rightharpoonup\hat{u}\quad\text{weakly-* in } L^\infty(0,T;L^2(\Omega))$$

$$f_m\rightharpoonup\hat{f}\quad\text{weakly in } L^2(Q)\,.$$

Let $v \in B$ defined by (3.1.7) be an arbitrary function. Consider

$$\begin{aligned}
\int_0^T \int_\Omega (u_m)_t \, v \, d\mathbf{x}dt &= -\int_0^T \int_\Omega A(\mathbf{x})\nabla u_m \cdot \nabla v \, d\mathbf{x}dt - \int_0^T \int_\Omega \Phi(u_m) v \, d\mathbf{x}dt \\
&\quad - \int_0^T \int_\Omega a(\mathbf{x}) u_m v \, d\mathbf{x}dt + \int_0^T \int_\Omega f_m v \, d\mathbf{x}dt \\
&\leq \|A(\mathbf{x})\nabla u_m\|_{L^2(Q)} \|\nabla v\|_{L^2(Q)} + C_2 \|u_m\|^{p_0-1}_{L^{p_0}(Q)} \|v\|_{L^{p_0}(Q)} \\
&\quad + \|a(\mathbf{x})\|_{L^\infty(\Omega)} \|u_m\|_{L^2(Q)} \|v\|_{L^2(Q)} + \|f_m\|_{L^2(Q)} \|v\|_{L^2(Q)} \\
&\leq C_3 \Big(\|A(\mathbf{x})\nabla u_m\|_{L^2(Q)} + \|u_m\|^{p_0-1}_{L^{p_0}(Q)} + \|a(\mathbf{x})\|_{L^\infty(\Omega)} \|u_m\|_{L^2(Q)} + \|f_m\|_{L^2(Q)} \Big) \|v\|_B.
\end{aligned}$$

Therefore, $\{(u_m)_t\}$ is bounded in B^*. Since B^* is reflexive we have

$$(u_m)_t \rightharpoonup \widetilde{u} \quad \text{weakly in } B^*,$$

and, as can easily be seen, $\widetilde{u} = \widehat{u}_t$. Now, we will show $u_m \longrightarrow \widehat{u}$ strongly in $L^2(Q)$. Let's consider $H_0^s(\Omega)$ where $s \geq 1$ is sufficiently large. We take s so large that $H_0^s(\Omega) \subset L^{p_0}(\Omega)$. Then we have $L^{p_0}(0,T; H_0^s(\Omega)) \subset B$, i.e., $B^* \subset L^{\frac{p_0}{p_0-1}}(0,T; H^{-s}(\Omega))$. Let

$$E = \{u \in L^2(0,T; H_0^1(\Omega)),\ u' \in L^{\frac{p_0}{p_0-1}}(0,T; H^{-s}(\Omega))\}.$$

We know that the injection $H_0^1(\Omega) \to L^2(\Omega)$ is compact by the Sobolev Embedding Theorem and

$$H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-s}(\Omega).$$

By a compactness theorem in Banach spaces [25, p.271], the injection of E into $L^2(Q)$ is compact. So $u_m \longrightarrow \widehat{u}$ strongly in $L^2(Q)$ which in turn implies $u_m \longrightarrow \widehat{u}$ a.e. in Q.

Let $\varphi \in C(0,T; H_0^1(\Omega) \cap L^{p_0}(\Omega))$ and $\epsilon > 0$ be given. Since $\{u_m\}$ is uniformly bounded in $L^{p_0}(Q)$, there is a constant $\tilde{C}_1 > 0$ such that

$$\|u_m\|^{p_0-1}_{L^{p_0}(Q)} + \|\widehat{u}\|^{p_0-1}_{L^{p_0}(Q)} \leq \tilde{C}_1. \tag{3.2.6}$$

Moreover, there exist constants $C_1(p_0)$, $C_2(p_0) > 0$ such that

$$\begin{aligned}\Big(\int_Q |\Phi(u)|^{\frac{p_0}{p_0-1}}\,d\mathbf{x}dt\Big)^{\frac{p_0-1}{p_0}} &\le \Big(\int_Q (M_1|u|^{p_0-1}+M_2)^{\frac{p_0}{p_0-1}}\,d\mathbf{x}dt\Big)^{\frac{p_0-1}{p_0}}\\ &\le \Big(\int_Q C(p_0)(M_1^{\frac{p_0}{p_0-1}}|u|^{p_0}+M_2^{\frac{p_0}{p_0-1}})\,d\mathbf{x}dt\Big)^{\frac{p_0-1}{p_0}} \qquad (3.2.7)\\ &\le C_1(p_0)M_1\|u\|^{p_0-1}_{L^{p_0}(Q)}+C_2(p_0)M_2\end{aligned}$$

where the constants M_1 and M_2 are defined by (1.11). Note that $\int_Q |\varphi|^{p_0}\,d\mathbf{x}dt < \infty$ so that by the absolute continuity of integrals, there exists a $\mu > 0$ such that if S is a measurable subset of Q and $meas(S) < \mu$ then

$$\Big(\int_S |\varphi|^{p_0}\,d\mathbf{x}dt\Big)^{\frac{1}{p_0}} < \frac{\epsilon}{2(C_1(p_0)M_1\tilde{C}_1+C_2(p_0)M_2)} \qquad (3.2.8)$$

where the constants $\tilde{C}_1$, $C_1(p_0)$ and $C_2(p_0)$ are defined by (3.2.6) and (3.2.7). Now by the Egoroff's Theorem we know that there exists a measurable subset $F \subset Q$ such that $meas(Q \setminus F) < \min(\epsilon, \mu)$ and $u_m \longrightarrow \hat{u}$ uniformly in F. Consider

$$\begin{aligned}&\Big|\int_Q (\Phi(u_m)-\Phi(\hat{u}))\,\varphi\,d\mathbf{x}dt\Big|\\ &\le \int_F |(\Phi(u_m)-\Phi(\hat{u}))\,\varphi|\,d\mathbf{x}dt + \int_{Q\setminus F} |(\Phi(u_m)-\Phi(\hat{u}))\,\varphi|\,d\mathbf{x}dt.\end{aligned}$$

Since $u_m \longrightarrow \hat{u}$ uniformly in F, there is a $M > 0$ such that for $m \ge M$,

$$\int_F |(\Phi(u_m)-\Phi(\hat{u}))\,\varphi|\,d\mathbf{x}dt < \frac{\epsilon}{2}.$$

On the other hand, by (3.2.6), (3.2.7) and (3.2.8), we obtain

$$\begin{aligned}\int_{Q\setminus F} |(\Phi(u_m)-\Phi(\hat{u}))\,\varphi|\,d\mathbf{x}dt &\le \int_{Q\setminus F}|\Phi(u_m)\,\varphi|\,d\mathbf{x}dt + \int_{Q\setminus F}|\Phi(\hat{u})\,\varphi|\,d\mathbf{x}dt\\ &\le \Big(\int_{Q\setminus F}|\Phi(u_m)|^{\frac{p_0}{p_0-1}}\,d\mathbf{x}dt\Big)^{\frac{p_0-1}{p_0}}\Big(\int_{Q\setminus F}|\varphi|^{p_0}\,d\mathbf{x}dt\Big)^{\frac{1}{p_0}}\\ &\quad+\Big(\int_{Q\setminus F}|\Phi(u)|^{\frac{p_0}{p_0-1}}\,d\mathbf{x}dt\Big)^{\frac{p_0-1}{p_0}}\Big(\int_{Q\setminus F}|\varphi|^{p_0}\,d\mathbf{x}dt\Big)^{\frac{1}{p_0}}\\ &\le (C_1(p_0)M_1\tilde{C}_1+C_2(p_0)M_2)\Big(\int_{Q\setminus F}|\varphi|^{p_0}\,d\mathbf{x}dt\Big)^{\frac{1}{p_0}}\\ &\le \frac{\epsilon}{2}.\end{aligned}$$

Combining the last two relations we have

$$\int_Q \Phi(u_m)\,\varphi\,d\mathbf{x}dt \longrightarrow \int_Q \Phi(\widehat{u})\,\varphi\,d\mathbf{x}dt \qquad \text{as } m \longrightarrow \infty.$$

Therefore $(\widehat{u}, \widehat{f})$ satisfies the equations (3.1.8), i.e., $(\widehat{u}, \widehat{f}) \in \mathcal{V}_{ad}$. Then the fact that the functional $\mathcal{J}(\cdot,\cdot)$ is lower semi-continuous implies that

$$\inf_{(u,f)\in\mathcal{V}_{ad}} \mathcal{J}(u,f) = \lim_{m\to\infty} \mathcal{J}(u_m, f_m) \geq \mathcal{J}(\widehat{u}, \widehat{f}).$$

Hence $(\widehat{u}, \widehat{f})$ is an optimal solution for (OP).

3.3 Dynamics of optimal control solutions

Theorem 3.3.1 *Assume that* $w \in H_0^1(\Omega)$, $W \in L^{p_0}(\Omega)$, $W^{(\gamma)} \in H^2(\Omega)\cap H_0^1(\Omega)\cap L^{p_0}(\Omega)$, $V^{(\gamma)}$ *satisfies (3.1.2) and* F *is defined by (3.1.5). If, in addition, hypotheses (3.1.1), (3.1.3) and (3.1.4) hold. Let* $(\widehat{u}, \widehat{f})$ *be the solution of* (OP). *Then the optimal solution* $\widehat{u}$ *as a function of the parameter* γ *satisfies*

$$\lim_{\gamma\to 0} \|\widehat{u}(T) - W\|_0 = 0\,.$$

proof: Let (u, f) be an arbitrary pair satisfying the parabolic equation (1.7)-(1.9). Writing V in place of $V^{(\gamma)}$ and setting $v = u - V$ we obtain:

$$\begin{cases} v_t - \operatorname{div}[A(\mathbf{x})\nabla v] + \Phi(u) - \Phi(V) + a(\mathbf{x})v & \\ \quad = f - F & (t,\mathbf{x}) \in (0,T)\times\Omega \\ v = 0 & (t,\mathbf{x}) \in (0,T)\times\partial\Omega \\ v(0,\mathbf{x}) = w - V(0,\mathbf{x}) & \mathbf{x}\in\Omega \end{cases} \tag{3.3.1}$$

By the results in [2, p.38, Theorem 3.1] we guarantee the existence of $\tilde{u}$ satisfying

$$\begin{cases} \tilde{u}_t - \operatorname{div}[A(\mathbf{x})\nabla\tilde{u}] + \Phi(\tilde{u}) + (a(\mathbf{x}) + k_0)\tilde{u} = g & (t,\mathbf{x}) \in (0,T)\times\Omega \\ \tilde{u} = 0 & (t,\mathbf{x}) \in (0,T)\times\partial\Omega \\ \tilde{u}(0,\mathbf{x}) = w & \mathbf{x}\in\Omega \end{cases} \tag{3.3.2}$$

where $g = V_t - \operatorname{div}[A(\mathbf{x})\nabla V] + \Phi(V) + (a(\mathbf{x}) + k_0)V$ and $k_0 > 0$ is a constant to be determined.

Denote

$$\widetilde{v} = \widetilde{u} - V \quad \text{and} \quad \widetilde{f} = F - k_0\widetilde{v}. \tag{3.3.3}$$

Then the pair $(\tilde{v}, \tilde{f})$ satisfies (3.3.1), i.e., we have

$$\widetilde{v}_t - \operatorname{div}[A(\mathbf{x})\nabla\widetilde{v}] + \Phi(\widetilde{u}) - \Phi(V) + (a(\mathbf{x}) + k_0)\widetilde{v} = 0 \tag{3.3.4}$$

By the same way in the proof of Theorem 3.2.2, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{v}(t)\|_0^2 + \int_\Omega A(\mathbf{x})\nabla\widetilde{v}\cdot\nabla\widetilde{v}\,d\mathbf{x} + \int_\Omega (\Phi(\widetilde{u}) - \Phi(V))(\widetilde{u} - V)\,d\mathbf{x}$$
$$+ \int_\Omega (a(\mathbf{x}) + k_0)\widetilde{v}^2\,d\mathbf{x} = 0.$$

By our assumption (1.10) and (1.11) we have

$$\frac{d}{dt}\|\widetilde{v}(t)\|_0^2 + 2(k_0 - C_1)\|\widetilde{v}(t)\|_0^2 \le 0.$$

Multiplying by $e^{2(k_0-C_1)t}$ and integrating with respect to t

$$\|\widetilde{v}(t)\|_0^2 \le e^{-2(k_0-C_1)t}\|\widetilde{v}(0)\|_0^2 \quad \text{and} \quad \|\widetilde{v}(T)\|_0^2 \le e^{-2(k_0-C_1)T}\|\widetilde{v}(0)\|_0^2. \tag{3.3.5}$$

Now we need an estimate for $\int_0^T \|\tilde{f} - F\|_0^2\,dt$.

$$\begin{aligned}\int_0^T \|\widetilde{f} - F\|_0^2\,dt &= k_0^2\int_0^T \|\widetilde{v}\|_0^2\,dt \\ &\le \frac{k_0^2}{2(k_0 - C_1)}\Big(1 - e^{-2(k_0-C_1)T}\Big)\|\widetilde{v}(0)\|_0^2.\end{aligned} \tag{3.3.6}$$

Let us consider the dynamics of optimal solution.

$$\begin{aligned}\|\hat{u}(T) - W\|_0^2 &\le \frac{2}{T}\Big(\frac{T}{2}\|\hat{u}(T) - W\|_0^2 + \frac{\gamma}{2}\int_0^T \|\hat{f} - F\|_0^2\,dt\Big) \\ &\le \frac{2}{T}\Big(\frac{T}{2}\|\widetilde{u}(T) - W\|_0^2 + \frac{\gamma}{2}\int_0^T \|\widetilde{f} - F\|_0^2\,dt\Big) \\ &\le \frac{2}{T}\Big(T\|\widetilde{u}(T) - V(T)\|_0^2 + T\|V(T) - W\|_0^2 + \frac{\gamma}{2}\int_0^T \|\widetilde{f} - F\|_0^2\,dt\Big) \\ &= 2\|\widetilde{u}(T) - V(T)\|_0^2 + 2\|V(T) - W\|_0^2 + \frac{\gamma}{T}\int_0^T \|\widetilde{f} - F\|_0^2\,dt\end{aligned} \tag{3.3.7}$$

Combining (3.3.5), (3.3.6) and (3.3.7) we finally arrive at

$$\begin{aligned} \|\widehat{u}(T) - W\|_0^2 &\le 2e^{-2(k_0 - C_1)T}\|w - V(0)\|_0^2 + 2\|V(T) - W\|_0^2 \\ &+ \frac{\gamma k_0^2}{2T(k_0 - C_1)}[1 - e^{-2(k_0 - C_1)T}]\|w - V(0)\|_0^2. \end{aligned} \tag{3.3.8}$$

It remains to prove $\lim_{\gamma \to 0} \|\widehat{u}(T) - W\|_0 = 0$. Let $\epsilon > 0$ be given. There exists a k_0 such that

$$2e^{-2(k_0 - C_1)T}\|w - V(0)\|_0^2 < \frac{\epsilon^2}{3}.$$

Holding this k_0 fixed and by (3.1.3), we may choose a γ_0 such that

$$2\|V^{(\gamma_0)}(T) - W\|_0^2 < \frac{\epsilon^2}{3}$$

and

$$\frac{\gamma_0 k_0^2}{2T(k_0 - C_1)}[1 - e^{-2(k_0 - C_1)T}]\|w - V(0)\|_0^2 < \frac{\epsilon^2}{3}.$$

Thus we obtain that $\|\hat{u}(T) - W\|_0 < \epsilon$ for each $\gamma \in [0, \gamma_0]$.

Proposition 3.3.1 *Assume that* $w \in H_0^1(\Omega)$, $W \in L^{p_0}(\Omega)$, $W^{(\gamma)} \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^{p_0}(\Omega)$, $V^{(\gamma)}$ *satisfies (3.1.2) and* F *is defined by (3.1.5). If, in addition, hypotheses (3.1.1), (3.1.3) and (3.1.4) hold. Let* $(\widehat{u}, \widehat{f})$ *be the solution of* (OP). *Then there exist positive constants* K_1 *and* K_2 *depending on* γ, T *and* C_1 *defined by (1.10) such that*

$$\|\widehat{u}(t) - V^{(\gamma)}(t)\|_0^2 \le K_1\|w - V^{(\gamma)}(0)\|_0^2 + K_2\|W - V^{(\gamma)}(T)\|_0^2 \quad \forall\, t \in [0, T].$$

proof: Writing V in place of $V^{(\gamma)}$ and setting $\widehat{v} = \widehat{u} - V$ we obtain:

$$\begin{cases} \widehat{v}_t - \operatorname{div}[A(\mathbf{x})\nabla\widehat{v}] + \Phi(\widehat{u}) - \Phi(V) + a(\mathbf{x})\widehat{v} & \\ \qquad = \widehat{f} - F & (t, \mathbf{x}) \in (0, T) \times \Omega \\ \widehat{v} = 0 & (t, \mathbf{x}) \in (0, T) \times \partial\Omega \\ \widehat{v}(0, \mathbf{x}) = w - V(0, \mathbf{x}) & \mathbf{x} \in \Omega \end{cases} \tag{3.3.9}$$

By the same way in the proof of Theorem 3.2.2, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\widehat{v}(t)\|_0^2+\int_\Omega A(\mathbf{x})\nabla\widehat{v}\cdot\nabla\widehat{v}\,d\mathbf{x}+\int_\Omega(\Phi(\widehat{u})-\Phi(V))(\widehat{u}-V)\,d\mathbf{x}+\int_\Omega a(\mathbf{x})\widehat{v}^2\,d\mathbf{x}$$
$$\leq\frac{1}{2}\|\widehat{f}-F\|_0^2+\frac{1}{2}\|\widehat{v}\|_0^2.$$

By our assumption (1.10) and (1.11) we have

$$\frac{d}{dt}\|\widehat{v}(t)\|_0^2\leq(1+2C_1)\|\widehat{v}\|_0^2+\|\widehat{f}-F\|_0^2.$$

By Gronwall's inequality,

$$\|\widehat{v}(t)\|_0^2\ \leq e^{(1+2C_1)t}\Big(\|\widehat{v}(0)\|_0^2+\int_0^t\|\widehat{f}-F\|_0^2\,dt\Big).\tag{3.3.10}$$

Now we need the estimate of last term in (3.3.10). Let $\widetilde{f}=F-\widetilde{v}$ where $\widetilde{v}$ is defined in the proof of Theorem 3.4.1. Using (3.3.5) we have

$$\begin{aligned}
\int_0^T\|\widehat{f}-F\|_0^2\,dt &\leq\frac{2}{\gamma}\Big(\frac{T}{2}\|\widehat{u}(T)-W\|_0^2+\frac{\gamma}{2}\int_0^T\|\widehat{f}-F\|_0^2\,dt\Big)\\
&\leq\frac{2}{\gamma}\Big(\frac{T}{2}\|\widetilde{u}(T)-W\|_0^2+\frac{\gamma}{2}\int_0^T\|\widetilde{f}-F\|_0^2\,dt\Big)\\
&\leq\frac{2T}{\gamma}\|\widetilde{u}(T)-V(T)\|_0^2+\frac{2T}{\gamma}\|V(T)-W\|_0^2+\int_0^T\|\widetilde{v}\|_0^2\,dt\\
&\leq\frac{2T}{\gamma}e^{-2(1-C_1)T}\|w-V(0)\|_0^2+\frac{2T}{\gamma}\|V(T)-W\|_0^2+\int_0^T\|\widetilde{v}\|_0^2\,dt\\
&\leq\frac{2T}{\gamma}e^{-2(1-C_1)T}\|w-V(0)\|_0^2+\frac{2T}{\gamma}\|V(T)-W\|_0^2\\
&\quad+\frac{1-e^{-2(1-C_1)T}}{2(1-C_1)}\|w-V(0)\|_0^2.
\end{aligned}\tag{3.3.11}$$

Combining (3.3.10) and (3.3.11) this proposition is proved.

3.4 The semidiscrete (spatially discrete) approximations

In order to compute the optimal solutions, we need to discretize this problem in both time and space. In this section, we will discretize the spatial variables by finite element methods. We choose a finite dimensional subspace $X_h\subset H_0^1(\Omega)$. This subspace is parameterized by the parameter h that tends to zero; commonly, this parameter is chosen to be some measure of the grid size in a subdivision of Ω into finite elements. One may choose subspace X_h that can be used for finding finite element solutions of parabolic equations. Thus, concerning these subspace, we make the following standard assumptions. First we have the approximation property: there exist an integer $k \geq 1$ and a constant $C' > 0$, independent of h and w such that

$$\begin{aligned} &\inf_{w_h \in X_h} \{\|w - w_h\|_0 + h\|\nabla(w - w_h)\|_0\} \\ &\leq C' h^m \|w\|_m \quad \forall w \in H^m(\Omega) \cap H_0^1(\Omega), \quad 1 \leq m \leq k. \end{aligned} \tag{3.4.1}$$

Now writing V in place of $V^{(\gamma)}$ we introduce an auxiliary element $V_h \in X_h$ determined by

$$\int_\Omega \nabla V_h \cdot \nabla \phi_h \, dx = \int_\Omega \nabla V \cdot \nabla \phi_h \, dx \quad \forall \phi_h \in X_h. \tag{3.4.2}$$

The existence of such a V_h follows form the well-known results of the finite element methods for parabolic equation [2, Thomee]. Furthermore, under the assumption that there is a $k \geq 1$ such that

$$V \in L^\infty(0, T; H^k(\Omega)) \cap C([0, T]; H^k(\Omega)), \tag{3.4.3}$$

the following error estimates hold;

$$\begin{aligned} &\|V_h(t) - V(t)\|_1 \leq \overline{C}_1 h^{k-1} \|V\|_k \leq \overline{C}_1 h^{k-1} \|V\|_{L^\infty(0,T;H^k(\Omega))} \\ &\|V_h(t) - V(t)\|_0 \leq \overline{C}_2 h^k \|V\|_k \leq \overline{C}_2 h^k \|V\|_{L^\infty(0,T;H^k(\Omega))}, \end{aligned} \tag{3.4.4}$$

where $\overline{C}_1$ and $\overline{C}_2$ are constants depending on Ω only. By differentiating (3.4.2) with respect t, we see that $\partial_t V_h(t)$ satisfies an equation similar to (3.4.2) so that under the assumption

$$\partial_t V \in L^\infty(0, T; H^k(\Omega)) \cap C([0, T]; H^k(\Omega)), \tag{3.4.5}$$

We have the error estimates

$$\begin{aligned}\|\partial_t V_h(t) - \partial_t V(t)\|_1 &\le \overline{C}_1 h^{k-1}\|\partial_t V\|_k \le \overline{C}_1 h^{k-1}\|\partial_t V\|_{L^\infty(0,T;H^k(\Omega))}\\ \|\partial_t V_h(t) - \partial_t V(t)\|_0 &\le \overline{C}_2 h^{k}\|\partial_t V\|_k \le \overline{C}_2 h^{k}\|\partial_t V\|_{L^\infty(0,T;H^k(\Omega))},\end{aligned} \tag{3.4.6}$$

By differentiating (3.4.2) twice with respect t, we see that $\partial_t V_h(t)$ also satisfies an equation similar to (3.4.2) so that under the assumption

$$\partial_{tt} V \in L^\infty(0,T;H^1(\Omega)) \cap C([0,T];H^1(\Omega)). \tag{3.4.7}$$

We have the error estimates

$$\begin{aligned}&\|\partial_{tt} V_h(t) - \partial_{tt} V(t)\|_1 \le \overline{C}_1\|\partial_{tt} V\|_1 \le \overline{C}_1\|\partial_{tt} V\|_{L^\infty(0,T;H^1(\Omega))}\\ &\|\partial_{tt} V_h(t) - \partial_{tt} V(t)\|_0 \le \overline{C}_2 h^s\|\partial_{tt} V\|_s \le \overline{C}_2 h^s\|\partial_{tt} V\|_{L^\infty(0,T;H^s(\Omega))} \quad \forall s \in [0,1];\end{aligned} \tag{3.4.8}$$

in particular,

$$\|\partial_{tt} V_h(t) - \partial_{tt} V(t)\|_0 \le \overline{C}_2\|\partial_{tt} V\|_0 \le \overline{C}_2\|\partial_{tt} V\|_{L^\infty(0,T;H^1(\Omega))}. \tag{3.4.9}$$

We can choose F_h as

$$\begin{aligned}\int_\Omega F_h(t)\,\phi_h\,d\mathbf{x} &= \langle \partial_t V_h(t), \phi_h\rangle + \int_\Omega [A(\mathbf{x})\nabla V_h(t)]\cdot\nabla\phi_h\,d\mathbf{x} + \int_\Omega \Phi(V_h)\,\phi_h\,d\mathbf{x}\\ &+ \int_\Omega a(\mathbf{x}) V_h\,\phi_h\,d\mathbf{x} \qquad \forall\,\phi_h \in X_h,\ \text{a.e. } t \in (0,T).\end{aligned} \tag{3.4.10}$$

We now define the space $Y_h = \{f_h = y_h + F_h : y_h \in X_h\}$ for the approximate distributed controls. Denote

$$\mathcal{J}_h(u_h, f_h) = \frac{T}{2}\|u_h(T) - W\|_0^2 + \frac{\gamma}{2}\int_0^T \|f_h(t) - F_h(t)\|_0^2\,dt\,. \tag{3.4.11}$$

Once the finite element space X_h has been chosen, we define the semidiscrete(spatially discrete) approximation of the optimal control problem as follows.

seek a pair $(\widehat{u}_h, \widehat{f}_h) \in X_h \times Y_h$ such that $\mathcal{J}_h(\widehat{u}_h, \widehat{f}_h)$ is minimized

subject to the semidiscrete parabolic equations;

$$\text{(SE – OP)} \qquad \langle \partial_t u_h(t), \phi_h \rangle + \int_\Omega [A(\mathbf{x})\nabla u_h(t)] \cdot \nabla \phi_h \, d\mathbf{x} + \int_\Omega \Phi(u_h)\, \phi_h \, d\mathbf{x}$$
$$+ \int_\Omega a(\mathbf{x}) u_h \, \phi_h \, d\mathbf{x} = \int_\Omega f_h(t)\, \phi_h \, d\mathbf{x} \qquad \forall\, \phi_h \in X_h, \text{ a.e. } t \in (0,T)$$

where the functional $\mathcal{J}_h$ is defined by (3.4.11).

Theorem 3.4.1 *Assume that* $w \in H_0^1(\Omega)$, $W \in L^{p_0}(\Omega)$, $W^{(\gamma)} \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^{p_0}(\Omega)$, $V^{(\gamma)}$ *satisfies (3.1.2),* $V_h^{(\gamma)}$ *is defined by (3.4.2) and* F_h *is defined by (3.4.10). If, in addition, hypotheses (3.1.1), (3.1.3), (3.1.4), (3.4.3),(3.4.5) and (3.4.7) hold. Let* $(\widehat{u}_h, \widehat{f}_h)$ *be the solution of* (SE-OP). *Then for all* $\epsilon > 0$, *there exist positive constants* γ_0 *such that if* $\gamma \le \gamma_0$ *then*

$$\|\widehat{u}_h(T) - W\|_0^2 \le \epsilon + 3\overline{C}_2 h^{2k} \|V^{(\gamma)}\|^2_{L^\infty(0,T;H^k(\Omega))}$$

where $\overline{C}_2$ *is defined by (3.4.4).*

proof: Let (u_h, f_h) be an arbitrary pair satisfying the following equation:

$$\begin{aligned} &\langle \partial_t u_h(t), \phi_h \rangle + \int_\Omega [A(\mathbf{x})\nabla u_h(t)] \cdot \nabla \phi_h \, d\mathbf{x} + \int_\Omega \Phi(u_h)\, \phi_h \, d\mathbf{x} \\ &+ \int_\Omega a(\mathbf{x}) u_h \, \phi_h \, d\mathbf{x} = \int_\Omega f_h(t)\, \phi_h \, d\mathbf{x} \qquad \forall\, \phi_h \in X_h, \text{ a.e. } t \in (0,T). \end{aligned} \tag{3.4.12}$$

Subtracting (3.4.12) from (3.4.10) and setting $v_h = u_h - V_h$ we obtain

$$\begin{aligned} &\langle \partial_t v_h(t), \phi_h \rangle + \int_\Omega [A(\mathbf{x})\nabla v_h(t)] \cdot \nabla \phi_h \, d\mathbf{x} + \int_\Omega (\Phi(u_h) - \Phi(V_h))\, \phi_h \, d\mathbf{x} \\ &+ \int_\Omega a(\mathbf{x}) v_h \, \phi_h \, d\mathbf{x} = \int_\Omega (f_h - F_h)\, \phi_h \, d\mathbf{x} \qquad \forall\, \phi_h \in X_h, \text{ a.e. } t \in (0,T). \end{aligned} \tag{3.4.13}$$

Let $f_h = F_h - k_0 v_h$ where $k_0 > 0$ is a constant to be determined and $\phi_h = v_h$ in (3.4.13) then we have

$$\frac{1}{2}\frac{d}{dt}\|v_h(t)\|_0^2 + \int_\Omega A(\mathbf{x})\nabla v_h \cdot \nabla v_h \, d\mathbf{x} + \int_\Omega (\Phi(u_h) - \Phi(V_h))(u_h - V_h)\, d\mathbf{x}$$
$$+ \int_\Omega (a(\mathbf{x}) + k_0) v_h^2 \, d\mathbf{x} = 0.$$

By our assumption (1.10) and (1.11) we have

$$\frac{d}{dt}\|v_h(t)\|_0^2 + 2(k_0 - C_1)\|v_h(t)\|_0^2 \le 0.$$

Multiplying by $e^{2(k_0-C_1)t}$ and integrating with respect to t

$$\|v_h(t)\|_0^2 \le e^{-2(k_0-C_1)t}\|v_h(0)\|_0^2 \quad \text{and} \quad \|v_h(T)\|_0^2 \le e^{-2(k_0-C_1)T}\|v_h(0)\|_0^2. \tag{3.4.14}$$

Now we need an estimate for $\int_0^T \|f_h - F_h\|_0^2\,dt$.

$$\begin{aligned}\int_0^T \|f_h - F_h\|_0^2\,dt &= k_0^2 \int_0^T \|v_h\|_0^2\,dt \\ &\le \frac{k_0^2}{2(k_0 - C_1)}\Big(1 - e^{-2(k_0-C_1)T}\Big)\|v_h(0)\|_0^2.\end{aligned} \tag{3.4.15}$$

Let us consider the dynamics of semidiscrete optimal solution.

$$\begin{aligned}\|\widehat{u}_h(T) - W\|_0^2 &\le \frac{2}{T}\Big(\frac{T}{2}\|\widehat{u}_h(T) - W\|_0^2 + \frac{\gamma}{2}\int_0^T \|\widehat{f}_h - F_h\|_0^2\,dt\Big) \\ &\le \frac{2}{T}\Big(\frac{T}{2}\|u_h(T) - W\|_0^2 + \frac{\gamma}{2}\int_0^T \|f_h - F_h\|_0^2\,dt\Big) \\ &\le 3\|u_h(T) - V_h(T)\|_0^2 + 3\|V_h(T) - V(T)\|_0^2 + 3\|V(T) - W\|_0^2 \\ &\quad + \frac{\gamma}{T}\int_0^T \|f_h - F_h\|_0^2\,dt.\end{aligned} \tag{3.4.16}$$

Combining (3.4.14), (3.4.15) and (3.4.16) we finally arrive at

$$\begin{aligned}\|\widehat{u}_h(T) - W\|_0^2 &\le 3e^{-2(k_0-C_1)T}\|w_h - V_h(0)\|_0^2 + 3\|V_h(T) - V(T)\|_0^2 \\ &\quad + 3\|V(T) - W\|_0^2 + \frac{\gamma k_0^2}{2T(k_0 - C_1)}[1 - e^{-2(k_0-C_1)T}]\|w_h - V_h(0)\|_0^2.\end{aligned} \tag{3.4.17}$$

Let $\epsilon > 0$ be given. There exists a k_0 such that

$$3e^{-2(k_0-C_1)T}\|w_h - V_h(0)\|_0^2 < \frac{\epsilon}{3}.$$

Holding this k_0 fixed and by (3.1.3), we may choose a γ_0 such that

$$3\|V^{(\gamma_0)}(T) - W\|_0^2 < \frac{\epsilon}{3}$$

and

$$\frac{\gamma_0 k_0^2}{2T(k_0 - C_1)}[1 - e^{-2(k_0-C_1)T}]\|w_h - V_h(0)\|_0^2 < \frac{\epsilon}{3}.$$

Thus the theorem is proved.

3.5 Two-dimensional numerical simulations

We now consider a gradient method to compute the optimal solution subject to following semilinear parabolic equations.

$$\begin{cases} u_t - \Delta u + u^3 - u = f & (t, \mathbf{x}) \in (0,T) \times \Omega, \\ u = 0 & (t, \mathbf{x}) \in (0,T) \times \partial\Omega, \\ u(0, \mathbf{x}) = w(\mathbf{x}) & \mathbf{x} \in \Omega \end{cases} \tag{3.5.1}$$

Let $\sigma_N = \{t_n\}_{n=0}^N$ be a partition of $[0,T]$ into N equal intervals, $\Delta t = T/N$, $t_0 = 0$ and $t_N = T$. Denote

$$\vec{u}_h = (u_h^1, u_h^2, \cdots, u_h^N) \quad \text{and} \quad \vec{f}_h = (f_h^1, f_h^2, \cdots, f_h^N).$$

The fully discretized functional is given by

$$\mathcal{J}_h^N(\vec{u}_h, \vec{f}_h) = \frac{T}{2}\|u_h^N - W\|_0^2 + \frac{\gamma}{2}\Delta t \sum_{n=1}^N \|f_h(t) - F_h(t)\|_0^2 . \tag{3.5.2}$$

Due to the forward-in-time nature of the state equations and the backward-in-time nature of the adjoint equations, any practical algorithm would involve a split of the optimality system into two parts. Thus fully discrete optimality system consists of

- *the state equation*

$$\begin{cases} \dfrac{1}{\Delta t}\displaystyle\int_\Omega (u_h^n - u_h^{n-1})\,\phi_h\,d\mathbf{x} + \int_\Omega \nabla u_h^n \cdot \nabla\phi_h\,d\mathbf{x} + \int_\Omega (u_h^n)^3\,\phi_h\,d\mathbf{x} \\ \qquad - \displaystyle\int_\Omega u_h^n\,\phi_h\,d\mathbf{x} = \int_\Omega f_h^n\,\phi_h\,d\mathbf{x} \qquad \forall\,\phi_h \in X_h \\ \text{with initial condition: } u_h^0 = w_h(\mathbf{x}) \end{cases} \tag{3.5.3}$$

- *the adjoint equation*

$$\begin{cases} -\dfrac{1}{\Delta t}\displaystyle\int_\Omega (\xi_h^n - \xi_h^{n-1})\,\phi_h\,d\mathbf{x} + \int_\Omega \nabla \xi_h^{n-1} \cdot \nabla\phi_h\,d\mathbf{x} + \int_\Omega 3(u_h^n)^2\,\xi_h^{n-1}\,\phi_h\,d\mathbf{x} \\ \qquad - \displaystyle\int_\Omega \xi_h^{n-1}\,\phi_h\,d\mathbf{x} = 0 \qquad \forall\,\phi_h \in X_h \\ \text{with terminal condition: } \xi_h^N = T(u_h^N - W) \end{cases} \tag{3.5.4}$$

The optimal control variable f_h^n is related to the adjoint state ξ_h^n by

$$f_h^n - F_h^n = \frac{1}{\gamma}\xi_h^{n-1}.$$

Let $\mathcal{J}_h^N(k) = \mathcal{J}_h^N(\vec{u}_h(k), \vec{f}_h(k))$, where $\mathcal{J}_h^N(\cdot,\cdot)$ is given by (3.5.2) and k is the iteration counter of the gradient algorithm. In the algorithm, τ will denote a prescribed tolerance used to test for the convergence of the functional. The *gradient algorithm* proceeds as follows.

- initialization:

  (i) choose τ and $\vec{f}_h(0)$; set $k = 0$ and $\epsilon = 1$;

  (ii) solve for the starting state $\vec{u}_h(0)$ from (3.5.3) with $\vec{f}_h = \vec{f}_h(0)$;

  (iii) evaluate $\mathcal{J}_h^N(0)$;

- main loop:

  (iv) set $k = k + 1$;

  (v) solve for $\vec{\xi}_h(k)$ from (3.5.4) with $\vec{u}_h = \vec{u}_h(k-1)$;

  (vi) set $\vec{f}_h(k) = \vec{f}_h(k-1) - \epsilon\,(\gamma(\vec{f}_h(k-1) - \vec{F}_h) + \vec{\xi}_h(k))$;

  (vii) solve for $\vec{u}_h(k)$ from (3.5.3) with $\vec{f}_h = \vec{f}_h(k)$;

  (viii) evaluate $\mathcal{J}_h^N(k)$;

  (ix) if $\mathcal{J}_h^N(k) \geq \mathcal{J}_h^N(k-1)$, set $\epsilon = 0.5\epsilon$ and go to (vi); otherwise, continue;

  (x) if $|\mathcal{J}_h^N(k) - \mathcal{J}_h^N(k-1)|/|\mathcal{J}_h^N(k)| > \tau$, set $\epsilon = min(1.5\epsilon, 1)$ and go to (iv); otherwise, stop.

The bulk of the computational costs are found in the backward-in-time solution of the adjoint equation in step (v) and the forward-in-time solution of the state equation in step (vii). We use time lag to linearize the semilinear parabolic equation, i.e., we use $\int_\Omega (u_h^{n-1})^2\, u_h^n\, \phi_h\, d\mathbf{x}$ instead of $\int_\Omega (u_h^n)^3\, \phi_h\, d\mathbf{x}$ in (3.5.3).

Here are some detailed data of the example. We choose the domain $\Omega = (0,1)\times(0,1)$ (i.e., the unit square). We consider two sets of data (the terminal time T, the initial condition w and the target function W):

DATA I.
$$\begin{cases} T = 1, \\ w = 0, \\ W = \sin(\pi x)\sin(\pi y). \end{cases}$$

DATA II.
$$\begin{cases} T = 1, \\ w = 0, \\ W = 1 = \displaystyle\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{nm}\sin(n\pi x)\sin(m\pi y) \\ \quad \text{where } \ A_{nm} = \dfrac{4[(-1)^n - 1][(-1)^m - 1]}{n\, m\, \pi^2}. \end{cases}$$

For each data set we consider two choices of $V^{(\gamma)}$. In the case of DATA I, first we choose

$$W^{(\gamma)} = W = \sin(\pi x)\sin(\pi y)$$

and

$$V^{(\gamma)} = t\, W.$$

Second we choose $V^{(\gamma)}$ as a steady-state function, i.e.,

$$V^{(\gamma)} = W.$$

In the case of DATA II, first we choose

$$W^{(\gamma)} = \sum_{m=1}^{M_\gamma}\sum_{n=1}^{N_\gamma} A_{nm}\sin(n\pi x)\sin(m\pi y)$$

where $M_\gamma, N_\gamma \to \infty$ as $\gamma \to 0$ (e.g., M_γ and N_γ are the integer part of the decimal number $[1000 + \ln(1/\gamma)]$.) and

$$V^{(\gamma)} = t\, W^{(\gamma)}.$$

Second we choose $V^{(\gamma)}$ as a steady-state function, i.e.,

$$V^{(\gamma)} = W^{(\gamma)}.$$

In all cases, it can be verified that $W^{(\gamma)}$ and $V^{(\gamma)}$ satisfy assumptions (3.1.1), (3.1.2), (3.1.3) and (3.1.4). For DATA I the admissible state and the target state have matching boundary conditions (both have homogeneous boundary conditions.) For DATA II the admissible state and the target function have nonmatching boundary conditions. For both data sets the optimal solutions did a good job of tracking the target functions in the interior at the terminal time T (Tables 3.1 and 3.2). The optimal solutions of second cases ($V^{(\gamma)}$ are steady-state functions) furnish good matchings to the target state even for $t = T/2$, $T/5$ and $T/10$ (Tables 3.3 and 3.4).

Table 3.1 $W = \sin(\pi x)\sin(\pi y)$ and $V^{(\gamma)} = t\,W^{(\gamma)}$.

	$\gamma = 100$	$\gamma = 10$	$\gamma = 1$	$\gamma = 0.1$	$\gamma = 0.01$
$\|u(T) - W\|_0 \cdot 10^3$	2.416931	2.411342	2.355052	1.754410	1.107353

Table 3.2 $W = 1$ and $V^{(\gamma)} = t\,W^{(\gamma)}$.

	$\gamma = 100$	$\gamma = 10$	$\gamma = 1$	$\gamma = 0.1$	$\gamma = 0.01$
$\|u(T) - W\|_0 \cdot 10^2$	9.234190	9.222508	9.170293	8.347707	4.141871

Table 3.3 $W = \sin(\pi x)\sin(\pi y)$ and $V^{(\gamma)}$ is a steady-state function.

	$\gamma = 10$	$\gamma = 1$	$\gamma = 0.1$	$\gamma = 0.01$
$\|u(T) - W\|_0 \cdot 10^5$	3.852836	3.821628	3.378629	2.933585
$\|u(T/2) - W\|_0 \cdot 10^3$	1.474637	1.474633	1.474747	1.974137
$\|u(T/5) - W\|_0 \cdot 10^2$	5.134306	5.134306	5.134319	5.179989
$\|u(T/10) - W\|_0 \cdot 10^1$	1.641943	1.641943	1.641944	1.645492

Table 3.4 $W = 1$ and $V^{(\gamma)}$ is a steady-state function.

	$\gamma = 10$	$\gamma = 1$	$\gamma = 0.1$	$\gamma = 0.01$
$\|u(T) - W\|_0 \cdot 10^2$	9.555906	9.489303	8.548089	4.184522
$\|u(T/2) - W\|_0 \cdot 10^2$	9.672858	9.672652	9.669354	12.37950
$\|u(T/5) - W\|_0 \cdot 10^1$	1.506499	1.506500	1.506514	1.842022
$\|u(T/10) - W\|_0 \cdot 10^1$	3.093821	3.093814	3.093856	3.424358

4 CONCLUSION

In this thesis, we studied terminal-state tracking optimal control problems for both linear and semilinear parabolic equations. We also explored the connection between optimal control problems and exact/approximate controllability problems. New achievements of this thesis in the linear case include:

- We constructed an reference function F that resulted in an optimal solution that approached the target state W more effectively.
- We derived and justified an explicit solution formulae for optimal control problems and exact controllability problems.
- We allowed the target state W and admissible state u to have nonmatching boundary conditions.

Our contributions in the semilinear case include:

- We proved that the optimal solutions as a function of the parameter γ provide a family of functions that solves the approximate controllability problem.
- We demonstrated similar approximate controllability results in the semidiscrete case (finite element approximations).
- We implemented a gradient algorithm for solving the optimal control problems and numerically investigated the controllability properties of the optimal solutions.

In the future, we will explore parallel algorithms for computing optimal solutions in two dimensions (with semilinear parobolic constraints). As we mentioned before, the bulk of the computational costs are found in the forward-in-time solution of the state equation and the backward-in-time solution of the adjoint equation in the gradient algorithm. It is possible that a parallel algorithm can be used to solve those equations. We will also study the approximate controllability problem for the semilinear parabolic equations wherein the control acts on an open and nonempty subset of Ω. We will also explore the case of boundary controls.

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