Optimization-based domain decomposition methods for multidisciplinary simulation

by

Jeehyun Lee

A dissertation submitted to the graduate faculty

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Major: Applied Mathematics

Major Professor: Max D. Gunzburger

Iowa State University

Ames, Iowa

2001

Graduate College
Iowa State University

This is to certify that the Doctoral dissertation of

Jeehyun Lee

has met the dissertation requirements of Iowa State University

Committee Member

Committee Member

Committee Member

Committee Member

Committee Member

Major Professor

For the Major Program

For the Graduate College

TABLE OF CONTENTS

ACKNOWLEDGMENTS

I first would like to thank Dr. Max D. Gunzburger for the expert guidance, almost interminable patience that he has provided throughout this research and the writing of this thesis. I am also grateful to my committee members for their efforts and contributions to this work : Dr. Qiang Du, Dr. L. Steven Hou, Dr. Justin Peters, Dr. Janet S. Peterson and Dr. Monlay Tidriri. Special thanks to Dr. Xioaming Wang who was generous to share his knowledge and to answer any questions about partial differential equations. Not a day of my life passes by without thoughts and fond memories of my family. A big thank to my parents, parents-in-law and my sister, Jeena for their encouragement and for believing in me. I want to express my deepest thanks to my husband, Wook for his love, friendship and unfailing support throughout my career in graduate school and to my lovely boy, Daniel who is the joy of my life.

ABSTRACT

We consider a domain decomposition method for a fluid-structure interaction problem. The fluid-structure interaction problem involves two mathematical models, each posed on a different domain, so that domain decomposition occurs naturally. Our approach to a domain decomposition method is based on a strategy in which unknown data at the interface is determined through an optimization process. We prove that the solution of the optimization problem exists. And we show that the Lagrange multiplier rule may be used to transform the constrained optimization problem into an unconstrained one and that rule is applied to derive an optimality system from which optimal solutions may be obtained. We then study a gradient method for solving optimization problem.

1 INTRODUCTION

Multidisciplinary simulation problems arise in a variety of settings in which more than one media, or more than one mathematical model, or more than one dominant effect are present. The direct solution of such problems presents a formidable challenge, since they usually involve large, coupled system of partial differential equations. For this reason, methods which uncouple the different disciplines are of interest. Here, we discuss uncoupling procedures which are based on using an optimization strategy.

A main virtue of our approach is that it allows for the user to use existing codes for each discipline as black boxes and only requires that the user write a simple code that effects the coupling between the disciplines. One reason we are able to do this is that our methodology allows for complete flexibility with regards to the boundary conditions imposed on each discipline. Another virtue of the optimizaion-based decoupling is that it allows for the use of efficient iterative strategies, e.g., quickly converging iterative processes by which solutions of the coupled, multidisciplinary problem are determined. Our methodology has other important virtues as well as allowing for the use of mismatched grids and different discretization methods for each discipline.

Here, for the sake of concreteness, we will describe the optimization-based domain decomposition method for a fluid-structure interaction problem. The subjects of fluid-structure interaction problem have been extensively studied in the past and continue to be the focus of much attention today. There are number of different types of mathematical models for fluid-structure interactions. We classify these models into three categories.

Elementary fluid. The fluid motion is governed by potential equations, e.g., Laplace equations or wave equations. In [34], a coupled system of a potential equation and a wave equation is considered. Elementary fluids coupled with rigid cavity or moving wall has been studied in [19] and with an elastic solid in [4].

Inviscid fluid. A few mathematical papers have appeared for fluid-structure interactions modeled using inviscid fluid models, e.g., the Euler equations. Interactions between linearized inviscid fluids and elastic solids have been analyzed in [2],[35]. An algorithm for an inviscid nonlinear fluid coupled with rigid walls was given in [3].

Viscous fluid. There is an extensive literature on linearized viscous fluids coupled with solids. Solids modeled using plate equations or shell equations are treated in [15],[16],[17],[32]. The Stokes equations coupled with a beam equation has been analyzed in [21]. In [11], [31], interactions between linearized viscous fluid and elastic solids are studied. See [7] interaction with rigid walls.

Also, there is a vast literature on fluid-structure interactions for which the fluid is modeled using nonlinear viscous fluid models, e.g., the Navier-Stokes equations. Rigid body motions of solids in a nonlinear viscous fluid have been studied in [8],[12], [22], [23]. In [18], a coupled problem of Navier-Stokes equations and a plate equation is studied. Some works have treated interactions between nonlinear viscous fluids and elastic solids. See, e.g., [8],[13],[14],[36],[37].

MODEL PROBLEM

In this paper, we consider elastic body motions in a fluid flow. Let Ω_f and Ω_s denote the regions occupied by the fluid and solid, respectively. Let Γ_0 denote the interface between the fluid and solid and let Γ_f and Γ_s denote the boundaries of the fluid and solid regions (other than the interface Γ_0). In the fluid region, we apply the Stokes System.

$$\begin{cases} \rho_f \mathbf{v}_t + \nabla p - \mu_f \Delta \mathbf{v} = \rho_f \mathbf{f} & \text{in } \Omega_f \\ \nabla \cdot \mathbf{v} = 0 & \text{in } \Omega_f \\ \mathbf{v} = 0 & \text{on } \Gamma_f \\ \mathbf{v}|_{t=0} = \mathbf{v}^0 & \text{in } \Omega_f \end{cases} \tag{1.1}$$

Here, ρ_f and μ_f denote the (constant) fluid density and viscosity, $\mathbf{v}$ the fluid velocity, p the fluid pressure, and $\mathbf{v}^0$ the initial velocity.

In the solid, we apply the equation of the linear elasticity.

$$\begin{cases} \rho_s \mathbf{u}_{tt} - \mu_s \Delta \mathbf{u} - (\lambda_s + \mu_s)\nabla(\nabla \cdot \mathbf{u}) = \rho_s \mathbf{b} & \text{in } \Omega_s \\ \mathbf{u} = 0 & \text{on } \Gamma_s \\ \mathbf{u}|_{t=0} = \mathbf{u}^0 & \text{in } \Omega_s \\ \mathbf{u}_t|_{t=0} = \mathbf{u}^1 & \text{in } \Omega_s \end{cases} \tag{1.2}$$

Here, μ_f and λ_f are the Lame constants and ρ_s the constant density of solid, $\mathbf{b}$ denotes a given loading force per unit mass, $\mathbf{u}$ the displacement of the solid, and $\mathbf{u}^0$ and $\mathbf{u}^1$ are given initial data.

Along the fixed interface Γ_0 between the fluid and solid, the velocity of the fluid and solid are equal, as are the stress vector in the fluid and solid. Thus, we have

$$\mathbf{u}_t = \mathbf{v} \quad \text{on } \Gamma_0 \tag{1.3}$$

and

$$\mu_s \nabla \mathbf{u} \cdot \mathbf{n} + (\lambda_s + \mu_s)(\nabla \cdot \mathbf{u})\mathbf{n} = p\mathbf{n} - \mu_f \nabla \mathbf{v} \cdot \mathbf{n} \quad \text{on } \Gamma_0 \tag{1.4}$$

Solving (1.1)-(1.4) is a formidable challenge. Fluid-structure interaction problems involve two different mathematical models, each posed on a different domain, so that domain decomposition occurs naturally. Our approach to domain decomposition is based on a strategy in which unknown data at the interface is determined through an optimization process. We consider the following interface conditions.

$$\mathbf{v} = \mathbf{g} \quad \text{on } \Gamma_0 \tag{1.5}$$

and

$$\mu_s \nabla \mathbf{u} \cdot \mathbf{n} + (\lambda_s + \mu_s)(\nabla \cdot \mathbf{u})\mathbf{n} = \mathbf{h} \quad \text{on } \Gamma_0 \tag{1.6}$$

Then we may solve (1.1) and (1.5) for $\mathbf{v}$ and p and solve (1.2) and (1.6) for $\mathbf{u}$. For an arbitrary choice of $\mathbf{g}$ and $\mathbf{h}$, (1.1) and (1.2) are satisfied. However, (1.3) and (1.4) will not be satisfied. On the other hand, we know that $\mathbf{g}$ and $\mathbf{h}$ exist such that solutions of (1.1),(1.5) and (1.2),(1.6) are solutions of (1.1)-(1.4). We merely have to choose $\mathbf{g} = \hat{\mathbf{v}}|_{\Gamma_0} = \hat{\mathbf{u}}_t|_{\Gamma_0}$ and $\mathbf{h} = \mu_s \nabla \hat{\mathbf{u}} \cdot \mathbf{n} + (\lambda_s + \mu_s)(\nabla \cdot \hat{\mathbf{u}})\mathbf{n} = \hat{p}\mathbf{n} - \mu_f \nabla \hat{\mathbf{v}} \cdot \mathbf{n}$ on Γ_0, where $(\hat{\mathbf{v}}, \hat{p}, \hat{\mathbf{u}})$ is a solution of (1.1)-(1.4).

The optimization-based domain decomposition algorithm finds such $\mathbf{g}$ and $\mathbf{h}$ by minimizing the functional

$$\begin{aligned} \mathcal{J}(\mathbf{v}, p, \mathbf{g}, \mathbf{u}, \mathbf{h}) &= \frac{1}{2} \int_0^T \int_{\Gamma_0} (\mathbf{u}_t - \mathbf{g})^2 \, d\Gamma \, dt \\ &+ \frac{1}{2} \int_0^T \int_{\Gamma_0} (p\mathbf{n} - \mu_f \nabla \mathbf{v} \cdot \mathbf{n} - \mathbf{h})^2 d\Gamma \, dt \end{aligned}$$

In the remainder of this chapter, we discuss the derivation of the model (1.1)-(1.4). This thesis is organized as follows. In chapter 2, a coupled system of fluid-structure interaction problem is studied. The existence of a weak solution is proved and finite element approximations are discussed. In chapter 3, we introduce an optimization problem to uncouple the system. We prove that the optimization problem has a solution and the solution converges to the solution of the coupled system. The Lagrange multiplier rule is used to derive an optimality system from which optimal solutions may be determined. Finally, we define a gradient method for the solution of the optimality system and show the convergence for the gradient method.

2 FLUID-STRUCTURE INTERACTION PROBLEMS

2.1 The model problems

We consider a coupled system of Stokes and elasticity problem that was introduced in the previous chapter.

$$\left\{\begin{aligned} \rho_f \mathbf{v}_t + \nabla p - \mu_f \Delta \mathbf{v} = \rho_f \mathbf{f} &\quad \text{in } \Omega_f \\ \nabla \cdot \mathbf{v} = 0 &\quad \text{in } \Omega_f \\ \mathbf{v} = 0 &\quad \text{on } \Gamma_f \\ \rho_s \mathbf{u}_{tt} - \mu_s \Delta \mathbf{u} - (\lambda_s + \mu_s)\nabla(\nabla \cdot \mathbf{u}) = \rho_s \mathbf{b} &\quad \text{in } \Omega_s \\ \mathbf{u} = 0 &\quad \text{on } \Gamma_s \\ \mathbf{u}_t = \mathbf{v} &\quad \text{on } \Gamma_0 \\ \mu_s \nabla \mathbf{u} \cdot \mathbf{n} + (\lambda_s + \mu_s)(\nabla \cdot \mathbf{u})\mathbf{n} = p\mathbf{n} - \mu_f \nabla \mathbf{v} \cdot \mathbf{n} &\quad \text{on } \Gamma_0 \\ \mathbf{v}|_{t=0} = \mathbf{v}^0 &\quad \text{in } \Omega_f \\ \mathbf{u}|_{t=0} = \mathbf{u}^0 &\quad \text{in } \Omega_s \\ \mathbf{u}_t|_{t=0} = \mathbf{u}^1 &\quad \text{in } \Omega_s \end{aligned}\right. \tag{2.1.1}$$

2.2 Notation

Throughout this paper, C will denote a positive constant whose meaning and value changes with context. $H^s(D)$, $s \in I\!R$, denotes the standard sobolev space of order s

with respect to the set D, equipped with the standard norm $\|\cdot\|_{s,D}$. Corresponding sobolev spaces of vector-valued functions will be denoted by $\mathbf{H}^s(D)$. Dual spaces will be denoted by $(.)^*$. We then define the subspaces

$$\begin{aligned} \mathbf{H}^1_f(\Omega_f) &= \{\mathbf{v} \in \mathbf{H}^1(\Omega_f) : \ \mathbf{v} = 0 \text{ on } \Gamma_f\} \\ J &= \{\mathbf{v} \in \mathcal{D}(\Omega_f) : \ \nabla \cdot \mathbf{v} = 0 \text{ on } \Omega_f\} \end{aligned}$$

where $\mathcal{D}(\Omega_f)$ be the space of $\mathcal{C}^\infty$ functions with compact support contained in Ω_f.

$$\begin{aligned} V &= \text{ the closure of } J \text{ in } \mathbf{H}^1_f(\Omega_f) \\ H &= \text{ the closure of } J \text{ in } \mathbf{L}^2(\Omega_f) \end{aligned}$$

and

$$\mathbf{H}^1_s(\Omega_s) = \{\mathbf{u} \in \mathbf{H}^1(\Omega_s) : \ \mathbf{u} = 0 \text{ on } \Gamma_s\}$$

We define, for $(pq) \in L^1(D)$ and $(\mathbf{u} \cdot \mathbf{v}) \in L^1(D)$,

$$(p,q)_D = \int_D pq \, dD \quad \text{and} \quad (\mathbf{u},\mathbf{v})_D = \int_D \mathbf{u} \cdot \mathbf{v} \, dD$$

respectively.

We define the bilinear forms

$$\begin{aligned} a_f(\mathbf{u},\mathbf{v}) &= \int_{\Omega_f} \mu_f \nabla \mathbf{u} : \nabla \mathbf{v} \, d\Omega \quad \forall \mathbf{u},\mathbf{v} \in \mathbf{H}^1(\Omega_f) \\ a_s(\mathbf{u},\mathbf{v}) &= \int_{\Omega_s} \{\mu_s \nabla \mathbf{u} : \nabla \mathbf{v} + (\lambda_s + \mu_s)(\nabla \cdot \mathbf{u})(\nabla \cdot \mathbf{v})\} \, d\Omega \quad \forall \mathbf{u},\mathbf{v} \in \mathbf{H}^1(\Omega_s) \\ a(\mathbf{u},\mathbf{v}) &= \int_{\Omega_s} \mu_s \nabla \mathbf{u} : \nabla \mathbf{v} \, d\Omega \quad \forall \mathbf{u},\mathbf{v} \in \mathbf{H}^1(\Omega_s) \end{aligned}$$

and

$$b(\mathbf{v},q) = -\int_{\Omega_f} q \ div \mathbf{v} \, d\Omega \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega_f), \quad \forall q \in L^2(\Omega_f)$$

It is well known that the forms $a_f(\cdot,\cdot)$, $a_s(\cdot,\cdot)$, $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are continuous, i.e. there exist positive constants k_a and k_b such that

$$|a_f(\mathbf{u},\mathbf{v})| \leq k_a \|\mathbf{u}\|_{1,\Omega_f} \|\mathbf{v}\|_{1,\Omega_f} \quad \forall \mathbf{u},\mathbf{v} \in \mathbf{H}^1(\Omega_f)$$

$$
\begin{aligned}
|a_s(\mathbf{u},\mathbf{v})| &\le k_a \|\mathbf{u}\|_{1,\Omega_s}\|\mathbf{v}\|_{1,\Omega_s} \quad \forall \mathbf{u},\mathbf{v} \in \mathbf{H}^1(\Omega_s) \\
|a(\mathbf{u},\mathbf{v})| &\le k_a \|\mathbf{u}\|_{1,\Omega_s}\|\mathbf{v}\|_{1,\Omega_s} \quad \forall \mathbf{u},\mathbf{v} \in \mathbf{H}^1(\Omega_s)
\end{aligned}
$$

and

$$
|b(\mathbf{v},q)| \le k_b \|\mathbf{v}\|_{1,\Omega_f}\|q\|_{0,\Omega_f} \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega_f), \quad \forall q \in L^2(\Omega_f)
$$

Also, $a_f(\cdot,\cdot)$, $a_s(\cdot,\cdot)$ and $a(\cdot,\cdot)$ satisfy the coercivity property and $b(\cdot,\cdot)$ satisfies the inf-sup condition, which means there exist positive constants K_a and K_b such that

$$
\begin{aligned}
|a_f(\mathbf{u},\mathbf{u})| &\ge K_a \|\mathbf{u}\|^2_{1,\Omega_f} \quad \forall \mathbf{u} \in \mathbf{H}^1(\Omega_f) \\
|a_s(\mathbf{u},\mathbf{u})| &\ge K_a \|\mathbf{u}\|^2_{1,\Omega_s} \quad \forall \mathbf{u} \in \mathbf{H}^1(\Omega_f) \\
|a(\mathbf{u},\mathbf{u})| &\ge K_a \|\mathbf{u}\|^2_{1,\Omega_s} \quad \forall \mathbf{u} \in \mathbf{H}^1(\Omega_f)
\end{aligned}
$$

and

$$
\inf_{0\neq q\in L^2(\Omega_f)} \sup_{0\neq \mathbf{v}\in \mathbf{H}^1_0(\Omega_f)} \frac{b(\mathbf{v},q)}{\|\mathbf{v}\|_{1,\Omega_f}\|q\|_{0,\Omega_f}} \ge K_b
$$

2.3 A weak formulation

We define a space of trial functions and test functions as

$$
\begin{aligned}
U = \{(\mathbf{v},\mathbf{u}) \ : \ & \mathbf{v} \in L^2(0,T;\mathbf{H}^1_f(\Omega_f)),\ \mathbf{u} \in L^2(0,T;\mathbf{H}^1_s(\Omega_s)), \\
& \mathbf{u}_t \in L^2(0,T;\mathbf{L}^2(\Omega_s)),\ \text{such that } \mathbf{v} = \mathbf{u}_t \text{ on } \Gamma_0\}
\end{aligned}
$$

A weak formulation corresponding to (2.1.1) is given by

For $\mathbf{f}, \mathbf{v}^0, \mathbf{b}, \mathbf{u}^0$ and $\mathbf{u}^1$ given,

$$\mathbf{f} \in L^2(0,T;\mathbf{H}^1_f(\Omega_f)^*) \tag{2.3.2}$$

$$\mathbf{v}^0 \in \mathbf{L}^2(\Omega_f) \tag{2.3.3}$$

$$\mathbf{b} \in L^2(0,T;\mathbf{L}^2(\Omega_s)) \tag{2.3.4}$$

$$\mathbf{u}^0 \in \mathbf{H}^1_s(\Omega_s) \tag{2.3.5}$$

$$\mathbf{u}^1 \in \mathbf{L}^2(\Omega_s) \tag{2.3.6}$$

to find $(\mathbf{v},\mathbf{u}) \in U$ satisfying

$$\begin{aligned} &\rho_f(\mathbf{v}_t,\mathbf{w})_{\Omega_f} + b(\mathbf{w},p) + a_f(\mathbf{v},\mathbf{w}) + \rho_s(\mathbf{u}_{tt},\theta_t)_{\Omega_s} \\ + \quad & a_s(\mathbf{u},\theta_t) = \rho_f(\mathbf{f},\mathbf{w})_{\Omega_f} + \rho_s(\mathbf{b},\theta_t)_{\Omega_s} \quad \forall(\mathbf{w},\theta) \in U \qquad (2.3.7) \\ & b(\mathbf{v},q) = 0 \quad \forall q \in L^2(\Omega_f) \qquad (2.3.8) \\ & \mathbf{v}|_{t=0} = \mathbf{v}^0 \qquad (2.3.9) \\ & \mathbf{u}|_{t=0} = \mathbf{u}^0 \qquad (2.3.10) \\ & \mathbf{u}_t|_{t=0} = \mathbf{u}^1 \qquad (2.3.11) \end{aligned}$$

We recall the definition of the spaces J, V and H, introduced in the previous section and which will be the basic spaces for W.

$$\begin{aligned} J &= \{\mathbf{v} \in \mathcal{D}(\Omega_f) : \nabla\cdot\mathbf{v} = 0 \text{ on } \Omega_f\} \\ V &= \text{the closure of } J \text{ in } \mathbf{H}^1_f(\Omega_f) \\ H &= \text{the closure of } J \text{ in } \mathbf{L}^2(\Omega_f) \end{aligned}$$

Now, we define a space of trial functions and test functions as

$$\begin{aligned} W = \{(\mathbf{v},\mathbf{u}) \ : \ & \mathbf{v} \in L^2(0,T;V),\ \mathbf{u} \in L^2(0,T;\mathbf{H}^1_s(\Omega_s)), \\ & \mathbf{u}_t \in L^2(0,T;\mathbf{L}^2(\Omega_s)),\ \text{such that } \mathbf{v} = \mathbf{u}_t \text{ on } \Gamma_0\} \end{aligned}$$

Then a weak formulation (2.3.7)-(2.3.11) is equivalent to the following:
To find $(\mathbf{v},\mathbf{u}) \in W$ satisfying

$$\begin{aligned} &\rho_f(\mathbf{v}_t,\mathbf{w})_{\Omega_f} + a_f(\mathbf{v},\mathbf{w}) + \rho_s(\mathbf{u}_{tt},\theta_t)_{\Omega_s} + a_s(\mathbf{u},\theta_t) \\ = \quad & \rho_f(\mathbf{f},\mathbf{w})_{\Omega_f} + \rho_s(\mathbf{b},\theta_t)_{\Omega_s} \quad \forall(\mathbf{w},\theta) \in W \qquad (2.3.12) \\ & \mathbf{v}|_{t=0} = \mathbf{v}^0 \qquad (2.3.13) \\ & \mathbf{u}|_{t=0} = \mathbf{u}^0 \qquad (2.3.14) \\ & \mathbf{u}_t|_{t=0} = \mathbf{u}^1 \qquad (2.3.15) \end{aligned}$$

2.4 The existence of a weak solution

To show the existence of weak solutions for the coupled system, we introduce an auxiliary problem involving an 'artificial viscosity' term. The auxiliary problem is defined as follows.

$$\begin{cases} \rho_f \mathbf{v}_t + \nabla p - \mu_f \Delta \mathbf{v} = \rho_f \mathbf{f} \quad \text{in } \Omega_f \\ \nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega_f \\ \mathbf{v} = 0 \quad \text{on } \Gamma_f \\ \rho_s \mathbf{u}_{tt} - \epsilon \mu_s \Delta \mathbf{u}_t - \mu_s \Delta \mathbf{u} - (\lambda_s + \mu_s) \nabla (\nabla \cdot \mathbf{u}) = \rho_s \mathbf{b} \quad \text{in } \Omega_s \\ \mathbf{u} = 0 \quad \text{on } \Gamma_s \\ \mathbf{u}_t = \mathbf{v} \quad \text{on } \Gamma_0 \\ \epsilon \mu_s \nabla \mathbf{u}_t \cdot \mathbf{n} + \mu_s \nabla \mathbf{u} \cdot \mathbf{n} + (\lambda_s + \mu_s)(\nabla \cdot \mathbf{u}) \mathbf{n} = p \mathbf{n} - \mu_f \nabla \mathbf{v} \cdot \mathbf{n} \quad \text{on } \Gamma_0 \\ \mathbf{v}|_{t=0} = \mathbf{v}^0 \quad \text{in } \Omega_f \\ \mathbf{u}|_{t=0} = \mathbf{u}^0 \quad \text{in } \Omega_s \\ \mathbf{u}_t|_{t=0} = \mathbf{u}^1 \quad \text{in } \Omega_s \end{cases} \tag{2.4.16}$$

We define a space of trial functions and test functions for the auxiliary problem as

$$\hat{U} = \{(\mathbf{v}, \mathbf{u}) \ : \ \mathbf{v} \in L^2(0,T; \mathbf{H}^1_f(\Omega_f)), \ \mathbf{u} \in L^2(0,T; \mathbf{H}^1_s(\Omega_s)),$$
$$\mathbf{u}_t \in L^2(0,T; \mathbf{H}^1_s(\Omega_s)), \text{ such that } \mathbf{v} = \mathbf{u}_t \text{ on } \Gamma_0\}$$

A weak formulation corresponding to (2.4.16) is given by

For $\mathbf{f}, \mathbf{v}^0, \mathbf{b}, \mathbf{u}^0$ and $\mathbf{u}^1$ given,

$$\mathbf{f} \in L^2(0,T; \mathbf{H}^1_f(\Omega_f)^*) \tag{2.4.17}$$

$$\mathbf{v}^0 \in \mathbf{L}^2(\Omega_f) \tag{2.4.18}$$

$$\mathbf{b} \in L^2(0,T;\mathbf{L}^2(\Omega_s)) \tag{2.4.19}$$

$$\mathbf{u}^0 \in \mathbf{H}^1_s(\Omega_s) \tag{2.4.20}$$

$$\mathbf{u}^1 \in \mathbf{L}^2(\Omega_s) \tag{2.4.21}$$

to find $(\mathbf{v},\mathbf{u}) \in \hat{U}$ satisfying

$$\begin{aligned} & \rho_f(\mathbf{v}_t,\mathbf{w})_{\Omega_f} + b(\mathbf{w},p) + a_f(\mathbf{v},\mathbf{w}) + \rho_s(\mathbf{u}_{tt},\theta_t)_{\Omega_s} + \epsilon a(\mathbf{u}_t,\theta_t) \\ + \quad & a_s(\mathbf{u},\theta_t) = \rho_f(\mathbf{f},\mathbf{w})_{\Omega_f} + \rho_s(\mathbf{b},\theta_t)_{\Omega_s} \quad \forall(\mathbf{w},\theta) \in \hat{U} \end{aligned} \tag{2.4.22}$$

$$b(\mathbf{v},q) = 0 \quad \forall q \in L^2(\Omega_f) \tag{2.4.23}$$

$$\mathbf{v}|_{t=0} = \mathbf{v}^0 \tag{2.4.24}$$

$$\mathbf{u}|_{t=0} = \mathbf{u}^0 \tag{2.4.25}$$

$$\mathbf{u}_t|_{t=0} = \mathbf{u}^1 \tag{2.4.26}$$

Now, we define a space of trial functions and test functions as

$$\begin{aligned} \hat{W} = \{(\mathbf{v},\mathbf{u}) \;:\; & \mathbf{v} \in L^2(0,T;V),\ \mathbf{u} \in L^2(0,T;\mathbf{H}^1_s(\Omega_s)), \\ & \mathbf{u}_t \in L^2(0,T;\mathbf{H}^1_s(\Omega_s)),\ \text{such that } \mathbf{v} = \mathbf{u}_t \text{ on } \Gamma_0\} \end{aligned}$$

Then a weak formulation (2.4.22)-(2.4.26) is equivalent to the following:
To find $(\mathbf{v},\mathbf{u}) \in \hat{W}$ satisfying

$$\begin{aligned} & \rho_f(\mathbf{v}_t,\mathbf{w})_{\Omega_f} + a_f(\mathbf{v},\mathbf{w}) + \rho_s(\mathbf{u}_{tt},\theta_t)_{\Omega_s} + \epsilon a(\mathbf{u}_t,\theta_t) \\ + \quad & a_s(\mathbf{u},\theta_t) = \rho_f(\mathbf{f},\mathbf{w})_{\Omega_f} + \rho_s(\mathbf{b},\theta_t)_{\Omega_s} \quad \forall(\mathbf{w},\theta) \in \hat{W} \end{aligned} \tag{2.4.27}$$

$$\mathbf{v}|_{t=0} = \mathbf{v}^0 \tag{2.4.28}$$

$$\mathbf{u}|_{t=0} = \mathbf{u}^0 \tag{2.4.29}$$

$$\mathbf{u}_t|_{t=0} = \mathbf{u}^1 \tag{2.4.30}$$

We show the existence of solutions of the auxiliary problem in the next theorem.

Theorem 2.4.1 *For given* $\mathbf{f}, \mathbf{v}^0, \mathbf{b}, \mathbf{u}^0$ *and* $\mathbf{u}^1$ *which satisfy (2.4.17)-(2.4.21), there exists a unique solution* $(\mathbf{v}, \mathbf{u}) \in \hat{W}$ *which satisfy (2.4.27)-(2.4.30). Moreover* $\mathbf{v} \in L^\infty(0,T;H)$, $\mathbf{u} \in L^\infty(0,T;\mathbf{H}_s^1(\Omega_s))$ *and* $\mathbf{u}_t \in L^\infty(0,T;\mathbf{L}^2(\Omega_s))$

proof: We use the Galerkin method. Let $(\phi_n)_{n\in\mathbf{N}}$ be a basis of V and $(\theta_n)_{n\in\mathbf{N}}$ be a basis of $\mathbf{H}_s^1(\Omega_s)$ such that $\phi_n = \theta_n$ on Γ_0. We define discrete spaces $M_n = span\{\phi_m,\ 1 \le m \le n\}$ and $N_n = span\{\theta_m,\ 1 \le m \le n\}$. Also define a discrete space of trial and test functions by

$$V_n = \{(\mathbf{v}, \mathbf{u}) \in \mathcal{V}_{n,f} \times \mathcal{V}_{n,s},\ \mathbf{u}_t = \mathbf{v} \text{ on } \Gamma_0\}$$

with

$$\begin{aligned} \mathcal{V}_{n,f} &= \{\mathbf{v} = \sum_{i=1}^{n} a_i(t)\phi_i,\ a_i(t) \in H^1(0,T)\} \\ \mathcal{V}_{n,s} &= \{\mathbf{u} = \sum_{i=1}^{n} b_i(t)\theta_i,\ b_i(t) \in H^1(0,T)\} \end{aligned}$$

The discrete problem is : Find $(\mathbf{v}_n, \mathbf{u}_n) \in V_n$ such that $\mathbf{v}_n(0) \in M_n$, $\mathbf{u}(0) \in N_n$ and $\mathbf{u}_{nt}(0) \in N_n$ with $\mathbf{v}_n(0) \to \mathbf{v}^0$ in $\mathbf{L}^2(\Omega_f)$, $\mathbf{u}_n(0) \to \mathbf{u}^0$ in $\mathbf{H}^1(\Omega_s)$ and $\mathbf{u}_{nt}(0) \to \mathbf{u}^1$ in $\mathbf{L}^2(\Omega_s)$ and

$$\begin{aligned} &\rho_f(\mathbf{v}_{nt}, \mathbf{w}_n)_{\Omega_f} + a_f(\mathbf{v}_n, \mathbf{w}_n) + \rho_s(\mathbf{u}_{ntt}, \mathbf{z}_{nt})_{\Omega_s} + \epsilon a(\mathbf{u}_{nt}, \mathbf{z}_{nt}) \\ &+ a_s(\mathbf{u}_n, \mathbf{z}_{nt}) = \rho_f(\mathbf{f}, \mathbf{w}_n)_{\Omega_f} + \rho_s(\mathbf{b}, \mathbf{z}_{nt})_{\Omega_s} \quad \forall(\mathbf{w}_n, \mathbf{z}_n) \in V_n \end{aligned} \tag{2.4.31}$$

Since this is a linear system of ordinary differential equations, there is a unique solution.

We will obtain a priori estimates independent of n for the functions $\mathbf{v}_n, \mathbf{u}_n$ and then pass to the limit. Set $\mathbf{w}_n = \mathbf{v}_n$ and $\mathbf{z}_n = \mathbf{u}_n$ in (2.4.31). We get

$$\begin{aligned} &\rho_f(\mathbf{v}_{nt}, \mathbf{v}_n)_{\Omega_f} + a_f(\mathbf{v}_n, \mathbf{v}_n) + \rho_s(\mathbf{u}_{ntt}, \mathbf{u}_{nt})_{\Omega_s} + \epsilon a(\mathbf{u}_{nt}, \mathbf{u}_{nt}) \\ &+ a_s(\mathbf{u}_n, \mathbf{u}_{nt}) = \rho_f(\mathbf{f}, \mathbf{v}_n)_{\Omega_f} + \rho_s(\mathbf{b}, \mathbf{u}_{nt})_{\Omega_s} \end{aligned} \tag{2.4.32}$$

and this gives

$$\begin{aligned} &\frac{\rho_f}{2}\frac{d}{dt}\|\mathbf{v}_n\|_{0,\Omega_f}^2 + K_a\|\mathbf{v}_n\|_{1,\Omega_f}^2 + \frac{\rho_s}{2}\frac{d}{dt}\|\mathbf{u}_{nt}\|_{0,\Omega_s}^2 + \epsilon K_a\|\mathbf{u}_{nt}\|_{1,\Omega_s}^2 \\ +\ &\frac{K_a}{2}\frac{d}{dt}\|\mathbf{u}_n\|_{1,\Omega_s}^2 \le \rho_f\|\mathbf{f}\|_{-1,\Omega_f}\|\mathbf{v}_n\|_{1,\Omega_f} + \rho_s\|\mathbf{b}\|_{0,\Omega_s}\|\mathbf{u}_{nt}\|_{0,\Omega_s} \end{aligned}$$

Integrating this from 0 to t, we obtain

$$\begin{aligned}
&\frac{\rho_f}{2}(\|\mathbf{v}_n\|^2_{0,\Omega_f} - \|\mathbf{v}_n(0)\|^2_{0,\Omega_f}) + K_a\int_0^t \|\mathbf{v}_n\|^2_{1,\Omega_f} + \frac{\rho_s}{2}(\|\mathbf{u}_{nt}\|^2_{0,\Omega_s} - \|\mathbf{u}_{nt}(0)\|^2_{0,\Omega_s}) \\
&+ \quad \epsilon K_a \int_0^t \|\mathbf{u}_{nt}\|^2_{1,\Omega_s} + \frac{K_a}{2}(\|\mathbf{u}_n\|^2_{1,\Omega_s} - \|\mathbf{u}_n(0)\|^2_{1,\Omega_s}) \\
&\le \quad \frac{\rho_f^2}{2K_a}\int_0^t \|\mathbf{f}\|^2_{-1,\Omega_f} + \frac{K_a}{2}\int_0^t \|\mathbf{v}_n\|^2_{1,\Omega_f} + T\rho_s\int_0^t \|\mathbf{b}\|^2_{0,\Omega_s} + \frac{\rho_s}{2T}\int_0^t \|\mathbf{u}_{nt}\|^2_{0,\Omega_s}
\end{aligned}$$

Gronwall's inequality may be used to conclude

$$\begin{aligned}
&\sup_t \rho_f\|\mathbf{v}_n\|^2_{0,\Omega_f} + K_a\int_0^T \|\mathbf{v}_n\|^2_{1,\Omega_f} + \sup_t \rho_s\|\mathbf{u}_{nt}\|^2_{0,\Omega_s} + \epsilon K_a\int_0^T \|\mathbf{u}_{nt}\|^2_{1,\Omega_s} \\
&+ \quad \sup_t K_a\|\mathbf{u}_n\|^2_{1,\Omega_s} \le C\int_0^T \|\mathbf{f}\|^2_{-1,\Omega_f} + \rho_s C\int_0^T \|\mathbf{b}\|^2_{0,\Omega_s} \\
&+ \quad \rho_f\|\mathbf{v}^0\|^2_{0,\Omega_f} + \rho_s\|\mathbf{u}^1\|^2_{0,\Omega_s} + K_a\|\mathbf{u}^0\|^2_{1,\Omega_s}
\end{aligned}$$

Hence $\{\mathbf{v}_n\}$ is uniformly bounded in $L^\infty(0,T;H)$ and $L^2(0,T;V)$; $\{\mathbf{u}_n\}$ is uniformly bounded in $L^\infty(0,T;\mathbf{H}^1_s(\Omega_s))$; $\{\mathbf{u}_{nt}\}$ is uniformly bounded in $L^\infty(0,T;\mathbf{L}^2(\Omega_s))$ and $L^2(0,T;\mathbf{H}^1_s(\Omega_s))$; Thus, there exist weakly convergent subsequences and by passing to the limit, a solution of (2.4.27)-(2.4.30) exists.

Uniqueness. Let $(\mathbf{v}_1,\mathbf{u}_1)$ and $(\mathbf{v}_2,\mathbf{u}_2)$ be two solutions of (2.4.27)-(2.4.30). Then energy estimates may be used to get

$$\begin{aligned}
&\sup_t \rho_f\|\mathbf{v}_1-\mathbf{v}_2\|^2_{0,\Omega_f} + K_a\int_0^T \|\mathbf{v}_1-\mathbf{v}_2\|^2_{1,\Omega_f} + \sup_t \rho_s\|\mathbf{u}_{1t}-\mathbf{u}_{2t}\|^2_{0,\Omega_s} \\
&+ \quad \epsilon K_a\int_0^T \|\mathbf{u}_{1t}-\mathbf{u}_{2t}\|^2_{1,\Omega_s} + \sup_t K_a\|\mathbf{u}_1-\mathbf{u}_2\|^2_{1,\Omega_s} \le 0
\end{aligned}$$

and hence $\mathbf{v}_1 = \mathbf{v}_2$ and $\mathbf{u}_1 = \mathbf{u}_2$.

Now we show the existence of the coupled system by taking the limit of solutions of the auxiliary problem as $\epsilon \to 0$

Theorem 2.4.2 *For given $\mathbf{f}, \mathbf{v}^0, \mathbf{b}, \mathbf{u}^0$ and $\mathbf{u}^1$ which satisfy (2.3.2)-(2.3.6), there exists a unique solution $(\mathbf{v},\mathbf{u}) \in W$ which satisfy (2.3.12)-(2.3.15). Moreover $\mathbf{v} \in L^\infty(0,T;H)$, $\mathbf{u} \in L^\infty(0,T;\mathbf{H}^1_s(\Omega_s))$ and $\mathbf{u}_t \in L^\infty(0,T;\mathbf{L}^2(\Omega_s))$*

proof: $\forall\epsilon$, there exists $\mathbf{v}_\epsilon$,$\mathbf{u}_\epsilon$ which are solutions of (2.4.27)-(2.4.30). Using a priori estimate obtained in the proof of Theorem 2.4.1,

$$\begin{aligned} &\sup_t \rho_f \|\mathbf{v}_\epsilon\|^2_{0,\Omega_f} + K_a \int_0^T \|\mathbf{v}_\epsilon\|^2_{1,\Omega_f} + \sup_t \rho_s \|(\mathbf{u}_\epsilon)_t\|^2_{0,\Omega_s} \\ + \quad &\epsilon K_a \int_0^t \|(\mathbf{u}_\epsilon)_t\|^2_{1,\Omega_s} + \sup_t K_a \|\mathbf{u}_\epsilon\|^2_{1,\Omega_s} \le C(\mathbf{v}^0, \mathbf{u}^0, \mathbf{u}^1, \mathbf{f}, \mathbf{b}) \end{aligned}$$

From which we deduce

$\mathbf{v}_\epsilon$ is uniformly bounded in $L^2(0,T;V) \cap L^\infty(0,T;H)$

$\mathbf{u}_\epsilon$ is uniformly bounded in $L^\infty(0,T;\mathbf{H}^1_s(\Omega_s))$

$(\mathbf{u}_\epsilon)_t$ is uniformly bounded in $L^\infty(0,T;\mathbf{L}^2(\Omega_s))$

$\sqrt{\epsilon}(\mathbf{u}_\epsilon)_t$ is uniformly bounded in $L^2(0,T;\mathbf{H}^1_s(\Omega_s))$

Therefore, we can extract a subsequence such that

$\mathbf{v}_\epsilon \rightharpoonup \mathbf{v}$ in $L^2(0,T;V) \cap L^\infty(0,T;H)$

$\mathbf{u}_\epsilon \rightharpoonup \mathbf{u}$ in $L^\infty(0,T;\mathbf{H}^1_s(\Omega_s))$

$(\mathbf{u}_\epsilon)_t \rightharpoonup \mathbf{u}_t$ in $L^\infty(0,T;\mathbf{L}^2(\Omega_s))$

Since $\sqrt{\epsilon}(\mathbf{u}_\epsilon)_t$ is uniformly bounded in $L^2(0,T;\mathbf{H}^1_s(\Omega_s))$ then $\sqrt{\epsilon}(\Delta\mathbf{u}_\epsilon)_t$ is uniformly bounded in $L^2(0,T;\mathbf{H}^1_s(\Omega_s)^*)$ and

$$\lim_{\epsilon\to 0} \epsilon a_s((\mathbf{u}_\epsilon)_t, \mathbf{w}) = 0 \quad \forall \mathbf{w} \in L^2(0,T;\mathbf{H}^1_s(\Omega_s))$$

From (2.4.27),

$$\begin{aligned} &\rho_f((\mathbf{v}_\epsilon)_t, \mathbf{w})_{\Omega_f} + \rho_s((\mathbf{u}_\epsilon)_{tt}, \theta_t)_{\Omega_s} \\ &= \quad -a_f(\mathbf{v}_\epsilon, \mathbf{w}) - \epsilon a((\mathbf{u}_\epsilon)_t, \theta_t) - a_s(\mathbf{u}_\epsilon, \theta_t) + \rho_f(\mathbf{f}, \mathbf{w})_{\Omega_f} + \rho_s(\mathbf{b}, \theta_t)_{\Omega_s} \\ &\to \quad -a_f(\mathbf{v}, \mathbf{w}) - a_s(\mathbf{u}, \theta_t) + \rho_f(\mathbf{f}, \mathbf{w})_{\Omega_f} + \rho_s(\mathbf{b}, \theta_t)_{\Omega_s} \quad \forall(\mathbf{w}, \theta) \in \hat{W} \end{aligned}$$

Moreover, since $\mathbf{v}_\epsilon \rightharpoonup \mathbf{v}$ in $L^\infty(0,T;H)$ and $(\mathbf{u}_\epsilon)_t \rightharpoonup \mathbf{u}_t$ in $L^\infty(0,T;\mathbf{L}^2(\Omega_s))$ we have

$$\begin{aligned} &\rho_f(\mathbf{v}_t, \mathbf{w})_{\Omega_f} + \rho_s(\mathbf{u}_{tt}, \theta_t)_{\Omega_s} \\ &= \quad -a_f(\mathbf{v}, \mathbf{w}) - a_s(\mathbf{u}, \theta_t) + \rho_f(\mathbf{f}, \mathbf{w})_{\Omega_f} + \rho_s(\mathbf{b}, \theta_t)_{\Omega_s} \quad \forall(\mathbf{w}, \theta) \in \hat{W} \end{aligned}$$

Further,

$$\mathbf{v}_\epsilon|_{t=0} \rightharpoonup \mathbf{v}|_{t=0} \text{ in } H$$

$$\mathbf{u}_\epsilon|_{t=0} \rightharpoonup \mathbf{u}|_{t=0} \text{ in } \mathbf{H}^1_s(\Omega_s)$$

$$\mathbf{u}_{\epsilon t}|_{t=0} \rightharpoonup \mathbf{u}_t|_{t=0} \text{ in } \mathbf{L}^2(\Omega_s)$$

Thus the existence theorem is proved. The proof of uniqueness is exactly the same as before.

2.5 Finite element approximations

Let h denote a discretization parameter tending to zero and, for each h, let $\mathbf{X}^h_f$, S^h and $\mathbf{X}^h_s$ be finite dimensional spaces such that $\mathbf{X}^h_f \subset H^1_f(\Omega_f)$, $S^h \subset L^2(\Omega_f)$ and $\mathbf{X}^h_s \subset H^1_s(\Omega_s)$. And

$$\begin{aligned} U^h = \{(\mathbf{v}^h, \mathbf{u}^h) \ : \ &\mathbf{v}^h \in L^2(0,T;\mathbf{X}^h_f),\ \mathbf{u}^h \in H^1(0,T;\mathbf{X}^h_s), \\ &\text{such that } \mathbf{v}^h = \mathbf{u}^h_t \text{ on } \Gamma_0\} \end{aligned}$$

We assume that the finite element spaces satisfy the standard approximation properties, i.e.,

$$\inf_{\mathbf{v}\in\mathbf{X}^h_f} \|\mathbf{v}-\mathbf{v}^h\|_{1,\Omega_f} \le \|\mathbf{v}\|_{m+1,\Omega_f} \quad \forall \mathbf{v} \in \mathbf{H}^{m+1}(\Omega_f)$$

$$\inf_{p\in S^h} \|p-p^h\|_{0,\Omega_f} \le \|p\|_{m,\Omega_f} \quad \forall p \in \mathbf{H}^{m}(\Omega_f)$$

and

$$\inf_{\mathbf{u}\in\mathbf{X}^h_s} \|\mathbf{u}-\mathbf{u}^h\|_{1,\Omega_s} \le \|\mathbf{u}\|_{m+1,\Omega_s} \quad \forall \mathbf{u} \in \mathbf{H}^{m+1}(\Omega_s)$$

We also assume the inf-sup condition (or LBB) condition,

$$\inf_{0\ne q^h\in S^h} \ \sup_{0\ne \mathbf{v}^h\in\mathbf{X}^h_f} \frac{b(\mathbf{v}^h, q^h)}{\|\mathbf{v}^h\|_{1,\Omega_f}\|q^h\|_{0,\Omega_f}} \ \ge \ K_b$$

where K_b is a positive constant independent of h.

Finite element approximations of solutions of the coupled system (2.3.7)-(2.3.11) are defined as follows: Seek $(\mathbf{v}^h, \mathbf{u}^h, p^h) \in U^h \times L^2(0,T;S^h)$ such that

$$\begin{aligned} & \rho_f(\mathbf{v}^h_t, \mathbf{w}^h)_{\Omega_f} + b(\mathbf{w}^h, p^h) + a_f(\mathbf{v}^h, \mathbf{w}^h) + \rho_s(\mathbf{u}^h_{tt}, \theta^h_t)_{\Omega_s} + a_s(\mathbf{u}^h, \theta^h_t) \\ = \quad & \rho_f(\mathbf{f}, \mathbf{w}^h)_{\Omega_f} + \rho_s(\mathbf{b}, \theta^h_t)_{\Omega_s} \quad \forall (\mathbf{w}^h, \theta^h) \in U^h \end{aligned} \tag{2.5.33}$$

$$b(\mathbf{v}^h, q^h) = 0 \quad \forall q^h \in L^2(0,T;S^h) \tag{2.5.34}$$

$$\mathbf{v}^h|_{t=0} = \mathbf{v}^{0h} \tag{2.5.35}$$

$$\mathbf{u}^h|_{t=0} = \mathbf{u}^{0h} \tag{2.5.36}$$

$$\mathbf{u}^h_t|_{t=0} = \mathbf{u}^{1h} \tag{2.5.37}$$

First, we show the convergence of finite element approximations. To show the convergence, we consider finite element approximations of the auxiliary problem (2.4.22)-(2.4.26): Seek $(\mathbf{v}^{\epsilon h}, \mathbf{u}^{\epsilon h}, p^{\epsilon h}) \in U^h \times L^2(0,T;S^h)$ such that

$$\begin{aligned} & \rho_f(\mathbf{v}^{\epsilon h}_t, \mathbf{w}^h)_{\Omega_f} + b(\mathbf{w}^h, p^{\epsilon h}) + a_f(\mathbf{v}^{\epsilon h}, \mathbf{w}^h) + \rho_s(\mathbf{u}^{\epsilon h}_{tt}, \theta^h_t)_{\Omega_s} \\ + \quad & \epsilon a(\mathbf{u}^{\epsilon h}_t, \theta^h_t) + a_s(\mathbf{u}^{\epsilon h}, \theta^h_t) \\ = \quad & \rho_f(\mathbf{f}, \mathbf{w}^h)_{\Omega_f} + \rho_s(\mathbf{b}, \theta^h_t)_{\Omega_s} \quad \forall (\mathbf{w}^h, \theta^h) \in U^h \end{aligned} \tag{2.5.38}$$

$$b(\mathbf{v}^{\epsilon h}, q^h) = 0 \quad \forall q^h \in L^2(0,T;S^h) \tag{2.5.39}$$

$$\mathbf{v}^{\epsilon h}|_{t=0} = \mathbf{v}^{0h} \tag{2.5.40}$$

$$\mathbf{u}^{\epsilon h}|_{t=0} = \mathbf{u}^{0h} \tag{2.5.41}$$

$$\mathbf{u}^{\epsilon h}_t|_{t=0} = \mathbf{u}^{1h} \tag{2.5.42}$$

Lemma 2.5.1 *For each $\epsilon \geq 0$, let $(\mathbf{v}^\epsilon, \mathbf{u}^\epsilon, p^\epsilon)$ denote solutions of auxiliary problem (2.4.22)-(2.4.26) and $(\mathbf{v}^{\epsilon h}, \mathbf{u}^{\epsilon h}, p^{\epsilon h})$ denote finite element approximations of solutions of auxiliary problem. Then $(\mathbf{v}^{\epsilon h}, \mathbf{u}^{\epsilon h}, p^{\epsilon h}) \to (\mathbf{v}^\epsilon, \mathbf{u}^\epsilon, p^\epsilon)$ in $L^2(0,T;\mathbf{H}^1(\Omega_f)) \times H^1(0,T;\mathbf{H}^1(\Omega_s)) \times H^{-1}(0,T;L^2(\Omega_f))$ as $h \to 0$.*

proof: Set $\mathbf{w}^h = \mathbf{v}^{\epsilon h}$ and $\theta^h = \mathbf{u}^{\epsilon h}$ in (2.5.38) and combine with (2.5.39), we obtain

$$\sup_t \rho_f \|\mathbf{v}^{\epsilon h}\|^2_{0,\Omega_f} + K_a \int_0^T \|\mathbf{v}^{\epsilon h}\|^2_{1,\Omega_f} + \sup_t \rho_s \|\mathbf{u}^{\epsilon h}_t\|^2_{0,\Omega_s}$$

$$
\begin{aligned}
&+ \quad \epsilon K_a \int_0^t \|\mathbf{u}_t^{\epsilon h}\|_{1,\Omega_s}^2 + \sup_t K_a \|\mathbf{u}^{\epsilon h}\|_{1,\Omega_s}^2 \\
&\leq \quad C \int_0^T \|\mathbf{f}\|_{-1,\Omega_f}^2 + \rho_s C \int_0^T \|\mathbf{b}\|_{0,\Omega_s}^2 \\
&+ \quad \rho_f \|\mathbf{v}^0\|_{0,\Omega_f}^2 + \rho_s \|\mathbf{u}^1\|_{0,\Omega_s}^2 + K_a \|\mathbf{u}^0\|_{1,\Omega_s}^2
\end{aligned}
$$

The same argument as in the proof of the Theorem 2.4.1 yields $(\mathbf{v}^{\epsilon h}, \mathbf{u}^{\epsilon h}) \rightarrow (\mathbf{v}^{\epsilon}, \mathbf{u}^{\epsilon})$ in $L^2(0,T;\mathbf{H}^1(\Omega_f)) \times H^1(0,T;\mathbf{H}^1(\Omega_s))$ as $h \rightarrow 0$. If we take $p^{\epsilon h}$ satisfying (2.5.38) and p^{ϵ} satisfying (2.4.22) then $p^{\epsilon h} \rightarrow p^{\epsilon}$ in $H^{-1}(0,T;L^2(\Omega_f))$ as $h \rightarrow 0$. Thus lemma is proved.

It was proved in Theorem 2.4.2 that $(\mathbf{v}^{\epsilon}, \mathbf{u}^{\epsilon}, p^{\epsilon}) \rightarrow (\mathbf{v}, \mathbf{u}, p)$ as $\epsilon \rightarrow 0$ and it can be shown $(\mathbf{v}^{\epsilon h}, \mathbf{u}^{\epsilon h}, p^{\epsilon h}) \rightarrow (\mathbf{v}^h, \mathbf{u}^h, p^h)$ as $\epsilon \rightarrow 0$ in the same manner. Combining this with Lemma 2.5.1 gives the following theorem.

Theorem 2.5.2 *Let $(\mathbf{v}, \mathbf{u}, p)$ denote solutions of the coupled system (2.3.7)-(2.3.11) and $(\mathbf{v}^h, \mathbf{u}^h, p^h)$ denote solutions of (2.5.33)-(2.5.37). Then $(\mathbf{v}^h, \mathbf{u}^h, p^h) \rightarrow (\mathbf{v}, \mathbf{u}, p)$ in $L^2(0,T;\mathbf{H}^1(\Omega_f)) \times (L^2(0,T;\mathbf{H}^1(\Omega_s) \cap H^1(0,T;\mathbf{L}^2(\Omega_s)) \times H^{-1}(0,T;L^2(\Omega_f))$ as $h \rightarrow 0$.*

2.6 Error estimates

For the purpose of the proof of next theorem, we introduce some spaces and a projection. Let Ω denote $\Omega_f \cap \Omega_s$ and $\mathbf{X}^h$ denote a finite dimensional space such that $\mathbf{X}^h \in \mathbf{H}^1(\Omega)$. We define a continous space G and a discrete space G^h by

$$
G = \{\mathbf{z} \in \mathbf{H}^1(\Omega) : \mathbf{z}_f \equiv \mathbf{z}|_{\Omega_f} \in V,\ \mathbf{z}_s \equiv \mathbf{z}|_{\Omega_s} \in \mathbf{H}^1(\Omega_s),\ \text{and } \mathbf{z}_f|_{\Gamma_0} = \mathbf{z}_s|_{\Gamma_0}\}
$$

$$
\begin{aligned}
G^h = \{\mathbf{z} \in \mathbf{X}^h : \psi_f^h \equiv \psi^h|_{\Omega_f} \in \mathbf{X}_f^h,\ \psi_s^h \equiv \psi^h|_{\Omega_s} \in \mathbf{X}_s^h, \\
b(\psi_f^h, q^h) = 0 \quad \forall q^h \in S^h,\ \text{and } \psi_f^h|_{\Gamma_0} = \psi_s^h|_{\Gamma_0}\}
\end{aligned}
$$

We define $P^h : G \longrightarrow G^h$ to be the projection with respect to the $L^2(\Omega)$ inner product, i.e. $P^h\mathbf{z} = \check{\mathbf{z}}$ if

$$
(\check{\mathbf{z}}, \psi^h)_\Omega = (\mathbf{z}, \psi^h)_\Omega \quad \forall \psi \in G^h, \quad \forall \mathbf{z} \in G
$$

Then

$$\|\mathbf{z} - \check{\mathbf{z}}\|_{1,\Omega} \leq Ch^m \|\mathbf{z}\|_{m+1,\Omega} \qquad (2.6.43)$$

$$\|\mathbf{z} - \check{\mathbf{z}}\|_{0,\Omega} \leq Ch^{m+1} \|\mathbf{z}\|_{m+1,\Omega} \qquad (2.6.44)$$

Now we prove the following estimate for the error between the solutions of the semi-discrete and continuous problems.

Let

$$\zeta = \begin{cases} \mathbf{v} \text{ on } \Omega_f \\ \mathbf{u}_t \text{ on } \Omega_s \end{cases}$$

And define $\check{\mathbf{v}} = P^h \zeta|_{\Omega_f}$, $\check{\mathbf{u}}_t = P^h \zeta|_{\Omega_s}$,

Theorem 2.6.1

$$\begin{aligned} & \rho_f \|\mathbf{v} - \mathbf{v}^h\|^2_{0,\Omega_f} + K_a \int_0^t \|\mathbf{v} - \mathbf{v}^h\|^2_{1,\Omega_f} dt \\ & + \rho_s \|\mathbf{u}_t - \mathbf{u}^h_t\|^2_{0,\Omega_s} + K_a \|\mathbf{u} - \mathbf{u}^h\|^2_{1,\Omega_s} \\ \leq \; & Ch^{2m+2} \|\mathbf{v}\|^2_{m+1,\Omega_f} + Ch^{2m} \int_0^t \|\mathbf{v}\|^2_{m+1,\Omega_f} dt \\ & + Ch^{2m} \|\mathbf{u}\|^2_{m+1,\Omega_s} + Ch^{2m+2} \|\mathbf{u}_t\|^2_{m+1,\Omega_s} \\ & + Ch^{2m} \int_0^t \|\mathbf{u}\|^2_{m+1,\Omega_s} dt + Ch^{2m} \int_0^t \|p\|^2_{m,\Omega_f} dt \end{aligned} \qquad (2.6.45)$$

proof: By subtracting (2.5.33)-(2.5.34) from the corresponding equations of (2.3.7)-(2.3.8) we obtain the "orthogonality conditions".

$$\begin{aligned} & \rho_f (\mathbf{v}_t - \mathbf{v}^h_t, \mathbf{w}^h)_{\Omega_f} + b(\mathbf{w}^h, p - p^h) + a_f(\mathbf{v} - \mathbf{v}^h, \mathbf{w}^h) \\ + \; & \rho_s (\mathbf{u}_{tt} - \mathbf{u}^h_{tt}, \theta^h_t)_{\Omega_s} + a_s(\mathbf{u} - \mathbf{u}^h, \theta^h_t) = 0 \quad \forall (\mathbf{w}^h, \theta^h) \in U^h \end{aligned} \qquad (2.6.46)$$

$$b(\mathbf{v} - \mathbf{v}^h, q^h) = 0 \quad \forall q^h \in L^2(0,T;S^h) \qquad (2.6.47)$$

Using (2.6.46)-(2.6.47) we deduce that

$$\rho_f (\mathbf{v}_t - \mathbf{v}^h_t, \mathbf{v} - \mathbf{v}^h)_{\Omega_f} + a_f(\mathbf{v} - \mathbf{v}^h, \mathbf{v} - \mathbf{v}^h)$$

$$
\begin{aligned}
& +\rho_s(\mathbf{u}_{tt} - \mathbf{u}^h_{tt}, \mathbf{u}_t - \mathbf{u}^h_t)_{\Omega_s} + a_s(\mathbf{u} - \mathbf{u}^h, \mathbf{u}_t - \mathbf{u}^h_t) \\
= \quad & \rho_f(\mathbf{v}_t - \mathbf{v}^h_t, \mathbf{v} - \check{\mathbf{v}})_{\Omega_f} + a_f(\mathbf{v} - \mathbf{v}^h, \mathbf{v} - \check{\mathbf{v}}) + \rho_s(\mathbf{u}_{tt} - \mathbf{u}^h_{tt}, \mathbf{u}_t - \check{\mathbf{u}}_t)_{\Omega_s} \\
& +a_s(\mathbf{u} - \mathbf{u}^h, \mathbf{u}_t - \check{\mathbf{u}}_t) - b(\check{\mathbf{v}} - \mathbf{v}^h, p - p^h) \\
& +\rho_f(\mathbf{v}_t - \mathbf{v}^h_t, \check{\mathbf{v}} - \mathbf{v}^h)_{\Omega_f} + a_f(\mathbf{v} - \mathbf{v}^h, \check{\mathbf{v}} - \mathbf{v}^h) + \rho_s(\mathbf{u}_{tt} - \mathbf{u}^h_{tt}, \check{\mathbf{u}}_t - \mathbf{u}^h_t)_{\Omega_s} \\
& +a_s(\mathbf{u} - \mathbf{u}^h, \check{\mathbf{u}}_t - \mathbf{u}^h_t) + b(\check{\mathbf{v}} - \mathbf{v}^h, p - p^h) \\
= \quad & \rho_f(\mathbf{v}_t - \mathbf{v}^h_t, \mathbf{v} - \check{\mathbf{v}})_{\Omega_f} + a_f(\mathbf{v} - \mathbf{v}^h, \mathbf{v} - \check{\mathbf{v}}) + \rho_s(\mathbf{u}_{tt} - \mathbf{u}^h_{tt}, \mathbf{u}_t - \check{\mathbf{u}}_t)_{\Omega_s} \\
& +a_s(\mathbf{u} - \mathbf{u}^h, \mathbf{u}_t - \check{\mathbf{u}}_t) - b(\check{\mathbf{v}} - \mathbf{v}^h, p - p^h) \qquad (2.6.48)
\end{aligned}
$$

Then using the facts that $(\psi^h_f, \mathbf{v} - \check{\mathbf{v}}) = 0 \quad \forall \psi^h_f \in \mathbf{X}^h_f$ and that

$$
b(\check{\mathbf{v}}, q^h) = 0 = b(\mathbf{v}^h, q^h) \quad \forall q^h \in S^h \qquad (2.6.49)
$$

We obtain (by also noting that (2.6.49) implies $\check{\mathbf{v}}_t \in \mathbf{X}^h_f$ and $\mathbf{v}^h_t \in \mathbf{X}^h_f$)

$$
\rho_f(\mathbf{v}_t - \mathbf{v}^h_t, \mathbf{v} - \check{\mathbf{v}})_{\Omega_f} = \rho_f(\mathbf{v}_t, \mathbf{v} - \check{\mathbf{v}})_{\Omega_f} = \rho_f(\mathbf{v}_t - \check{\mathbf{v}}_t, \mathbf{v} - \check{\mathbf{v}})_{\Omega_f} \qquad (2.6.50)
$$

Similarly,

$$
\rho_s(\mathbf{u}_{tt} - \mathbf{u}^h_{tt}, \mathbf{u}_t - \check{\mathbf{u}}_t)_{\Omega_s} = \rho_s(\mathbf{u}_{tt} - \check{\mathbf{u}}_{tt}, \mathbf{u}_t - \check{\mathbf{u}}_t)_{\Omega_s} \qquad (2.6.51)
$$

Combining (2.6.48)-(2.6.51) we deduce that for all $q^h \in L^2(0, T; S^h)$

$$
\begin{aligned}
& \frac{\rho_f}{2}\frac{d}{dt}\|\mathbf{v} - \mathbf{v}^h\|^2_{0,\Omega_f} + K_a\|\mathbf{v} - \mathbf{v}^h\|^2_{1,\Omega_f} \\
& \quad +\frac{\rho_s}{2}\frac{d}{dt}\|\mathbf{u}_t - \mathbf{u}^h_t\|^2_{0,\Omega_s} + \frac{K_a}{2}\frac{d}{dt}\|\mathbf{u} - \mathbf{u}^h\|^2_{1,\Omega_s} \\
\leq \quad & \frac{\rho_f}{2}\frac{d}{dt}\|\mathbf{v} - \check{\mathbf{v}}\|^2_{0,\Omega_f} + k_a\|\mathbf{v} - \mathbf{v}^h\|_{1,\Omega_f}\|\mathbf{v} - \check{\mathbf{v}}\|_{1,\Omega_f} \\
& +\frac{\rho_s}{2}\frac{d}{dt}\|\mathbf{u}_t - \check{\mathbf{u}}_t\|^2_{0,\Omega_s} + k_a\|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega_s}\|\mathbf{u}_t - \check{\mathbf{u}}_t\|_{1,\Omega_s} \\
& +k_b\|\mathbf{v} - \check{\mathbf{v}}\|_{1,\Omega_f}\|p - q^h\|_{0,\Omega_f} + k_b\|\mathbf{v} - \mathbf{v}^h\|_{1,\Omega_f}\|p - q^h\|_{0,\Omega_f}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \rho_f\|\mathbf{v} - \mathbf{v}^h\|^2_{0,\Omega_f} + K_a\int_0^t \|\mathbf{v} - \mathbf{v}^h\|^2_{1,\Omega_f}dt + \rho_s\|\mathbf{u}_t - \mathbf{u}^h_t\|^2_{0,\Omega_s} + K_a\|\mathbf{u} - \mathbf{u}^h\|^2_{1,\Omega_s} \\
\leq \quad & C(\|\mathbf{v} - \check{\mathbf{v}}\|^2_{0,\Omega_f} + \int_0^t \|\mathbf{v} - \check{\mathbf{v}}\|^2_{1,\Omega_f}dt + \|\mathbf{u}(t_1) - \check{\mathbf{u}}(t_1)\|^2_{1,\Omega_s} \\
& +\|\mathbf{u}_t - \check{\mathbf{u}}_t\|^2_{0,\Omega_s} + \int_0^t \|\mathbf{u}_t - \check{\mathbf{u}}_t\|^2_{1,\Omega_s}dt + \int_0^t \|p - q^h\|^2_{0,\Omega_f}dt) \qquad (2.6.52)
\end{aligned}
$$

for all $q^h \in L^2(0,T;S^h)$, where $\|\mathbf{u}(t_1)-\check{\mathbf{u}}(t_1)\| = \max_{t\in[0,T]} \|\mathbf{u}(t)-\check{\mathbf{u}}(t)\|$. Hence (2.6.45) follows from (2.6.52), (2.6.43) and (2.6.44).

3 OPTIMIZATION-BASED DOMAIN DECOMPOSITION

In this chapter, we present an optimization-based domain decomposition method to uncouple the computations.

3.1 The model problems

Let $\mathbf{v}, p, \mathbf{g}, \mathbf{u}$ and $\mathbf{h}$ satisfy the coupled system (2.1.1). Instead of constraints (2.1.1), we consider the uncoupled system

$$\left\{ \begin{aligned} \rho_f \mathbf{v}_t + \nabla p - \mu_f \Delta \mathbf{v} = \rho_f \mathbf{f} &\quad \text{in } \Omega_f \\ \nabla \cdot \mathbf{v} = 0 &\quad \text{in } \Omega_f \\ \mathbf{v} = 0 &\quad \text{on } \Gamma_f \\ \mathbf{v} = \mathbf{g} &\quad \text{on } \Gamma_0 \\ \rho_s \mathbf{u}_{tt} - \mu_s \Delta \mathbf{u} - (\lambda_s + \mu_s) \nabla (\nabla \cdot \mathbf{u}) = \rho_s \mathbf{b} &\quad \text{in } \Omega_s \\ \mathbf{u} = 0 &\quad \text{on } \Gamma_s \\ \mu_s \nabla \mathbf{u} \cdot \mathbf{n} + (\lambda_s + \mu_s)(\nabla \cdot \mathbf{u}) \mathbf{n} = \mathbf{h} &\quad \text{on } \Gamma_0 \\ \mathbf{v}|_{t=0} = \mathbf{v}^0 &\quad \text{in } \Omega_f \\ \mathbf{u}|_{t=0} = \mathbf{u}^0 &\quad \text{in } \Omega_s \\ \mathbf{u}_t|_{t=0} = \mathbf{u}^1 &\quad \text{in } \Omega_s \end{aligned} \right. \tag{3.1.1}$$

In this paper, we refer to $\mathbf{g}$ and $\mathbf{h}$ as controls. Our goal is to find $\mathbf{g}$ and $\mathbf{h}$ such that the solutions of (3.1.1) coincide with solutions of (2.1.1). The optimization-based domain decomposition algorithm finds such $\mathbf{g}$ and $\mathbf{h}$ by minimizing the functional

$$\mathcal{J}(\mathbf{v}, p, \mathbf{g}, \mathbf{u}, \mathbf{h}) = \frac{1}{2}\int_0^T\int_{\Gamma_0} (\mathbf{u}_t - \mathbf{g})^2 \, d\Gamma \, dt + \frac{1}{2}\int_0^T\int_{\Gamma_0} (p\mathbf{n} - \mu_f \nabla\mathbf{v}\cdot\mathbf{n} - \mathbf{h})^2 d\Gamma \, dt$$

In order to regularize the optimization problem, we instead minimize the penalized functional

$$\begin{aligned}\mathcal{J}_\delta(\mathbf{v}, p, \mathbf{g}, \mathbf{u}, \mathbf{h}) &= \frac{1}{2}\int_0^T\int_{\Gamma_0} (\mathbf{u}_t - \mathbf{g})^2 \, d\Gamma \, dt \\ &+ \frac{1}{2}\int_0^T\int_{\Gamma_0} (p\mathbf{n} - \mu_f \nabla\mathbf{v}\cdot\mathbf{n} - \mathbf{h})^2 d\Gamma \, dt + \frac{\delta}{2}\int_0^T\int_{\Gamma_0} \mathbf{g}^2 \, d\Gamma \, dt \\ &+ \frac{\delta}{2}\int_0^T\int_{\Gamma_0} (\nabla_\Gamma \mathbf{g})^2 \, d\Gamma \, dt + \frac{\delta}{2}\int_0^T\int_{\Gamma_0} \mathbf{g}_t^2 \, d\Gamma \, dt + \frac{\delta}{2}\int_0^T\int_{\Gamma_0} \mathbf{h}^2 \, d\Gamma \, dt\end{aligned}$$

where the penalty parameter δ is a positive constant that can be chosen to change the relative importance of penalty terms in $\mathcal{J}_\delta$ and ∇_Γ denotes tangential gradient. Thus the optimization problem we propose to solve is given by

Problem 1

Find ($\mathbf{v}$,p,$\mathbf{g}$,$\mathbf{u}$,$\mathbf{h}$) such that the functional $\mathcal{J}_\delta$ is minimized subject to (3.1.1).

3.2 The existence of an optimal solution

We recall the definition of the spaces

$$\begin{aligned}\mathbf{H}^1_f(\Omega_f) &= \{\mathbf{v} \in \mathbf{H}^1(\Omega_f) : \ \mathbf{v} = 0 \text{ on } \Gamma_f\} \\ J &= \{\mathbf{v} \in \mathcal{D}(\Omega_f) : \ \nabla\cdot\mathbf{v} = 0 \text{ on } \Omega_f\} \\ V &= \text{ the closure of } J \text{ in } \mathbf{H}^1_f(\Omega_f) \\ H &= \text{ the closure of } J \text{ in } \mathbf{L}^2(\Omega_f)\end{aligned}$$

and

$$\mathbf{H}^1_s(\Omega_s) = \{\mathbf{u} \in \mathbf{H}^1(\Omega_s) : \ \mathbf{u} = 0 \text{ on } \Gamma_s\}$$

For functions defined on Γ_0, we will use the subspaces

$$\begin{aligned} Y &= \{\mathbf{g} \in L^2(0,T;\mathbf{H}^1(\Gamma_0)) \cap H^1(0,T;\mathbf{L}^2(\Gamma_0)) : \ \mathbf{g} = 0 \text{ on } \partial\Gamma_0\} \\ Z &= L^2(0,T;\mathbf{L}^2(\Gamma_0)) \end{aligned}$$

with norms

$$\begin{aligned} \|\mathbf{g}\|_Y^2 &= \int_0^T\int_{\Gamma_0} \mathbf{g}^2 \, d\Gamma \, dt + \int_0^T\int_{\Gamma_0} (\nabla_\Gamma \mathbf{g})^2 \, d\Gamma \, dt + \int_0^T\int_{\Gamma_0} \mathbf{g}_t^2 \, d\Gamma \, dt \\ \|\mathbf{h}\|_Z^2 &= \int_0^T\int_{\Gamma_0} \mathbf{h}^2 \, d\Gamma \, dt \end{aligned}$$

A weak formulation corresponding to (3.1.1) is given by : Seek $\mathbf{v} \in L^2(0,T;\mathbf{H}^1_f(\Omega_f)),\ p \in L^2(0,T;L^2(\Omega_f)),\ \mathbf{g} \in Y,\ \mathbf{u} \in L^2(0,T;\mathbf{H}^1_s(\Omega_s)) \cap H^1(0,T;\mathbf{L}^2(\Omega_s))$ and $\mathbf{h} \in Z$ satisfying

$$\rho_f(\mathbf{v}_t,\mathbf{w})_{\Omega_f} + b(\mathbf{w},p) + a_f(\mathbf{v},\mathbf{w}) + (\mathbf{w}, p\mathbf{n} - \mu_f \nabla\mathbf{v}\cdot\mathbf{n})_{\Gamma_0}$$

$$= \rho_f(\mathbf{f},\mathbf{w})_{\Omega_f} \quad \forall\ \mathbf{w} \in \mathbf{H}^1_f(\Omega_f) \tag{3.2.2}$$

$$b(\mathbf{v},q) = 0 \quad \forall\ q \in L^2(\Omega_f) \tag{3.2.3}$$

$$(\mathbf{v},\mathbf{s})_{\Gamma_0} - (\mathbf{g},\mathbf{s})_{\Gamma_0} = 0 \quad \forall\ \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma_0) \tag{3.2.4}$$

$$\rho_s(\mathbf{u}_{tt},\theta)_{\Omega_s} + a_s(\mathbf{u},\theta) = \rho_s(\mathbf{b},\theta)_{\Omega_s} + (\mathbf{h},\theta)_{\Gamma_0}$$

$$\forall\ \theta \in \mathbf{H}^1_s(\Omega_s) \tag{3.2.5}$$

$$\mathbf{v}|_{t=0} = \mathbf{v}^0 \quad \text{in } \Omega_f \tag{3.2.6}$$

$$\mathbf{u}|_{t=0} = \mathbf{u}^0 \quad \text{in } \Omega_s \tag{3.2.7}$$

$$\mathbf{u}_t|_{t=0} = \mathbf{u}^1 \quad \text{in } \Omega_s \tag{3.2.8}$$

Next, we give a precise definition of an optimal solution. Let the admissibility set is defined by

$$\begin{aligned} \mathcal{U}_{ad} = \{(\mathbf{v},p,\mathbf{g},\mathbf{u},\mathbf{h}) &\in L^2(0,T;\mathbf{H}^1_f(\Omega_f)) \times L^2(0,T;L^2(\Omega_f)) \times \\ & Y \times L^2(0,T;\mathbf{H}^1_s(\Omega_s)) \cap H^1(0,T;\mathbf{L}^2(\Omega_s)) \times Z \ : \\ & \mathcal{J}_\delta(\mathbf{v},p,\mathbf{g},\mathbf{u},\mathbf{h}) < \infty \text{ and (3.2.2)-(3.2.8) is satisfied } \} \end{aligned} \tag{3.2.9}$$

Then $(\hat{\mathbf{v}}, \hat{p}, \hat{\mathbf{g}}, \hat{\mathbf{u}}, \hat{\mathbf{h}})$ is called an optimal solution if there exists $\alpha \geq 0$ such that

$$\mathcal{J}_\delta(\hat{\mathbf{v}}, \hat{p}, \hat{\mathbf{g}}, \hat{\mathbf{u}}, \hat{\mathbf{h}}) \leq \mathcal{J}_\delta(\mathbf{v}, p, \mathbf{g}, \mathbf{u}, \mathbf{h})$$

for all $(\mathbf{v}, p, \mathbf{g}, \mathbf{u}, \mathbf{h}) \in \mathcal{U}_{ad}$ satisfying

$$\|\hat{\mathbf{g}} - \mathbf{g}\|_Y + \|\hat{\mathbf{h}} - \mathbf{h}\|_Z \leq \alpha$$

To show the existence of optimal solutions, we introduce an auxiliary problem again.

$$\begin{cases} \rho_f \mathbf{v}_t + \nabla p - \mu_f \Delta \mathbf{v} = \rho_f \mathbf{f} \quad \text{in } \Omega_f \\ \nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega_f \\ \mathbf{v} = 0 \quad \text{on } \Gamma_f \\ \mathbf{v} = \mathbf{g} \quad \text{on } \Gamma_0 \\ \rho_s \mathbf{u}_{tt} - \epsilon\mu_s \Delta \mathbf{u}_t - \mu_s \Delta \mathbf{u} - (\lambda_s + \mu_s)\nabla(\nabla \cdot \mathbf{u}) = \rho_s \mathbf{b} \quad \text{in } \Omega_s \\ \mathbf{u} = 0 \quad \text{on } \Gamma_s \\ \epsilon\mu_s \nabla \mathbf{u}_t \cdot \mathbf{n} + \mu_s \nabla \mathbf{u} \cdot \mathbf{n} + (\lambda_s + \mu_s)(\nabla \cdot \mathbf{u})\mathbf{n} = \mathbf{h} \quad \text{on } \Gamma_0 \\ \mathbf{v}|_{t=0} = \mathbf{v}^0 \quad \text{in } \Omega_f \\ \mathbf{u}|_{t=0} = \mathbf{u}^0 \quad \text{in } \Omega_s \\ \mathbf{u}_t|_{t=0} = \mathbf{u}^1 \quad \text{in } \Omega_s \end{cases} \tag{3.2.10}$$

Problem 2

Find ($\mathbf{v}$,p,$\mathbf{g}$,$\mathbf{u}$,$\mathbf{h}$) such that the functional $\mathcal{J}_\delta$ is minimized subject to (3.2.10).

A weak formulation corresponding to (3.2.10) is given by: Seek $\mathbf{v} \in L^2(0, T; \mathbf{H}^1_f(\Omega_f))$, $p \in L^2(0, T; L^2(\Omega_f))$, $\mathbf{g} \in Y$, $\mathbf{u} \in H^1(0, T; \mathbf{H}^1_s(\Omega_s))$ and $\mathbf{h} \in Z$ satisfying

$$\begin{aligned} &\rho_f(\mathbf{v}_t, \mathbf{w})_{\Omega_f} + b(\mathbf{w}, p) + a_f(\mathbf{v}, \mathbf{w}) + (\mathbf{w}, p\mathbf{n} - \mu_f \nabla \mathbf{v} \cdot \mathbf{n})_{\Gamma_0} \\ = \;& \rho_f(\mathbf{f}, \mathbf{w})_{\Omega_f} \quad \forall\, \mathbf{w} \in \mathbf{H}^1_f(\Omega_f) \end{aligned} \tag{3.2.11}$$

$$b(\mathbf{v}, q) = 0 \quad \forall\, q \in L^2(\Omega_f) \tag{3.2.12}$$

$$(\mathbf{v}, \mathbf{s})_{\Gamma_0} - (\mathbf{g}, \mathbf{s})_{\Gamma_0} = 0 \quad \forall\, \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma_0) \tag{3.2.13}$$

$$\begin{aligned} &\rho_s(\mathbf{u}_{tt}, \theta)_{\Omega_s} + \epsilon a(\mathbf{u}_t, \theta) + a_s(\mathbf{u}, \theta) \\ = \quad &\rho_s(\mathbf{b}, \theta)_{\Omega_s} + (\mathbf{h}, \theta)_{\Gamma_0} \quad \forall\, \theta \in \mathbf{H}^1_s(\Omega_s) \end{aligned} \tag{3.2.14}$$

$$\mathbf{v}|_{t=0} = \mathbf{v}^0 \quad \text{in } \Omega_f \tag{3.2.15}$$

$$\mathbf{u}|_{t=0} = \mathbf{u}^0 \quad \text{in } \Omega_s \tag{3.2.16}$$

$$\mathbf{u}_t|_{t=0} = \mathbf{u}^1 \quad \text{in } \Omega_s \tag{3.2.17}$$

And admissibility set be defined by

$$\begin{aligned} \mathcal{U}^\epsilon_{ad} = \{ &(\mathbf{v}, p, \mathbf{g}, \mathbf{u}, \mathbf{h}) \in L^2(0, T; \mathbf{H}^1_f(\Omega_f)) \times L^2(0, T; L^2(\Omega_f)) \times \\ &Y \times H^1(0, T; \mathbf{H}^1_s(\Omega_s)) \times Z \ : \ \mathcal{J}_\delta(\mathbf{v}, p, \mathbf{g}, \mathbf{u}, \mathbf{h}) < \infty \\ &\text{and (3.2.11)-(3.2.17) is satisfied } \} \end{aligned} \tag{3.2.18}$$

Using the properties of the bilinear forms we can obtain an a priori bounds for solutions of the weak formulation (3.2.11)-(3.2.17). Let $(\mathbf{v}, p, \mathbf{g}, \mathbf{u}, \mathbf{h})$ satisfy (3.2.11)-(3.2.17). From (3.2.11) and (3.2.14), we obtain

$$\begin{aligned} &\rho_f(\mathbf{v}_t, \mathbf{w})_{\Omega_f} + b(\mathbf{w}, p) + a_f(\mathbf{v}, \mathbf{w}) \\ &\quad + \rho_s(\mathbf{u}_{tt}, \theta)_{\Omega_s} + \epsilon a(\mathbf{u}_t, \theta) + a_s(\mathbf{u}, \theta) \\ = \quad &\rho_f(\mathbf{f}, \mathbf{w})_{\Omega_f} + \rho_s(\mathbf{b}, \theta)_{\Omega_s} - (\mathbf{w}, p\mathbf{n} - \mu_f \nabla \mathbf{v} \cdot \mathbf{n})_{\Gamma_0} + (\mathbf{h}, \theta)_{\Gamma_0} \\ &\quad \forall\, \mathbf{w} \in \mathbf{H}^1_f(\Omega_f), \ \forall\, \theta \in \mathbf{H}^1_s(\Omega_s) \end{aligned}$$

or

$$\begin{aligned} &\rho_f(\mathbf{v}_t, \mathbf{w})_{\Omega_f} + b(\mathbf{w}, p) + a_f(\mathbf{v}, \mathbf{w}) \\ &\quad + \rho_s(\mathbf{u}_{tt}, \theta)_{\Omega_s} + \epsilon a(\mathbf{u}_t, \theta) + a_s(\mathbf{u}, \theta) \\ = \quad &\rho_f(\mathbf{f}, \mathbf{w})_{\Omega_f} + \rho_s(\mathbf{b}, \theta)_{\Omega_s} - (\mathbf{w}, p\mathbf{n} - \mu_f \nabla \mathbf{v} \cdot \mathbf{n})_{\Gamma_0} \\ &\quad + (\mathbf{w}, \mathbf{h})_{\Gamma_0} + (\mathbf{h}, \theta)_{\Gamma_0} - (\mathbf{w}, \mathbf{h})_{\Gamma_0} \end{aligned}$$

$$\forall\ \mathbf{w} \in \mathbf{H}^1_f(\Omega_f),\ \forall\ \theta \in \mathbf{H}^1_s(\Omega_s) \tag{3.2.19}$$

Taking $\mathbf{w} = \mathbf{v}$ and $\theta = \mathbf{u}_t$ in (3.2.19). Then because of (3.2.12)-(3.2.13) we have

$$\begin{aligned}
&\rho_f(\mathbf{v}_t, \mathbf{v})_{\Omega_f} + a_f(\mathbf{v}, \mathbf{v}) + \rho_s(\mathbf{u}_{tt}, \mathbf{u}_t)_{\Omega_s} + \epsilon a(\mathbf{u}_t, \mathbf{u}_t) + a_s(\mathbf{u}, \mathbf{u}_t) \\
&= \rho_f(\mathbf{f}, \mathbf{v})_{\Omega_f} + \rho_s(\mathbf{b}, \mathbf{u}_t)_{\Omega_s} - (\mathbf{g}, p\mathbf{n} - \mu_f \nabla\mathbf{v}\cdot\mathbf{n})_{\Gamma_0} + (\mathbf{g}, \mathbf{h})_{\Gamma_0} \\
&\quad +(\mathbf{h}, \mathbf{u}_t)_{\Gamma_0} - (\mathbf{g}, \mathbf{h})_{\Gamma_0}
\end{aligned}$$

This may be reduced to

$$\begin{aligned}
&\frac{\rho_f}{2}\frac{d}{dt}\|\mathbf{v}\|^2_{0,\Omega_f} + K_a\|\mathbf{v}\|^2_{1,\Omega_f} + \frac{\rho_s}{2}\frac{d}{dt}\|\mathbf{u}_t\|^2_{0,\Omega_s} + \epsilon K_a\|\mathbf{u}_t\|^2_{1,\Omega_s} + \frac{K_a}{2}\frac{d}{dt}\|\mathbf{u}\|^2_{1,\Omega_s} \\
&= -(\mathbf{g}, p\mathbf{n} - \mu_f\nabla\mathbf{v}\cdot\mathbf{n} - \mathbf{h})_{\Gamma_0} + (\mathbf{h}, \mathbf{u}_t - \mathbf{g})_{\Gamma_0} + \rho_f(\mathbf{f}, \mathbf{v})_{\Omega_f} + \rho_s(\mathbf{b}, \mathbf{u}_t)_{\Omega_s} \\
&\le \|\mathbf{g}\|_{0,\Gamma_0}\|p\mathbf{n} - \mu_f\nabla\mathbf{v}\cdot\mathbf{n} - \mathbf{h}\|_{0,\Gamma_0} + \|\mathbf{h}\|_{0,\Gamma_0}\|\mathbf{u}_t - \mathbf{g}\|_{0,\Gamma_0} \\
&\quad +\rho_f\|\mathbf{f}\|_{-1,\Omega_f}\|\mathbf{v}\|_{1,\Omega_f} + \rho_s\|\mathbf{b}\|_{0,\Omega_s}\|\mathbf{u}_t\|_{0,\Omega_s}
\end{aligned}$$

Integrating from 0 to t yields

$$\begin{aligned}
&\rho_f \sup_t \|\mathbf{v}\|^2_{0,\Omega_f} + K_a\int_0^T \|\mathbf{v}\|^2_{1,\Omega_f}dt + \rho_s\sup_t\|\mathbf{u}_t\|^2_{0,\Omega_s} \\
&\quad +\epsilon K_a\int_0^T\|\mathbf{u}_t\|^2_{1,\Omega_s}dt + K_a\sup_t\|\mathbf{u}\|^2_{1,\Omega_s} \\
&\le \int_0^T\|\mathbf{g}\|^2_{0,\Gamma_0}dt + \int_0^T\|p\mathbf{n} - \mu_f\nabla\mathbf{v}\cdot\mathbf{n} - \mathbf{h}\|^2_{0,\Gamma_0}dt + \int_0^T\|\mathbf{h}\|^2_{0,\Gamma_0}dt \\
&\quad + \int_0^T\|\mathbf{u}_t - \mathbf{g}\|^2_{0,\Gamma_0}dt + \rho_f C\int_0^T\|\mathbf{f}\|^2_{-1,\Omega_f}dt + \rho_s C\int_0^T\|\mathbf{b}\|^2_{0,\Omega_s}dt \\
&\quad +\rho_f\|\mathbf{v}^0\|^2_{0,\Omega_f} + \rho_s\|\mathbf{u}^1\|^2_{0,\Omega_s} + K_a\|\mathbf{u}^0\|^2_{1,\Omega_s}
\end{aligned} \tag{3.2.20}$$

We take test functions $\mathbf{w}$ with $\nabla\cdot\mathbf{w} = 0$ in equation (3.2.11), we get

$$\rho_f(\mathbf{v}_t, \mathbf{w})_{\Omega_f} = -a_f(\mathbf{v}, \mathbf{w}) - (\mathbf{w}, p\mathbf{n} - \mu_f\nabla\mathbf{v}\cdot\mathbf{n})_{\Gamma_0} + \rho_f(\mathbf{f}, \mathbf{w})_{\Omega_f} \quad \forall\mathbf{w}\in V$$

and this gives

$$\begin{aligned}
&|\rho_f(\mathbf{v}_t, \mathbf{w})_{\Omega_f}| \le k_a\|\mathbf{v}\|_{1,\Omega_f}\|\mathbf{w}\|_{1,\Omega_f} \\
&\quad + \|\mathbf{w}\|_{1,\Omega_f}\|p\mathbf{n} - \mu_f\nabla\mathbf{v}\cdot\mathbf{n}\|_{0,\Gamma_0} + \rho_f\|\mathbf{f}\|_{-1,\Omega_f}\|\mathbf{w}\|_{1,\Omega_f}
\end{aligned}$$

Therefore,

$$\rho_f \int_0^T \|\mathbf{v}_t\|^2_{V^*} dt \le k_a \int_0^T \|\mathbf{v}\|^2_{1,\Omega_f} dt + \int_0^T \|p\mathbf{n} - \mu_f \nabla \mathbf{v}\cdot \mathbf{n} - \mathbf{h}\|^2_{0,\Gamma_0} dt$$
$$+ \int_0^T \|\mathbf{h}\|^2_{0,\Gamma_0} dt + \rho_f \int_0^T \|\mathbf{f}\|^2_{-1,\Omega_f} dt \qquad (3.2.21)$$

From (3.2.11),

$$b(\mathbf{w}, p) = -\rho_f(\mathbf{v}_t, \mathbf{w})_{\Omega_f} - a_f(\mathbf{v}, \mathbf{w})$$
$$- \quad (\mathbf{w}, p\mathbf{n} - \mu_f \nabla \mathbf{v}\cdot\mathbf{n})_{\Gamma_0} + \rho_f(\mathbf{f}, \mathbf{w})_{\Omega_f} \quad \forall \mathbf{w} \in \mathbf{H}^1_f(\Omega_f)$$

Inf-sup condition may be used to get,

$$K_b \int_0^T \|p\|^2_{0,\Omega_f} dt \le \rho_f \int_0^T \|\mathbf{v}_t\|^2_{V^*} dt + k_a \int_0^T \|\mathbf{v}\|^2_{1,\Omega_f} dt$$
$$+ \quad \int_0^T \|p\mathbf{n} - \mu_f \nabla \mathbf{v}\cdot \mathbf{n}\|^2_{0,\Gamma_0} dt + \rho_f \int_0^T \|\mathbf{f}\|^2_{-1,\Omega_f} dt \qquad (3.2.22)$$

Theorem 3.2.1 *There exists an optimal solution* $(\hat{\mathbf{v}}, \hat{p}, \hat{\mathbf{g}}, \hat{\mathbf{u}}, \hat{\mathbf{h}}) \in \mathcal{U}^{\epsilon}_{ad}$ *for Problem 2.*

proof: It is clear that $\mathcal{U}^{\epsilon}_{ad}$ is not empty. Let $(\mathbf{v}^n, p^n, \mathbf{g}^n, \mathbf{u}^n, \mathbf{h}^n)$ be a minimizing sequence in $\mathcal{U}_{ad}$. i.e.

$$\lim_{n\to\infty} \mathcal{J}_\delta(\mathbf{v}^n, p^n, \mathbf{g}^n, \mathbf{u}^n, \mathbf{h}^n) = \inf_{(\mathbf{v},p,\mathbf{g},\mathbf{u},\mathbf{h})\in\mathcal{U}_{ad}} \mathcal{J}_\delta(\mathbf{v}, p, \mathbf{g}, \mathbf{u}, \mathbf{h})$$

Thus from (3.2.18), we have that $\|\mathbf{g}^n\|$ and $\|\mathbf{h}^n\|$ are uniformly bounded in $Y \times Z$. Then since by (3.2.20) and (3.2.22) $(\mathbf{v}^n, p^n, \mathbf{u}^n) \in L^2(0,T;\mathbf{H}^1_f(\Omega_f)) \times L^2(0,T;L^2(\Omega_f)) \times H^1(0,T;\mathbf{H}^1_s(\Omega_s))$ is uniformly bounded. Thus, there exist subsequences, denoted by $(\mathbf{v}^n, p^n, \mathbf{g}^n, \mathbf{u}^n, \mathbf{h}^n)$ for simplicity, such that

$$\mathbf{v}_n \rightharpoonup \mathbf{v} \text{ in } L^2(0,T;\mathbf{H}^1_f(\Omega_f))$$

$$p_n \rightharpoonup p \text{ in } L^2(0,T;\text{Ł}^2(\Omega_f))$$

$$\mathbf{g}_n \rightharpoonup \mathbf{g} \text{ in } Y$$

$$\mathbf{u}_n \rightharpoonup \mathbf{u} \text{ in } H^1(0,T;\mathbf{H}^1_s(\Omega_s))$$

$$\mathbf{h}_n \rightharpoonup \mathbf{h} \text{ in } Z$$

for some $(\hat{\mathbf{v}}, \hat{p}, \hat{\mathbf{g}}, \hat{\mathbf{u}}, \hat{\mathbf{h}}) \in \mathcal{U}_{ad}^{\epsilon}$.

Now, by passing to the limit, we have that $(\hat{\mathbf{v}}, \hat{p}, \hat{\mathbf{g}}, \hat{\mathbf{u}}, \hat{\mathbf{h}}) \in \mathcal{U}_{ad}^{\epsilon}$ satisfies (3.2.11)-(3.2.17). Then by the fact that functional $\mathcal{J}_\delta(\cdot,\cdot,\cdot,\cdot,\cdot)$ is lower semi-continuous, we conclude that

$$\mathcal{J}_\delta(\hat{\mathbf{v}}, \hat{p}, \hat{\mathbf{g}}, \hat{\mathbf{u}}, \hat{\mathbf{h}}) = \inf_{(\mathbf{v},p,\mathbf{g},\mathbf{u},\mathbf{h})\in\mathcal{U}_{ad}^{\epsilon}} \mathcal{J}_\delta(\mathbf{v}, p, \mathbf{g}, \mathbf{u}, \mathbf{h})$$

i.e. $(\hat{\mathbf{v}}, \hat{p}, \hat{\mathbf{g}}, \hat{\mathbf{u}}, \hat{\mathbf{h}})$ is an optimal solution.

Now we take the limit as $\epsilon \to 0$ of optimal solutions of Problem 2 to get optimal solutions of Problem 1.

Theorem 3.2.2 *There exists an optimal solution* $(\hat{\mathbf{v}}, \hat{p}, \hat{\mathbf{g}}, \hat{\mathbf{u}}, \hat{\mathbf{h}}) \in \mathcal{U}_{ad}$ *for Problem 1.*

The proof is the same as in the Theorem 2.2.

3.3 Convergence with vanishing penalty parameter

In the next theorem we show that optimal solutions of Problem 2 converges to the weak solution of (2.4.16) as $\delta \to 0$.

Theorem 3.3.1 *For each* δ*, let* $(\mathbf{v}^\delta, p^\delta, \mathbf{g}^\delta, \mathbf{u}^\delta, \mathbf{h}^\delta)$ *denote an optimal solution of Problem 2 and* $(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}})$ *denote a solution of (2.4.22)-(2.4.26). Then* $\|\tilde{\mathbf{v}} - \mathbf{v}^\delta\|_{1,\Omega_f} \to 0$*,* $\|\tilde{p} - p^\delta\|_{0,\Omega_f} \to 0$*,* $\|\tilde{\mathbf{u}} - \mathbf{u}^\delta\|_{1,\Omega_s} \to 0$*,* $\|\tilde{\mathbf{u}}_t - \mathbf{u}_t^\delta\|_{1,\Omega_s} \to 0$ *as* $\delta \to 0$*.*

proof: Let $\tilde{\mathbf{g}} = \tilde{\mathbf{v}}|_{\Gamma_0}$ and $\tilde{\mathbf{h}} = \epsilon\mu_s \nabla\tilde{\mathbf{u}}_t \cdot \mathbf{n} + \mu_s \nabla\tilde{\mathbf{u}} \cdot \mathbf{n} + (\lambda_s + \mu_s)(\nabla \cdot \tilde{\mathbf{u}})\mathbf{n}$ on Γ_0. For any solution $(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}})$ of (2.4.16), we have

$$\mathcal{J}_\delta(\mathbf{v}^\delta, p^\delta, \mathbf{g}^\delta, \mathbf{u}^\delta, \mathbf{h}^\delta) \le \mathcal{J}_\delta(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{g}}, \tilde{\mathbf{u}}, \tilde{\mathbf{h}})$$

i.e.

$$\begin{aligned} &\frac{1}{2}\int_0^T\int_{\Gamma_0} (\mathbf{u}_t^\delta - \mathbf{g})^2 \, d\Gamma \, dt + \frac{1}{2}\int_0^T\int_{\Gamma_0} (p^\delta \mathbf{n} - \mu_f \nabla\mathbf{v}^\delta \cdot \mathbf{n} - \mathbf{h}^\delta)^2 d\Gamma \, dt \\ + &\frac{\delta}{2}\int_0^T\int_{\Gamma_0} \|\mathbf{g}^\delta\|_Y^2 d\Gamma \, dt + \frac{\delta}{2}\int_0^T\int_{\Gamma_0} \|\mathbf{h}^\delta\|_Z^2 d\Gamma \, dt \\ \le &\mathcal{J}(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{g}}, \tilde{\mathbf{u}}, \tilde{\mathbf{h}}) + \frac{\delta}{2}\int_0^T\int_{\Gamma_0} \|\tilde{\mathbf{g}}\|_Y^2 d\Gamma \, dt + \frac{\delta}{2}\int_0^T\int_{\Gamma_0} \|\tilde{\mathbf{h}}\|_Z^2 d\Gamma \, dt \quad \forall\delta \end{aligned}$$

Then $\|\mathbf{g}^\delta\|_Y$ and $\|\mathbf{h}^\delta\|_Z$ are uniformly bounded, $\|\mathbf{u}_t^\delta - \mathbf{g}\|_{0,\Gamma_0} \to 0$ and $\|p^\delta\mathbf{n} - \mu_f\nabla\mathbf{v}^\delta\cdot\mathbf{n} - \mathbf{h}^\delta\|_{0,\Gamma_0} \to 0$ as $\delta \to 0$. We then obtain $\|\mathbf{v}^\delta\|_{1,\Omega_f}, \|p^\delta\|_{0,\Omega_f}, \|\mathbf{u}^\delta\|_{1,\Omega_s}$ and $\|\mathbf{u}_t^\delta\|_{1,\Omega_s}$ are uniformly bounded by (3.2.20) and (3.2.22). Hence, as $\delta \to 0$, there exists a subsequence of $\{(\mathbf{v}^\delta, p^\delta, \mathbf{g}^\delta, \mathbf{u}^\delta, \mathbf{h}^\delta)\}$ that converges to some $(\check{\mathbf{v}}, \check{p}, \check{\mathbf{g}}, \check{\mathbf{u}}, \check{\mathbf{h}}) \in L^2(0,T;\mathbf{H}_f^1(\Omega_f)) \times L^2(0,T;L^2(\Omega_f)) \times Y \times H^1(0,T;\mathbf{H}_s^1(\Omega_s)) \times Z$. The fact that $\|\mathbf{u}_t^\delta - \mathbf{g}\|_{0,\Gamma_0} \to 0$ yields $\check{\mathbf{v}} = \check{\mathbf{u}}_t$ on Γ_0, also $\|p^\delta\mathbf{n} - \mu_f\nabla\mathbf{v}^\delta\cdot\mathbf{n} - \mathbf{h}^\delta\|_{0,\Gamma_0} \to 0$ as $\delta \to 0$ implies that $\epsilon\mu_s\nabla\check{\mathbf{u}}_t\cdot\mathbf{n} + \mu_s\nabla\check{\mathbf{u}}\cdot\mathbf{n} + (\lambda_s+\mu_s)(\nabla\cdot\check{\mathbf{u}})\mathbf{n} = \check{p}\mathbf{n} - \mu_f\nabla\check{\mathbf{v}}\cdot\mathbf{n}$ on Γ_0.

By passing to the limit, we have that $(\check{\mathbf{v}}, \check{p}, \check{\mathbf{u}})$ satisfy (2.4.22)-(2.4.26). By the uniqueness of solutions of (2.4.22)-(2.4.26), $(\check{\mathbf{v}}, \check{p}, \check{\mathbf{u}}) = (\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}})$ and hence theorem is proved.

The following theorem is achieved due to Theorem 2.4.2, Theorem 3.2.2 and Theorem 3.3.1.

Theorem 3.3.2 *For each δ, let $(\mathbf{v}^\delta, p^\delta, \mathbf{g}^\delta, \mathbf{u}^\delta, \mathbf{h}^\delta)$ denote an optimal solution of Problem 1 and $(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}})$ denote a solution of (2.3.7)-(2.3.11). Then $\|\tilde{\mathbf{v}} - \mathbf{v}^\delta\|_{1,\Omega_f} \to 0$, $\|\tilde{p} - p^\delta\|_{0,\Omega_f} \to 0$, $\|\tilde{\mathbf{u}} - \mathbf{u}^\delta\|_{1,\Omega_s} \to 0$, $\|\tilde{\mathbf{u}}_t - \mathbf{u}_t^\delta\|_{1,\Omega_s} \to 0$ as $\delta \to 0$.*

3.4 Optimality system

We use the Lagrange multiplier rule to derive the first order necessary conditions that optimal solutions must satisfy.

Let $B_1 = L^2(0,T;H_f^1(\Omega_f)) \times L^2(0,T;L^2(\Omega_f)) \times Y \times H^1(0,T;H_s^1(\Omega_s)) \times Z$ and $B_2 = L^2(0,T;H_f^1(\Omega_f))^* \times L^2(0,T;H_f^1(\Omega_f)) \times Y \times H^1(0,T;H_s^1(\Omega_s))^*$.
where, $(\cdot)$ denotes the dual space. Suppose the linear operator $M : B_1 \to B_2$ denotes the constraint operator, i.e., $M : B_1 \to B_2$ is defined by $M(\mathbf{v}, p, \mathbf{g}, \mathbf{u}, \mathbf{h}) = (\mathbf{f}, \mathbf{z}, \mathbf{d}, \mathbf{b})$ if and only if

$$\begin{aligned}
&\int_0^T [\rho_f(\mathbf{v}_t, \mathbf{w})_{\Omega_f} + b(\mathbf{w}, p) + a_f(\mathbf{v}, \mathbf{w}) + (\mathbf{w}, p\mathbf{n} - \mu_f\nabla\mathbf{v}\cdot\mathbf{n})_{\Gamma_0}]\, dt \\
&= \int_0^T \rho_f(\mathbf{f}, \mathbf{w})_{\Omega_f}\, dt \quad \forall\ \mathbf{w} \in H_f^1(\Omega_f)
\end{aligned}$$

$$\int_0^T b(\mathbf{v},q)\,dt = \int_0^T b(\mathbf{z},q)\,dt \quad \forall\, q \in L^2(\Omega_f)$$
$$\int_0^T [(\mathbf{v},\mathbf{s})_{\Gamma_0} - (\mathbf{g},\mathbf{s})_{\Gamma_0}]\,dt = \int_0^T (\mathbf{d},\mathbf{s})_{\Gamma_0}\,dt \quad \forall\, \mathbf{s} \in H^{-1/2}(\Gamma_0)$$

and

$$\int_0^T [\rho_s(\mathbf{u}_{tt},\theta)_{\Omega_s} + \epsilon a(\mathbf{u}_t,\theta) + a_s(\mathbf{u},\theta) - (\mathbf{h},\theta)_{\Gamma_0}]\,dt$$
$$= \int_0^T \rho_s(\mathbf{b},\theta)_{\Omega_s}\,dt \quad \forall\, \theta \in H^1_s(\Omega_s)$$

Note that the constraint equations (3.2.11)-(3.2.14) can simply be expressed as $M(\mathbf{v},p,\mathbf{g},\mathbf{u},\mathbf{h}) = (\mathbf{f},0,0,\mathbf{b})$

The operator $M'(\mathbf{v},p,\mathbf{g},\mathbf{u},\mathbf{h}) \in \mathcal{L}(B_1,B_2)$ may be defined as follows:

$M'(\mathbf{v},p,\mathbf{g},\mathbf{u},\mathbf{h}) \cdot (\mathbf{\Lambda},y,\mathbf{k},\chi,\mathbf{l}) = (\bar{\mathbf{f}},\bar{\mathbf{z}},\bar{\mathbf{d}},\bar{\mathbf{b}})$ if and only if

$$\int_0^T [\rho_f(\mathbf{\Lambda}_t,\mathbf{w})_{\Omega_f} + b(\mathbf{w},y) + a_f(\mathbf{\Lambda},\mathbf{w}) + (\mathbf{w}, y\mathbf{n} - \mu_f \nabla\mathbf{\Lambda}\cdot\mathbf{n})_{\Gamma_0}]\,dt$$
$$= \int_0^T \rho_f(\bar{\mathbf{f}},\mathbf{w})_{\Omega_f}\,dt \quad \forall\, \mathbf{w} \in H^1_f(\Omega_f)$$
$$\int_0^T b(\mathbf{\Lambda},q)\,dt = \int_0^T b(\bar{\mathbf{z}},q)\,dt \quad \forall\, q \in L^2(\Omega_f)$$
$$\int_0^T [(\mathbf{\Lambda},\mathbf{s})_{\Gamma_0} - (\mathbf{k},\mathbf{s})_{\Gamma_0}]\,dt = \int_0^T (\bar{\mathbf{d}},\mathbf{s})_{\Gamma_0}\,dt \quad \forall\, \mathbf{s} \in H^{-1/2}(\Gamma_0)$$

and

$$\int_0^T [\rho_s(\chi_{tt},\theta)_{\Omega_s} + \epsilon a(\chi_t,\theta) + a_s(\chi,\theta) - (\mathbf{l},\theta)_{\Gamma_0}]\,dt$$
$$= \int_0^T \rho_s(\bar{\mathbf{b}},\theta)_{\Omega_s}\,dt \quad \forall\, \theta \in H^1_s(\Omega_s)$$

We also have that the operator $\mathcal{J}_\delta \in \mathcal{L}(B_1,B_2)$ may be defined by

$\mathcal{J}_\delta{}'(\mathbf{v},p,\mathbf{g},\mathbf{u},\mathbf{h}) \cdot (\mathbf{\Lambda},\mathbf{y},\mathbf{k},\chi,\mathbf{l}) = \tilde{a}$ for $(\mathbf{\Lambda},\mathbf{y},\mathbf{k},\chi,\mathbf{l}) \in B_1$

if and only if

$$\int_0^T (\mathbf{u}_t - \mathbf{g}, \chi_t - \mathbf{k})_{\Gamma_0}\,dt$$
$$+ \int_0^T (p\mathbf{n} - \mu_f\nabla\mathbf{v}\cdot\mathbf{n} - \mathbf{h}, y\mathbf{n} - \mu_f\nabla\mathbf{\Lambda}\cdot\mathbf{n} - \mathbf{l})_{\Gamma_0}\,dt$$
$$+\delta \int_0^T (\mathbf{g},\mathbf{k})_{\Gamma_0}\,dt + \delta\int_0^T (\nabla_\Gamma\mathbf{g},\nabla_\Gamma\mathbf{k})_{\Gamma_0}\,dt + \delta\int_0^T (\mathbf{g}_t,\mathbf{k}_t)_{\Gamma_0}\,dt$$
$$+\delta\int_0^T (\mathbf{h},\mathbf{l})_{\Gamma_0}\,dt = \tilde{a}$$

Suppose $(\hat{\mathbf{v}}, \hat{p}, \hat{\mathbf{g}}, \hat{\mathbf{u}}, \hat{\mathbf{h}}) \in B_1$ is an optimal solution of Problem1. Then there exists a nonzero Lagrange multiplier $(\mu, \phi, \tau, \eta) \in B_2^*$ satisfying the Euler equation.

$$\begin{aligned} &-\mathcal{J}_\delta{}'(\mathbf{v}, p, \mathbf{g}, \mathbf{u}, \mathbf{h}) \cdot (\mathbf{\Lambda}, \mathbf{y}, \mathbf{k}, \chi, \mathbf{l}) \\ &\quad + \quad < (\mu, \phi, \tau, \eta), M'(\mathbf{v}, p, \mathbf{g}, \mathbf{t}, \mathbf{u}, \mathbf{h}) \cdot (\mathbf{\Lambda}, \mathbf{y}, \mathbf{k}, \chi, \mathbf{l}) >= 0 \\ &\qquad \forall (\mathbf{\Lambda}, \mathbf{y}, \mathbf{k}, \chi, \mathbf{l}) \in B_1 \end{aligned} \tag{3.4.23}$$

where, $< \cdot, \cdot >$ denotes the duality pairing between B_2 and the dual space B_2^*.

$$\begin{aligned} &\int_0^T [\rho_f(\mathbf{\Lambda}_t, \mu)_{\Omega_f} + b(\mu, y) + a_f(\mathbf{\Lambda}, \mu) + (\mu, y\mathbf{n} - \mu_f \nabla \mathbf{\Lambda} \cdot \mathbf{n})_{\Gamma_0} + b(\mathbf{\Lambda}, \phi) \\ &\quad + (\mathbf{\Lambda}, \tau)_{\Gamma_0} - (\mathbf{k}, \tau)_{\Gamma_0} + \rho_s(\chi_{tt}, \eta)_{\Omega_s} + \epsilon a(\chi_t, \eta) + a_s(\chi, \eta) - (\mathbf{l}, \eta)_{\Gamma_0}]\, dt \\ = \quad & \int_0^T [(\mathbf{u}_t - g, \chi_t - \mathbf{k})_{\Gamma_0} \\ &\quad + (p\mathbf{n} - \mu_f \nabla \mathbf{v} \cdot \mathbf{n} - \mathbf{h}, y\mathbf{n} - \mu_f \nabla \mathbf{\Lambda} \cdot \mathbf{n} - \mathbf{l})_{\Gamma_0}]\, dt \\ &\quad + \delta \int_0^T (\mathbf{g}, \mathbf{k})_{\Gamma_0}\, dt + \delta \int_0^T (\nabla_\Gamma \mathbf{g}, \nabla_\Gamma \mathbf{k})_{\Gamma_0}\, dt + \delta \int_0^T (\mathbf{g}_t, \mathbf{k}_t)_{\Gamma_0}\, dt \\ &\quad + \delta \int_0^T (\mathbf{h}, \mathbf{l})_{\Gamma_0}\, dt \quad \forall (\mathbf{\Lambda}, \mathbf{y}, \mathbf{k}, \chi, \mathbf{l}) \in B_1 \end{aligned} \tag{3.4.24}$$

We may rewrite (3.4.24) in the form

$$-\rho_f(\mu_t, \mathbf{\Lambda})_{\Omega_f} + a_f(\mathbf{\Lambda}, \mu) + b(\mathbf{\Lambda}, \phi) \tag{3.4.25}$$

$$\begin{aligned} &+ (\mathbf{\Lambda}, \tau)_{\Gamma_0} - (\mu_f \nabla \mathbf{\Lambda} \cdot \mathbf{n}, \mu) \\ &= -(p\mathbf{n} - \mu_f \nabla \mathbf{v} \cdot \mathbf{n} - \mathbf{h}, \mu_f \nabla \mathbf{\Lambda} \cdot \mathbf{n}) \quad \forall\ \mathbf{\Lambda} \in H_f^1(\Omega_f) \end{aligned}$$

$$b(\mu, y) = 0 \quad \forall\ y \in L^2(\Omega_f) \tag{3.4.26}$$

$$(\mu, y\mathbf{n})_{\Gamma_0} = (p\mathbf{n} - \mu_f \nabla \mathbf{v} \cdot \mathbf{n} - \mathbf{h}, y\mathbf{n})_{\Gamma_0} \quad \forall\ y \in L_0^2(\Omega_f) \tag{3.4.27}$$

$$\mu|_{t=T} = 0 \tag{3.4.28}$$

$$\begin{aligned} &\rho_s(\eta_{tt}, \chi)_{\Omega_s} + \epsilon a(\chi_t, \eta) + a_s(\chi, \eta) \\ &\quad = -(\mathbf{u}_{tt} - \mathbf{g}_t, \chi)_{\Gamma_0} \quad \forall\ \chi \in H_s^1(\Omega_s) \end{aligned} \tag{3.4.29}$$

$$\eta|_{t=T} = 0 \tag{3.4.30}$$

$$\eta_t|_{t=T} = 0 \tag{3.4.31}$$

$$(\mathbf{k},\tau)_{\Gamma_0} = (\mathbf{u}_t - \mathbf{g},\mathbf{k})_{\Gamma_0} - \delta(\mathbf{g},\mathbf{k})_{\Gamma_0} \qquad (3.4.32)$$

$$-\delta(\Delta_\Gamma \mathbf{g}, \Delta_\Gamma \mathbf{k})_{\Gamma_0} + \delta(\mathbf{g}_{tt},\mathbf{k})_{\Gamma_0} \quad \forall\, \mathbf{k} \in Y$$

$$(\mathbf{l},\eta)_{\Gamma_0} = (p\mathbf{n} - \mu_f \nabla \mathbf{v}\cdot\mathbf{n} - \mathbf{h},\mathbf{l})_{\Gamma_0} - \delta(\mathbf{h},\mathbf{l})_{\Gamma_0} \quad \forall\, \mathbf{l} \in Z \qquad (3.4.33)$$

Thus, solutions of the optimal problem are determined by solving the system (3.2.11)-(3.2.17) and (3.4.25)-(3.4.33). This system of equations is called the optimality system. We may replace (3.4.25) and (3.4.32) by

$$-\rho_f(\mu_t,\mathbf{\Lambda})_{\Omega_f} + a_f(\mathbf{\Lambda},\mu) + b(\mathbf{\Lambda},\phi) \qquad (3.4.34)$$

$$+(\mathbf{\Lambda}, \phi\mathbf{n} - \mu_f\nabla\mu\cdot\mathbf{n})_{\Gamma_0} = 0 \quad \forall\, \mathbf{\Lambda} \in H^1_f(\Omega_f)$$

$$(\mathbf{k}, \phi\mathbf{n} - \mu_f\nabla\mu\cdot\mathbf{n})_{\Gamma_0} = (\mathbf{u}_t - \mathbf{g},\mathbf{k})_{\Gamma_0} - \delta(\mathbf{g},\mathbf{k})_{\Gamma_0} \qquad (3.4.35)$$

$$-\delta(\Delta_\Gamma \mathbf{g}, \Delta_\Gamma \mathbf{k})_{\Gamma_0} + \delta(\mathbf{g}_{tt},\mathbf{k})_{\Gamma_0} \quad \forall\, \mathbf{k} \in Y$$

The optimality system is a weak formulation corresponding to

$$
\left\{
\begin{aligned}
\rho_f \mathbf{v}_t + \nabla p - \mu_f \Delta \mathbf{v} = \rho_f \mathbf{f} \quad & \text{in } \Omega_f \\
\nabla \cdot \mathbf{v} = 0 \quad & \text{in } \Omega_f \\
\mathbf{v} = 0 \quad & \text{on } \Gamma_f \\
\mathbf{v} = \mathbf{g} \quad & \text{on } \Gamma_0 \\
\rho_s \mathbf{u}_{tt} - \epsilon \mu_s \Delta \mathbf{u}_t - \mu_s \Delta \mathbf{u} - (\lambda_s + \mu_s) \nabla (\nabla \cdot \mathbf{u}) = \rho_s \mathbf{b} \quad & \text{in } \Omega_s \\
\mathbf{u} = 0 \quad & \text{on } \Gamma_s \\
\epsilon \mu_s \nabla \mathbf{u}_t \cdot \mathbf{n} + \mu_s \nabla \mathbf{u} \cdot \mathbf{n} + (\lambda_s + \mu_s)(\nabla \cdot \mathbf{u}) \mathbf{n} = \mathbf{h} \quad & \text{on } \Gamma_0 \\
\mathbf{v}|_{t=0} = \mathbf{v}^0 \quad & \text{in } \Omega_f \\
\mathbf{u}|_{t=0} = \mathbf{u}^0 \quad & \text{in } \Omega_s \\
\mathbf{u}_t|_{t=0} = \mathbf{u}^1 \quad & \text{in } \Omega_s \\
-\rho_f \mu_t + \nabla \phi - \mu_f \Delta \mu = 0 \quad & \text{in } \Omega_f \\
\nabla \cdot \mu = 0 \quad & \text{in } \Omega_f \\
\mu = 0 \quad & \text{on } \Gamma_f \\
\mu = p\mathbf{n} - \mu_f \nabla \mathbf{v} \cdot \mathbf{n} - \mathbf{h} \quad & \text{on } \Gamma_0 \\
\rho_s \eta_{tt} - \epsilon \mu_s \Delta \eta_t - \mu_s \Delta \eta - (\lambda_s + \mu_s) \nabla (\nabla \cdot \eta) = 0 \quad & \text{in } \Omega_s \\
\eta = 0 \quad & \text{on } \Gamma_s \\
\epsilon \mu_s \nabla \eta_t \cdot \mathbf{n} + \mu_s \nabla \eta \cdot \mathbf{n} + (\lambda_s + \mu_s)(\nabla \cdot \eta) \mathbf{n} = -(\mathbf{u}_{tt} - \mathbf{g}_t) \quad & \text{on } \Gamma_0 \\
\mu|_{t=T} = 0 \quad & \text{in } \Omega_f \\
\eta|_{t=T} = 0 \quad & \text{in } \Omega_s \\
\eta_t|_{t=T} = 0 \quad & \text{in } \Omega_s \\
(1+\delta)\mathbf{g} - \delta \Delta_\Gamma \mathbf{g} - \delta \mathbf{g}_{tt} = -\phi \mathbf{n} + \mu_f \nabla \mu \cdot \mathbf{n} + \mathbf{u}_t \quad & \text{on } \Gamma_0 \\
(1+\delta)\mathbf{h} = (p\mathbf{n} - \mu_f \nabla \mathbf{v} \cdot \mathbf{n} - \eta) \quad & \text{on } \Gamma_0
\end{aligned}
\right.
\tag{3.4.36}
$$

3.5 Sensitivity derivatives

The optimality system (3.2.11)-(3.2.17) and (3.4.25)-(3.4.33) may also be derived using sensitivity derivatives instead of the Lagrange multiplier rule. The first derivatives $\frac{\partial \mathcal{J}_\delta}{\partial \mathbf{g}}, \frac{\partial \mathcal{J}_\delta}{\partial \mathbf{h}}$ of $\mathcal{J}_\delta$ are defined through their actions on variations $\tilde{\mathbf{g}}$ and $\tilde{\mathbf{h}}$ as follows:

$$< \frac{\partial \mathcal{J}_\delta}{\partial \mathbf{g}}, \tilde{\mathbf{g}} >= \int_0^T (\mathbf{u}_t - \mathbf{g}, -\tilde{\mathbf{g}})_{\Gamma_0} dt \tag{3.5.37}$$

$$+ \int_0^T (p\mathbf{n} - \mu_f \nabla \mathbf{v} \cdot \mathbf{n} - \mathbf{h}, \tilde{p}\mathbf{n} - \mu_f \nabla \tilde{\mathbf{v}} \cdot \mathbf{n})_{\Gamma_0} dt$$

$$+\delta \int_0^T (\mathbf{g}, \tilde{\mathbf{g}})_{\Gamma_0} dt + \delta \int_0^T (\nabla_\Gamma \mathbf{g}, \nabla_\Gamma \tilde{\mathbf{g}})_{\Gamma_0} dt$$

$$+\delta \int_0^T (\mathbf{g}_t, \tilde{\mathbf{g}}_t)_{\Gamma_0} dt \tag{3.5.38}$$

where $\tilde{p}, \tilde{\mathbf{v}}$ are solutions of

$$\rho_f (\tilde{\mathbf{v}}_t, \mathbf{w})_{\Omega_f} + b(\mathbf{w}, \tilde{p}) + a_f(\tilde{\mathbf{v}}, \mathbf{w}) \tag{3.5.39}$$

$$+(\mathbf{w}, \tilde{p}\mathbf{n} - \mu_f \nabla \tilde{\mathbf{v}} \cdot \mathbf{n})_{\Gamma_0} = 0 \quad \forall\ \mathbf{w} \in H^1_f(\Omega_f)$$

$$b(\tilde{\mathbf{v}}, q) = 0 \quad \forall\ q \in H^(\Omega_f) \tag{3.5.40}$$

$$(\tilde{\mathbf{v}}, s)_{\Gamma_0} - (\tilde{\mathbf{g}}, s)_{\Gamma_0} = 0 \quad \forall\ s \in H^{-1/2}(\Gamma_0) \tag{3.5.41}$$

$$\tilde{\mathbf{v}}|_{t=0} = 0 \tag{3.5.42}$$

Setting $\boldsymbol{\Lambda} = \tilde{\mathbf{v}}$, $y\mathbf{n} = \tilde{p}\mathbf{n} - \mu_f \nabla \tilde{\mathbf{v}} \cdot \mathbf{n}$ in (3.4.34),(3.4.27) and $\mathbf{w} = \mu$ in (3.5.39) and from (3.5.37)

$$< \frac{\partial \mathcal{J}_\delta}{\partial \mathbf{g}}, \tilde{\mathbf{g}} >= \int_0^T (\mathbf{u}_t - \mathbf{g}, -\tilde{\mathbf{g}})_{\Gamma_0} dt$$

$$+ \int_0^T (\tilde{\mathbf{g}}, \phi\mathbf{n} - \mu_f \nabla \mu \cdot \mathbf{n})_{\Gamma_0} dt + \delta \int_0^T (\mathbf{g}, \tilde{\mathbf{g}})_{\Gamma_0} dt \tag{3.5.43}$$

$$-\delta \int_0^T (\Delta_\Gamma \mathbf{g}, \tilde{\mathbf{g}})_{\Gamma_0} dt - \delta \int_0^T (\mathbf{g}_{tt}, \tilde{\mathbf{g}})_{\Gamma_0} dt$$

$$< \frac{\partial \mathcal{J}_\delta}{\partial \mathbf{h}}, \tilde{\mathbf{h}} >= \int_0^T (\mathbf{u}_t - \mathbf{g}, \tilde{\mathbf{u}}_t)_{\Gamma_0} dt$$

$$+ \quad \int_0^T (p\mathbf{n} - \mu_f \nabla \mathbf{v} \cdot \mathbf{n} - \mathbf{h}, -\tilde{\mathbf{h}})_{\Gamma_0} dt + \delta \int_0^T (\mathbf{h}, \tilde{\mathbf{h}})_{\Gamma_0} dt \tag{3.5.44}$$

where $\tilde{\mathbf{u}}$ is a solution of

$$\rho_s(\tilde{\mathbf{u}}_{tt},\theta)_{\Omega_s} + \epsilon a(\tilde{\mathbf{u}}_t,\theta) + a_s(\tilde{\mathbf{u}},\theta) = (\tilde{\mathbf{h}},\theta)_{\Gamma_0} \quad \forall\ \theta \in H_s^1(\Omega_s) \tag{3.5.45}$$

$$\tilde{\mathbf{u}}|_{t=0} = 0 \tag{3.5.46}$$

$$\tilde{\mathbf{u}}_t|_{t=0} = 0 \tag{3.5.47}$$

Setting $\chi = \tilde{\mathbf{u}}$ in (3.4.29) and $\theta = \eta$ in (3.5.45) and from (3.5.44)

$$\begin{aligned} < \frac{\partial \mathcal{J}_\delta}{\partial \mathbf{h}}, \tilde{\mathbf{h}} > &= \int_0^T (\tilde{\mathbf{h}},\eta)_{\Gamma_0} dt \\ &+ \int_0^T (p\mathbf{n} - \mu_f \nabla \mathbf{v}\cdot\mathbf{n} - \mathbf{h}, -\tilde{\mathbf{h}})_{\Gamma_0} dt + \delta \int_0^T (\mathbf{h},\tilde{\mathbf{h}})_{\Gamma_0} dt \end{aligned} \tag{3.5.48}$$

Thus the first order necessary conditions $\frac{\partial \mathcal{J}_\delta}{\partial \mathbf{g}} = 0$ and $\frac{\partial \mathcal{J}_\delta}{\partial \mathbf{h}} = 0$ yield that

$$(1+\delta)\mathbf{g} - \delta\Delta_\Gamma \mathbf{g} - \delta \mathbf{g}_{tt} = -\phi\mathbf{n} + \mu_f \nabla\mu\cdot\mathbf{n} + \mathbf{u}_t \quad \text{on } \Gamma_0$$

$$(1+\delta)\mathbf{h} = p\mathbf{n} - \mu_f\nabla\mathbf{v}\cdot\mathbf{n} - \eta \quad \text{on } \Gamma_0$$

which are the same as in (3.4.36).

Note that equations (3.5.43) and (3.5.48) give an explicit formula for the gradient of $\mathcal{J}_\delta$, i.e.,

$$\frac{\partial \mathcal{J}_\delta}{\partial \mathbf{g}} = \int_0^T [(1+\delta)\mathbf{g} - \delta\Delta_\Gamma\mathbf{g} - \delta\mathbf{g}_{tt} + \phi\mathbf{n} - \mu_f\nabla\mu\cdot\mathbf{n} - \mathbf{u}_t] dt \tag{3.5.49}$$

$$\frac{\partial \mathcal{J}_\delta}{\partial \mathbf{h}} = \int_0^T [(1+\delta)\mathbf{h} - p\mathbf{n} + \mu_f\nabla\mathbf{v}\cdot\mathbf{n} + \eta] dt \tag{3.5.50}$$

3.6 Gradient method

In this section, we study a gradient method to solve the optimization system (3.2.11)-(3.2.17) and (3.4.25)-(3.4.33). The simple gradient method we consider is defined as follows.

Given a starting guess $\mathbf{g}^{(0)}$ and $\mathbf{h}^{(0)}$

$$\begin{pmatrix} \mathbf{g}^{(n+1)} \\ \mathbf{h}^{(n+1)} \end{pmatrix} = \begin{pmatrix} \mathbf{g}^{(n)} \\ \mathbf{h}^{(n)} \end{pmatrix} - \alpha \begin{pmatrix} \frac{\partial \mathcal{J}_\delta}{\partial \mathbf{g}} \\ \frac{\partial \mathcal{J}_\delta}{\partial \mathbf{h}} \end{pmatrix} \quad for \quad n = 1, 2, \cdots,$$

where α is a step size. Combining with (3.5.49) and (3.5.50) yields, for $n = 1, 2, \cdots$

$$\begin{pmatrix} \mathbf{g}^{(n+1)} \\ \mathbf{h}^{(n+1)} \end{pmatrix} = \begin{pmatrix} \mathbf{g}^{(n)} \\ \mathbf{h}^{(n)} \end{pmatrix} - \alpha \begin{pmatrix} \int_0^T [(1+\delta)\mathbf{g} - \delta\Delta_\Gamma \mathbf{g} - \delta \mathbf{g}_{tt} + \phi \mathbf{n} - \mu_f \nabla \mu \cdot \mathbf{n} - \mathbf{u}_t] dt \\ \int_0^T [(1+\delta)\mathbf{h} - p\mathbf{n} + \mu_f \nabla \mathbf{v} \cdot \mathbf{n} + \eta] dt \end{pmatrix}$$

Algorithm 3.6.1 *1. Choose* $\mathbf{g}^{(0)}$ *and* $\mathbf{h}^{(0)}$.

2. For n=0,1,2,$\cdots$,

(a) determine $\mathbf{v}^{(n)}, p^{(n)}$ *and* $\mathbf{u}^{(n)}$ *from*

$$\rho_f(\mathbf{v}_t^{(n)}, \mathbf{w})_{\Omega_f} + b(\mathbf{w}, p^{(n)}) + a_f(\mathbf{v}^{(n)}, \mathbf{w})$$

$$+(\mathbf{w}, p^{(n)}\mathbf{n} - \mu_f \nabla \mathbf{v}^{(n)} \cdot \mathbf{n})_{\Gamma_0} = \rho_f(\mathbf{f}, \mathbf{w})_{\Omega_f} \quad \forall\ \mathbf{w} \in H_f^1(\Omega_f)$$

$$b(\mathbf{v}^{(n)}, q) = 0 \quad \forall\ q \in H^2(\Omega_f)$$

$$(\mathbf{v}^{(n)}, \mathbf{s})_{\Gamma_0} - (\mathbf{g}^{(n)}, \mathbf{s})_{\Gamma_0} = 0 \quad \forall\ \mathbf{s} \in H^{-1/2}(\Gamma_0)$$

$$\mathbf{v}^{(n)}|_{t=0} = \mathbf{v}^0 \ \ in\ \Omega_f$$

$$\rho_s(\mathbf{u}_{tt}^{(n)}, \theta)_{\Omega_s} + \epsilon a(\mathbf{u}_t^{(n)}, \theta) + a_s(\mathbf{u}^{(n)}, \theta) = \rho_s(\mathbf{b}, \theta)_{\Omega_s} + (\mathbf{h}^{(n)}, \theta)_{\Gamma_0}$$

$$\forall\ \theta \in H_s^1(\Omega_s)$$

$$\mathbf{u}^{(n)}|_{t=0} = \mathbf{u}^0 \ \ in\ \Omega_s$$

$$\mathbf{u}_t^{(n)}|_{t=0} = \mathbf{u}^1 \ \ in\ \Omega_s$$

(b) determine $\mu^{(n)}$,$\phi^{(n)}$ *and* $\eta^{(n)}$ *from*

$$-\rho_f(\mu_t^{(n)}, \mathbf{\Lambda})_{\Omega_f} + a_f(\mathbf{\Lambda}, \mu^{(n)}) + b(\mathbf{\Lambda}, \phi^{(n)})$$

$$+(\mathbf{\Lambda}, \phi^n \mathbf{n} - \mu_f \nabla \mu^{(n)} \cdot \mathbf{n})_{\Gamma_0} = 0 \quad \forall\ \mathbf{\Lambda} \in H^1_f(\Omega_f)$$

$$b(\mu^{(n)}, q) = 0 \quad \forall\ q \in L^2(\Omega_f)$$

$$(\mu^{(n)}, s)_{\Gamma_0} = (p^{(n)}\mathbf{n} - \mu_f \nabla \mathbf{v}^{(n)} \cdot \mathbf{n} - \mathbf{h}^{(n)}, s)_{\Gamma_0} \quad \forall\ s \in H^{-1/2}(\Gamma_0)$$

$$\mu^{(n)}|_{t=T} = 0$$

$$\rho_s(\eta_{tt}^{(n)}, \chi)_{\Omega_s} + \epsilon a(\chi_t, \eta^{(n)}) + a_s(\chi, \eta^{(n)}) = -(\mathbf{u}_{tt}^{(n)} - \mathbf{g}_t^{(n)}, \chi^{(n)})_{\Gamma_0}$$

$$\forall\ \chi \in H^1_s(\Omega_s)$$

$$\eta^{(n)}|_{t=T} = 0$$

$$\eta_t^{(n)}|_{t=T} = 0$$

(c) determine $\mathbf{g}^{(n+1)}$ *and* $\mathbf{h}^{(n+1)}$ *from*

$$\mathbf{g}^{(n+1)} = \mathbf{g}^{(n)}$$

$$-\alpha \textstyle\int_0^T [(1+\delta)\mathbf{g}^{(n)} - \delta \Delta_\Gamma \mathbf{g}^{(n)} - \delta \mathbf{g}_{tt}^{(n)} + \phi^{(n)}\mathbf{n} - \mu_f \nabla \mu^{(n)} \cdot \mathbf{n} - \mathbf{u}_t^{(n)}]dt$$

$$\mathbf{h}^{(n+1)} = \mathbf{h}^{(n)} - \alpha \textstyle\int_0^T [(1+\delta)\mathbf{h}^{(n)} - p^{(n)}\mathbf{n} + \mu_f \nabla \mathbf{v}^{(n)} \cdot \mathbf{n} + \eta^{(n)}]dt$$

The following result is useful to determining sufficient conditions for the convergence of the gradient method.

Theorem 3.6.1 *Let X be a Hilbert space equipped with the inner product $(\cdot,\cdot)_X$ and norm $\|\cdot\|_X$. Suppose $\mathcal{M}$ is a functional on x such that*

1. *$\mathcal{M}$ has a local minimum at $\hat{x}$ and is twice differentiable in an open ball B centered at $\hat{x}$;*

2. $| < \mathcal{M}''(u), (x,y) > | \leq M\|x\|_X \|y\|_Y,\ \forall u \in B, x \in X, y \in Y$;

3. $| < \mathcal{M}''(u), (x,x) > | \geq m\|x\|_X^2,\ \forall u \in B, x \in X$,

where M and m are positive constants. Let R denote the Riesz map. Choose $x^{(0)}$ sufficiently close to $\hat{x}$ and choose a sequence ρ_n such that $0 < \rho_* \leq \rho_n \leq \rho^* < 2m/M^2$. Then the sequence $x^{(n)}$ defined by

$$x^{(n)} = x^{(n-1)} - \rho_n R\mathcal{M}'(x^{(n-1)}), \quad \text{for } n = 1, 2, \cdots,$$

converges to $\hat{x}$.

We examine the second derivatives of $\mathcal{J}$ to determine the constants M and m.

$$
\begin{aligned}
&< \frac{\partial^2 \mathcal{J}}{\partial \mathbf{g}^2}, (\tilde{\mathbf{g}}, \mathbf{g}) >= \int_0^T (\mathbf{g}, \tilde{\mathbf{g}})_{\Gamma_0} dt \\
&\quad + \int_0^T (p\mathbf{n} - \mu_f \nabla \mathbf{v} \cdot \mathbf{n}, \tilde{p}\mathbf{n} - \mu_f \nabla \tilde{\mathbf{v}} \cdot \mathbf{n})_{\Gamma_0} dt \\
&\quad + \delta \int_0^T (\mathbf{g}, \tilde{\mathbf{g}})_{\Gamma_0} dt + \delta \int_0^T (\nabla_\Gamma \mathbf{g}, \nabla_\Gamma \tilde{\mathbf{g}})_{\Gamma_0} dt + \delta \int_0^T (\mathbf{g}_t, \tilde{\mathbf{g}}_t)_{\Gamma_0} dt \\
&< \frac{\partial^2 \mathcal{J}}{\partial \mathbf{g} \partial \mathbf{h}}, (\tilde{\mathbf{g}}, \mathbf{h}) >= \int_0^T (-\tilde{\mathbf{g}}, \mathbf{u}_t)_{\Gamma_0} dt \\
&\quad + \int_0^T (\tilde{p}\mathbf{n} - \mu_f \nabla \tilde{\mathbf{v}} \cdot \mathbf{n}, -\mathbf{h})_{\Gamma_0} dt \\
&< \frac{\partial^2 \mathcal{J}}{\partial \mathbf{h} \partial \mathbf{g}}, (\tilde{\mathbf{h}}, \mathbf{g}) >= \int_0^T (\tilde{\mathbf{u}}_t, -\mathbf{g})_{\Gamma_0} dt \\
&\quad + \int_0^T (-\tilde{\mathbf{h}}, p\mathbf{n} - \mu_f \nabla \mathbf{v} \cdot \mathbf{n})_{\Gamma_0} dt \\
&< \frac{\partial^2 \mathcal{J}}{\partial \mathbf{h}^2}, (\tilde{\mathbf{h}}, \mathbf{h}) >= \int_0^T (\mathbf{u}_t, \tilde{\mathbf{u}}_t)_{\Gamma_0} dt + (1 + \delta) \int_0^T (\mathbf{h}, \tilde{\mathbf{h}})_{\Gamma_0} dt
\end{aligned}
$$

where, $\mathbf{v}, p, \mathbf{u}, \tilde{\mathbf{v}}, \tilde{p}$ and $\tilde{\mathbf{u}}$ are solutions of

$$
\left\{
\begin{array}{c}
\rho_f(\mathbf{v}_t, \mathbf{w})_{\Omega_f} + b(\mathbf{w}, p) + a_f(\mathbf{v}, \mathbf{w}) \\
+(\mathbf{w}, p\mathbf{n} - \mu_f \nabla \mathbf{v} \cdot \mathbf{n})_{\Gamma_0} = 0 \quad \forall\, \mathbf{w} \in H^1_f(\Omega_f) \\
b(\mathbf{v}, q) = 0 \quad \forall\, q \in L^2(\Omega_f) \\
(\mathbf{v}, \mathbf{s})_{\Gamma_0} - (\mathbf{g}, \mathbf{s})_{\Gamma_0} = 0 \quad \forall\, \mathbf{s} \in H^{-1/2}(\Gamma_0) \\
\mathbf{v}|_{t=0} = 0
\end{array}
\right.
\tag{3.6.51}
$$

$$\left\{\begin{array}{c} \rho_f(\tilde{\mathbf{v}}_t,\mathbf{w})_{\Omega_f}+b(\mathbf{w},\tilde{p})+a_f(\tilde{\mathbf{v}},\mathbf{w}) \\ +(\mathbf{w},\tilde{p}\mathbf{n}-\mu_f\nabla\tilde{\mathbf{v}}\cdot\mathbf{n})_{\Gamma_0}=0 \quad \forall\ \mathbf{w}\in H^1_f(\Omega_f) \\ b(\tilde{\mathbf{v}},q)=0 \quad \forall\ q\in L^2(\Omega_f) \\ (\tilde{\mathbf{v}},\mathbf{s})_{\Gamma_0}-(\tilde{\mathbf{g}},\mathbf{s})_{\Gamma_0}=0 \quad \forall\ \mathbf{s}\in H^{-1/2}(\Gamma_0) \\ \tilde{\mathbf{v}}|_{t=0}=0 \end{array}\right. \tag{3.6.52}$$

$$\left\{\begin{array}{c} \rho_s(\mathbf{u}_{tt},\theta)_{\Omega_s}+\epsilon a(\mathbf{u}_t,\theta)+a_s(\mathbf{u},\theta)=(\mathbf{h},\theta)_{\Gamma_0} \quad \forall\ \theta\in H^1_s(\Omega_s) \\ \mathbf{u}|_{t=0}=0 \\ \mathbf{u}_t|_{t=0}=0 \end{array}\right. \tag{3.6.53}$$

$$\left\{\begin{array}{c} \rho_s(\tilde{\mathbf{u}}_{tt},\theta)_{\Omega_s}+\epsilon a(\tilde{\mathbf{u}}_t,\theta)+a_s(\tilde{\mathbf{u}},\theta)=(\tilde{\mathbf{h}},\theta)_{\Gamma_0} \quad \forall\ \theta\in H^1_s(\Omega_s) \\ \tilde{\mathbf{u}}|_{t=0}=0 \\ \tilde{\mathbf{u}}_t|_{t=0}=0 \end{array}\right. \tag{3.6.54}$$

Then,

$$\begin{aligned} &(\tilde{\mathbf{g}}\ \tilde{\mathbf{h}})\begin{pmatrix} \frac{\partial^2\mathcal{J}}{\partial\mathbf{g}^2} & \frac{\partial^2\mathcal{J}}{\partial\mathbf{g}\partial\mathbf{h}} \\ \frac{\partial^2\mathcal{J}}{\partial\mathbf{h}\partial\mathbf{g}} & \frac{\partial^2\mathcal{J}}{\partial\mathbf{h}^2} \end{pmatrix}\begin{pmatrix}\mathbf{g}\\ \mathbf{h}\end{pmatrix} \\ = \ & <\frac{\partial^2\mathcal{J}}{\partial\mathbf{g}^2},(\tilde{\mathbf{g}},\mathbf{g})>+<\frac{\partial^2\mathcal{J}}{\partial\mathbf{g}\partial\mathbf{h}},(\tilde{\mathbf{g}},\mathbf{h})>+<\frac{\partial^2\mathcal{J}}{\partial\mathbf{h}\partial\mathbf{g}},(\tilde{\mathbf{h}},\mathbf{g})>+<\frac{\partial^2\mathcal{J}}{\partial\mathbf{h}^2},(\tilde{\mathbf{h}},\mathbf{h})> \\ = \ & \int_0^T(\mathbf{g},\tilde{\mathbf{g}})_{\Gamma_0}dt+\int_0^T(p\mathbf{n}-\mu_f\nabla\mathbf{v}\cdot\mathbf{n},\tilde{p}\mathbf{n}-\mu_f\nabla\tilde{\mathbf{v}}\cdot\mathbf{n})_{\Gamma_0}dt \\ & +\delta\int_0^T(\mathbf{g},\tilde{\mathbf{g}})_{\Gamma_0}dt+\delta\int_0^T(\nabla_\Gamma\mathbf{g},\nabla_\Gamma\tilde{\mathbf{g}})_{\Gamma_0}dt+\delta\int_0^T(\mathbf{g}_t,\tilde{\mathbf{g}}_t)_{\Gamma_0}dt \\ & +\int_0^T(-\tilde{\mathbf{g}},\mathbf{u}_t)_{\Gamma_0}dt+\int_0^T(\tilde{p}\mathbf{n}-\mu_f\nabla\tilde{\mathbf{v}}\cdot\mathbf{n},-\mathbf{h})_{\Gamma_0}dt \\ & +\int_0^T(\tilde{\mathbf{u}}_t,-\mathbf{g})_{\Gamma_0}dt+\int_0^T(-\tilde{\mathbf{h}},p\mathbf{n}-\mu_f\nabla\mathbf{v}\cdot\mathbf{n})_{\Gamma_0}dt \\ & +\int_0^T(\mathbf{u}_t,\tilde{\mathbf{u}}_t)_{\Gamma_0}dt+(1+\delta)\int_0^T(\mathbf{h},\tilde{\mathbf{h}})_{\Gamma_0}dt \\ = \ & \int_0^T(\mathbf{u}_t-\mathbf{g},\tilde{\mathbf{u}}_t-\tilde{\mathbf{g}})_{\Gamma_0}dt \end{aligned}$$

$$+\int_0^T (p\mathbf{n} - \mu_f \nabla\mathbf{v}\cdot\mathbf{n} - \mathbf{h}, \tilde{p}\mathbf{n} - \mu_f \nabla\tilde{\mathbf{v}}\cdot\mathbf{n} - \tilde{\mathbf{h}})_{\Gamma_0} dt$$
$$+\delta\int_0^T (\mathbf{g}, \tilde{\mathbf{g}})_{\Gamma_0} dt + \delta\int_0^T (\nabla_\Gamma\mathbf{g}, \nabla_\Gamma\tilde{\mathbf{g}})_{\Gamma_0} dt + \delta\int_0^T (\mathbf{g}_t, \tilde{\mathbf{g}}_t)_{\Gamma_0} dt$$
$$+\delta\int_0^T (\mathbf{h}, \tilde{\mathbf{h}})_{\Gamma_0} dt$$

We show that

$$\left| (\widetilde{\mathbf{g}}, \widetilde{\mathbf{h}}) \begin{pmatrix} \frac{\partial^2\mathcal{J}}{\partial\mathbf{g}^2} & \frac{\partial^2\mathcal{J}}{\partial\mathbf{g}\partial\mathbf{h}} \\ \frac{\partial^2\mathcal{J}}{\partial\mathbf{h}\partial\mathbf{g}} & \frac{\partial^2\mathcal{J}}{\partial\mathbf{h}^2} \end{pmatrix} \begin{pmatrix} \mathbf{g} \\ \mathbf{h} \end{pmatrix} \right| \leq M\|\mathbf{x}\|\|\mathbf{y}\|$$

where $\mathbf{x} = (\widetilde{\mathbf{g}}, \widetilde{\mathbf{h}})^T,\ \mathbf{y} = (\mathbf{g}, \mathbf{h})^T,\ \|\mathbf{x}\| = \sqrt{\|\widetilde{\mathbf{g}}\|_Y^2 + \|\widetilde{\mathbf{h}}\|_Z^2}$ and $\|\mathbf{y}\| = \sqrt{\|\mathbf{g}\|_Y^2 + \|\mathbf{h}\|_Z^2}$

First, we obtain bounds for $\mathbf{v}$ and p. Let $\mathbf{v}, p$ satisfy (3.6.51), which is a weak formulation of

$$\begin{cases} \rho_f \mathbf{v}_t + \nabla p - \mu_f \Delta\mathbf{v} = 0 & \text{in } \Omega_f \\ \nabla\cdot\mathbf{v} = 0 & \text{in } \Omega_f \\ \mathbf{v} = \hat{\mathbf{g}} & \text{on } \Gamma \\ \mathbf{v}|_{t=0} = 0 & \text{in } \Omega_f \end{cases} \tag{3.6.55}$$

where, $\Gamma = \Gamma_f \cup \Gamma_0$ and

$$\hat{\mathbf{g}} = \begin{cases} \mathbf{g} \text{ on } \Gamma_0 \\ 0 \text{ on } \Gamma_f \end{cases}$$

We introduce a space

$$\mathbf{H}^s_\Delta(\Omega_f) = \{\mathbf{v} \in \mathbf{H}^s(\Omega_f) : \Delta\mathbf{v} \in \mathbf{L}^2(\Omega_f)\}$$

with a norm

$$\|\mathbf{v}\|_{s\Delta,\Omega_f} = \|\mathbf{v}\|_{s,\Omega_f} + \|\Delta\mathbf{v}\|_{0,\Omega_f}$$

Then, for $\hat{\mathbf{g}} \in L^2(0,T;\mathbf{H}^1(\Gamma)) \cap H^1(0,T;\mathbf{L}^2(\Gamma))$, there is an extension $R_0\hat{\mathbf{g}} \in L^2(0,T;\mathbf{H}^{3/2}_\Delta(\Omega_f)) \cap H^1(0,T;\mathbf{H}^{1/2}_\Delta(\Omega_f))$ such that $\nabla\cdot(R_0\hat{\mathbf{g}}) = 0$ in Ω_f and $R_0\hat{\mathbf{g}} = \hat{\mathbf{g}}$ on Γ.

And

$$\|R_0\hat{\mathbf{g}}\|_{3/2\Delta,\Omega_f} + \|R_0\hat{\mathbf{g}}_t\|_{1/2\Delta,\Omega_f} \leq \|\hat{\mathbf{g}}\|_{1,\Gamma} + \|\hat{\mathbf{g}}_t\|_{0,\Gamma}. \tag{3.6.56}$$

Write $\mathbf{v} = \bar{\mathbf{v}} + R_0\hat{\mathbf{g}}$, then we have

$$\begin{cases} \rho_f\bar{\mathbf{v}}_t + \nabla p - \mu_f\Delta\bar{\mathbf{v}} = -\rho_f R_0\hat{\mathbf{g}}_t + \mu_f\Delta R_0\hat{\mathbf{g}} \quad \text{in } \Omega_f \\ \nabla\cdot\bar{\mathbf{v}} = 0 \quad \text{in } \Omega_f \\ \bar{\mathbf{v}} = 0 \quad \text{on } \Gamma \\ \bar{\mathbf{v}}|_{t=0} = -\rho_f R_0\hat{\mathbf{g}}|_{t=0} \quad \text{in } \Omega_f \end{cases} \tag{3.6.57}$$

Notice that $-\rho_f R_0\hat{\mathbf{g}}_t + \mu_f\Delta R_0\hat{\mathbf{g}} \in \mathbf{L}^2(\Omega_f)$ and we show that $R_0\hat{\mathbf{g}}|_{t=0} \in \mathbf{H}^1(\Omega_f)$. $[X,Y]_\theta$ denotes the interpolation space equipped with a norm

$$\|u\|_{[X,Y]_\theta} = \|\Lambda^{1-\theta}u\|_Y$$

where $\Lambda : Y \to Y$ is an operator such that $\|u\|_X = \|\Lambda u\|_Y$. And define a space $W^1(0,T)$ as

$$W^1(0,T) = \{u \in L^2(0,T;X), u_t \in L^2(0,T;Y)\}$$

Theorem 3.6.2 *(Trace theorem) There exists a bounded operator* $\gamma_0 : W^1(0,T) \to [X,Y]_{1/2}$ *such that* $\gamma_0 u = u|_{t=0}$ *and* $\|\gamma_0 u\|_{[X,Y]_{1/2}} \leq C\|u\|_{W^1(0,T)}$

proof: See Vivsik, Fursikov [40].

Lemma 3.6.3 *If* $X = \mathbf{H}^{3/2}(\Omega_f)$ *and* $Y = \mathbf{H}^{1/2}(\Omega_f)$ *then* $[X,Y]_{1/2} = \mathbf{H}^1(\Omega_f)$

proof:

$$(S\mathbf{u}_1, \mathbf{u}_2)_{1/2,\Omega_f} = \int_{\Omega_f} (1+|\xi|^2)^{1/2} S\mathbf{u}_1\bar{\mathbf{u}}_2 d\xi$$

$$(\mathbf{u}_1, \mathbf{u}_2)_{3/2,\Omega_f} = \int_{\Omega_f} (1+|\xi|^2)^{3/2} \mathbf{u}_1\bar{\mathbf{u}}_2 d\xi$$

$(S\mathbf{u}_1, \mathbf{u}_2)_{1/2,\Omega_f} = (\mathbf{u}_1, \mathbf{u}_2)_{3/2,\Omega_f}$ implies $S = 1 + |\xi|^2$ and hence $\Lambda = (1 + |\xi|^2)^{1/2}$ i.e. $\|\Lambda\mathbf{u}\|_{1/2,\Omega_f} = \|\mathbf{u}\|_{3/2,\Omega_f}$. By the definition of the interpolation space,

$$\|\mathbf{u}\|^2_{[X,Y]_{1/2}} = \|\Lambda^{1/2}\mathbf{u}\|^2_{1/2,\Omega_f} = \|\mathbf{u}\|^2_{1,\Omega_f}$$

Therefore, $[X, Y]_{1/2} = \mathbf{H}^1(\Omega_f)$

If $X = \mathbf{H}^{3/2}(\Omega_f)$ and $Y = \mathbf{H}^{1/2}(\Omega_f)$ then we may rewrite the Trace Theorem as follows :

There exists a bounded operator $\gamma_0 : W^1(0, T) \to \mathbf{H}^1(\Omega_f)$ such that $\gamma_0\mathbf{u} = \mathbf{u}|_{t=0}$ and $\|\gamma_0\mathbf{u}\|^2_{1,\Omega_f} \le C(\int_0^T \|\mathbf{u}\|^2_{3/2,\Omega_f} dt + \int_0^T \|\mathbf{u}_t\|^2_{1/2,\Omega_f} dt)$

Since $R_0\hat{\mathbf{g}} \in L^2(0, T; \mathbf{H}^{3/2}_\Delta(\Omega_f)) \cap H^1(0, T; \mathbf{H}^{1/2}_\Delta(\Omega_f))$, we have $R_0\hat{\mathbf{g}}|_{t=0} \in \mathbf{H}^1(\Omega_f))$ and

$$\|R_0\hat{\mathbf{g}}|_{t=0}\|^2_{1,\Omega} \le C(\int_0^T \|R_0\hat{\mathbf{g}}\|^2_{3/2,\Omega_f} dt + \int_0^T \|R_0\hat{\mathbf{g}}_t\|^2_{1/2,\Omega_f} dt) \tag{3.6.58}$$

Then (3.6.57) is an evolution Stokes equation and hence we obtain

$$\bar{\mathbf{v}} \in L^2(0, T; \mathbf{H}^2(\Omega_f))$$
$$\bar{\mathbf{v}}_t \in L^2(0, T; \mathbf{L}^2(\Omega_f))$$
$$\bar{p} \in L^2(0, T; H^1(\Omega_f))$$

Moreover,

$$\begin{aligned} \int_0^T \|\bar{\mathbf{v}}_t\|^2_{0,\Omega_f} dt &\le \rho_f^2 \int_0^T \|R_0\hat{\mathbf{g}}_t\|^2_{0,\Omega_f} dt \\ &\quad \mu_f^2 \int_0^T \|\Delta R_0\hat{\mathbf{g}}\|^2_{0,\Omega_f} dt + \|R_0\hat{\mathbf{g}}|_{t=0}\|^2_{1,\Omega} \\ &\le \quad C(\|\hat{\mathbf{g}}\|^2_{1,\Gamma} + \|\hat{\mathbf{g}}_t\|^2_{0,\Gamma}) \end{aligned}$$

by (3.6.56) and (3.6.58).

We then come back to (3.6.57) and we apply the regularity theorem of stationary case. For almost every t in $[0,T]$,

$$\begin{cases} \nabla p - \mu_f \Delta \bar{\mathbf{v}} = -\rho_f \bar{\mathbf{v}}_t - \rho_f R_0 \hat{\mathbf{g}}_t + \mu_f \Delta R_0 \hat{\mathbf{g}} & \text{in } \Omega_f \\ \nabla \cdot \bar{\mathbf{v}} = 0 & \text{in } \Omega_f \\ \bar{\mathbf{v}} = 0 & \text{on } \Gamma \end{cases}$$

so that $\bar{\mathbf{v}}(t) \in \mathbf{H}^2(\Omega)$ and $p(t) \in H^1(\Omega)$. Moreover,

$$\begin{aligned} & \int_0^T \|\bar{\mathbf{v}}\|^2_{2,\Omega_f} dt + \int_0^T \|p\|^2_{1,\Omega_f} dt \\ \leq \;& C(\rho_f^2 \int_0^T \|\bar{\mathbf{v}}_t\|^2_{0,\Omega_f} dt + \rho_f^2 \int_0^T \|R_0 \hat{\mathbf{g}}_t\|^2_{0,\Omega_f} dt + \mu_f^2 \int_0^T \|\Delta R_0 \hat{\mathbf{g}}\|^2_{0,\Omega_f} dt) \\ \leq \;& C(\|\hat{\mathbf{g}}\|^2_{1,\Gamma} + \|\hat{\mathbf{g}}_t\|^2_{0,\Gamma}) \end{aligned}$$

Now we get bound for $\mathbf{v}$.

$$\begin{aligned} \int_0^T \|\mathbf{v}\|^2_{3/2\Delta,\Omega_f} dt \leq & \int_0^T \|\bar{\mathbf{v}}\|^2_{3/2\Delta,\Omega_f} dt + \int_0^T \|R_0 \hat{\mathbf{g}}\|^2_{3/2\Delta,\Omega_f} dt \\ \leq \;& 2(\|\bar{\mathbf{v}}\|^2_{3/2,\Omega_f} dt + \|\Delta \bar{\mathbf{v}}\|^2_{0,\Omega_f} dt) + \int_0^T \|R_0 \hat{\mathbf{g}}\|^2_{3/2\Delta,\Omega_f} dt \\ \leq \;& C(\|\hat{\mathbf{g}}\|^2_{1,\Gamma} + \|\hat{\mathbf{g}}_t\|^2_{0,\Gamma}) \end{aligned}$$

Then

$$\begin{aligned} \int_0^T \|p\mathbf{n}\|^2_{0,\Gamma_0} dt \leq & \int_0^T \|p\|^2_{1,\Omega_f} dt \\ \leq \;& C(\|\hat{\mathbf{g}}\|^2_{1,\Gamma} + \|\hat{\mathbf{g}}_t\|^2_{0,\Gamma}) = C\|\mathbf{g}\|^2_Y \end{aligned}$$

and

$$\begin{aligned} \int_0^T \|\nabla \mathbf{v} \cdot \mathbf{n}\|^2_{0,\Gamma_0} dt \leq & \int_0^T \|\mathbf{v}\|^2_{3/2\Delta,\Omega_f} dt \\ \leq \;& C(\|\hat{\mathbf{g}}\|^2_{1,\Gamma} + \|\hat{\mathbf{g}}_t\|^2_{0,\Gamma}) = C\|\mathbf{g}\|^2_Y \end{aligned}$$

Now, we obtain a bound for $\mathbf{u}$. Suppose $\mathbf{u}$ is a solution of (3.6.53). Then

$$\rho_s(\mathbf{u}_{tt}, \theta)_{\Omega_s} + \epsilon a(\mathbf{u}_t, \theta) + a_s(\mathbf{u}, \theta) = (\mathbf{h}, \theta)_{\Gamma_0} \quad \forall\, \theta \in H^1_s(\Omega_s) \tag{3.6.59}$$

Setting $\theta = \mathbf{u}_t$ in (3.6.59) to get

$$\begin{aligned}
&\sup_t \rho_s \|\mathbf{u}_t\|^2_{0,\Omega_s} + \epsilon K_a \int_0^T \|\mathbf{u}_t\|^2_{1,\Omega_s} dt + \sup_t K_a \|\mathbf{u}\|^2_{1,\Omega_s} \\
&\leq \;\; C \int_0^T \|\mathbf{h}\|^2_{0,\Gamma_0} dt = C\|\mathbf{h}\|^2_Z
\end{aligned}$$

We determine the constants M using bounds we have obtained.

$$\begin{aligned}
&\left| (\widetilde{\mathbf{g}}, \widetilde{\mathbf{h}}) \begin{pmatrix} \frac{\partial^2 \mathcal{J}}{\partial \mathbf{g}^2} & \frac{\partial^2 \mathcal{J}}{\partial \mathbf{g} \partial \mathbf{h}} \\ \frac{\partial^2 \mathcal{J}}{\partial \mathbf{h} \partial \mathbf{g}} & \frac{\partial^2 \mathcal{J}}{\partial \mathbf{h}^2} \end{pmatrix} \begin{pmatrix} \mathbf{g} \\ \mathbf{h} \end{pmatrix} \right| \\
= \;\; & | \int_0^T (\mathbf{u}_t - \mathbf{g}, \tilde{\mathbf{u}}_t - \tilde{\mathbf{g}})_{\Gamma_0} dt \\
& + \int_0^T (p\mathbf{n} - \mu_f \nabla \mathbf{v} \cdot \mathbf{n} - \mathbf{h}, \tilde{p}\mathbf{n} - \mu_f \nabla \tilde{\mathbf{v}} \cdot \mathbf{n} - \tilde{\mathbf{h}})_{\Gamma_0} dt \\
& + \delta \int_0^T (\mathbf{g}, \tilde{\mathbf{g}})_{\Gamma_0} dt + \delta \int_0^T (\nabla_\Gamma \mathbf{g}, \nabla_\Gamma \tilde{\mathbf{g}})_{\Gamma_0} dt + \delta \int_0^T (\mathbf{g}_t, \tilde{\mathbf{g}}_t)_{\Gamma_0} dt \\
& + \delta \int_0^T (\mathbf{h}, \tilde{\mathbf{h}})_{\Gamma_0} dt | \\
\leq \;\; & 2 (\int_0^T \|\mathbf{u}_t\|^2_{1,\Omega_s} dt + \int_0^T \|\mathbf{g}\|^2_{0,\Gamma_0} dt)^{1/2} (\int_0^T \|\tilde{\mathbf{u}}_t\|^2_{1,\Omega_s} dt + \int_0^T \|\tilde{\mathbf{g}}\|^2_{0,\Gamma_0} dt)^{1/2} \\
& + 2 (\int_0^T \|p\mathbf{n}\|^2_{0,\Gamma_0} dt + \mu_f^2 \int_0^T \|\nabla \mathbf{v} \cdot \mathbf{n}\|^2_{0,\Gamma_0} dt + \int_0^T \|\mathbf{h}\|^2_{0,\Gamma_0} dt)^{1/2} \\
& (\int_0^T \|\tilde{p}\mathbf{n}\|^2_{0,\Gamma_0} dt + \mu_f^2 \int_0^T \|\nabla \tilde{\mathbf{v}} \cdot \mathbf{n}\|^2_{0,\Gamma_0} dt + \int_0^T \|\tilde{\mathbf{h}}\|^2_{0,\Gamma_0} dt)^{1/2} \\
& + \delta (\int_0^T \|\mathbf{g}\|^2_{0,\Gamma_0} dt)^{1/2} (\int_0^T \|\tilde{\mathbf{g}}\|^2_{0,\Gamma_0} dt)^{1/2} \\
& + \delta (\int_0^T \|\nabla_\Gamma \mathbf{g}\|^2_{0,\Gamma_0} dt)^{1/2} (\int_0^T \|\nabla_\Gamma \tilde{\mathbf{g}}\|^2_{0,\Gamma_0} dt)^{1/2} \\
& + \delta (\int_0^T \|\mathbf{g}_t\|^2_{0,\Gamma_0} dt)^{1/2} (\int_0^T \|\tilde{\mathbf{g}}_t\|^2_{0,\Gamma_0} dt)^{1/2} \\
& + \delta (\int_0^T \|\mathbf{h}\|^2_{0,\Gamma_0} dt)^{1/2} (\int_0^T \|\tilde{\mathbf{h}}\|^2_{0,\Gamma_0} dt)^{1/2} \\
\leq \;\; & 2 (C\|\mathbf{h}\|^2_Z + \|\mathbf{g}\|^2_Y)^{1/2} (C\|\tilde{\mathbf{h}}\|^2_Z + \|\mathbf{g}\|^2_Y)^{1/2} \\
& + 2 (C\|\mathbf{g}\|^2_Y + \mu_f^2 C\|\mathbf{g}\|^2_Y + \|\mathbf{h}\|^2_Z)^{1/2} (C\|\tilde{\mathbf{g}}\|^2_Y + \mu_f^2 C\|\tilde{\mathbf{g}}\|^2_Y + \|\tilde{\mathbf{h}}\|^2_Z)^{1/2} \\
& + \delta \|\mathbf{g}\|_Y \|\tilde{\mathbf{g}}\|_Y + \delta \|\mathbf{h}\|_Y \|\tilde{\mathbf{h}}\|_Y \\
\leq \;\; & M \|\mathbf{x}\| \|\mathbf{y}\|
\end{aligned}$$

where, $M = 2C + 2(1 + \mu_f^2)C + \delta$.

Setting $\widetilde{\mathbf{g}} = \mathbf{g}$ and $\widetilde{\mathbf{h}} = \mathbf{h}$ to determine the constant m.

$$
\begin{aligned}
&(\mathbf{g}, \mathbf{h}) \begin{pmatrix} \frac{\partial^2 \mathcal{J}}{\partial \mathbf{g}^2} & \frac{\partial^2 \mathcal{J}}{\partial \mathbf{g} \partial \mathbf{h}} \\ \frac{\partial^2 \mathcal{J}}{\partial \mathbf{h} \partial \mathbf{g}} & \frac{\partial^2 \mathcal{J}}{\partial \mathbf{h}^2} \end{pmatrix} \begin{pmatrix} \mathbf{g} \\ \mathbf{h} \end{pmatrix} \\
= \;& \int_0^T \int_{\Gamma_0} (\mathbf{u}_t - \mathbf{g})^2 \, d\Gamma \, dt + \int_0^T \int_{\Gamma_0} (p\mathbf{n} - \mu_f \nabla \mathbf{v} \cdot \mathbf{n} - \mathbf{h})^2 d\Gamma \, dt \\
& + \delta \int_0^T \int_{\Gamma_0} \mathbf{g}^2 \, d\Gamma \, dt + \delta \int_0^T \int_{\Gamma_0} (\nabla_\Gamma \mathbf{g})^2 \, d\Gamma \, dt \\
& + \delta \int_0^T \int_{\Gamma_0} \mathbf{g}_t^2 \, d\Gamma \, dt + \delta \int_0^T \int_{\Gamma_0} \mathbf{h}^2 \, d\Gamma \, dt \\
= \;& \mathcal{J}(\mathbf{v}, p, \mathbf{g}, \mathbf{u}, \mathbf{h}) + \delta \|\mathbf{g}\|_Y^2 + \delta \|\mathbf{h}\|_Z^2 \geq \delta \|\mathbf{x}\|^2
\end{aligned}
$$

where $m = \delta$.

This proves the convergence of the Gradient method.

BIBLIOGRAPHY

[1] R. Adams, *Sobolev Spaces*, Academic, New York, 1975.

[2] A. Bermúdez, R. Durán and R. Rodríguez, Finite element analysis of compressible and incompressible fluid-solid systems. *Math. Comp.* 67(1998), no.221, pp.111-136.

[3] F.Blom, A monolithical fluid-structure interaction algorithm applied to the piston problem. *Comput. Methods Appl. Mech. Engrg.*167(1998), no.3-4, pp.369-391

[4] J. Boujot, Mathematical formulation of fluid-structure interaction problems. *RAIRO Modél. Math. Anal. Numér.* 21(1987), no.2. pp.239-260.

[5] K. Chrysafinos, M. D. Gunzburger and L. S. Hou, Error estimates for semidiscrete finite element approximations of linear and semilinear parabolic equations under minimal regularity assumptions, *to appear*.

[6] P. Ciarlet, *The finite element method for elliptic problems*, North-Holland, Amsterdam, 1978.

[7] C. Conca and M. Durán, A numerical study of a spectral problem in solid-fluid type structures. *Numer. Methods Partial Differential Equations* 11(1995), no.4, pp.423-444.

[8] C. Conca, J. Martín and J. Tucsnak, Motion of a rigid body in a viscous fluid. *C. R. Acad. Sci. Paris Sér. I Math.* 328(1999), no.6, pp.473-478.

[9] R. Dautray and J.-L. Lions, *Mathematical analysis and numerical methods for science and technology, Volume 1 Physical origins and classical methods*, Springer, Berlin, 1990.

[10] R. Dautray and J.-L. Lions, *Mathematical analysis and numerical methods for science and technology, Volume 6 Evolution problems II*, Springer, Berlin, 1988.

[11] S. Dasser, A Penalization method for the homogenization of a mixed fluid-structure problem. *C. R. Acad. Sci. Paris Sér. I Math.* 320(1995), n0.6, pp.759-764.

[12] B. Desjardins and M. J. Esteban, On weak solutions for fluid-rigid structure interaction: compressible and incompressible models. *Comm. Partial Differential Equations* 25(2000), no.7-8, pp.1399-1413

[13] D. Errate, M. J. Esteban and Y. Maday, Couplage fluid-structure. Un modèle simplifié en dimension 1, *C. R. Acad. Sci. Paris Sér. I Math.* 318(1994), pp.275-281.

[14] C. Farhat, M. Lesoinne and P. LeTallec, Load and motion transfer algorithms for fluid/structure interaction problems with non-matching discrete interfaces: momentum and energy conservation, optimal discretization and application to aeroelasticity. *Comput. Methods Appl. Mech. Engrg.* 157(1998), no.1-2, pp.95-114.

[15] F. Flori and P. Orenga, Fluid-structure interaction:analysis of a 3-D compressible model. *Ann. Inst. H. Poincaré anl. Non Linéaire* 17(2000), no.6, pp.753-777.

[16] F. Flori and P. Orenga, On a fluid-structure interaction problem. *Trends in applications of mathematics to mechanics (Nice, 1998)*, pp.293-305

[17] F. Flori and P. Orenga, On a nonlinear fluid-structure interaction problem defined on a domain depending on time. *Nonlinear Anal.* 38(1999), no.5, pp.549-569

[18] F. Flori and P. Orenga, Analysis of a nonlinear fluid-structure interaction problem in velocity-displacement formulation. *Nonlinear Anal.* 35(1999) , no.5, pp.561-587.

[19] L. Gastaldi, Mixed finite element methods in fluid structure system. *Numer. Math.* 74(1996), no.2, pp.153-176.

[20] V. Girault and P. Raviart, *Finite element methods for Navier-Stokes equations*, Springer, Berlin, 1986.

[21] C. Grandmont, Existence and uniqueness for a two-dimensional steady-state fluid-structure interaction problem. *C. R. Acad. Sci. Paris Sér. I Math.* 326(1998), no.5, pp.651-656.

[22] C. Grandmont and Y. Maday, Existence for an unsteady fluid-structure interaction problem. *M2AN Math. Model. Numer. Anal.* 34(2000), no.3, pp.609-636.

[23] C. Grandmont and Y. Maday, Existence of solutions for a two-dimensional unsteady fluid-structure interaction problem. *C. R. Acad. Sci. Paris Sér. I Math.* 326(1998), no.4, pp.525-530 no.3, pp.609-636.

[24] M. D. Gunzburger, L. S. Hou and T. Svobodny, Analysis and finite element approximation of optimal control problems for the stationary Navier-Stokes equations with Dirichlet control, *Math. Model. Numer. Anal.* 25(1991).

[25] M. D. Gunzburger, L. S. Hou and T. Svobodny, Optimal control problems for a class of nonlinear equations with an application to the control of fluids; *Optimal control of viscous flows*, ed. by S. Sritharan, SIAM, Philadelphia, 1998, pp.43-62.

[26] M. D. Gunzburger and J. Lee, A domain decomposition method for optimization problems for partial differential equations, *Comput. Math. Appl.* 40(2000), pp.177-192.

[27] M. D. Gunzburger and H. K. Lee, An optimization based domain decomposition method for the Navier-Stokes equations, *SIAM J. Numer. Anal.* 37(2000), pp.1455-1480.

[28] M. D. Gunzburger, J. S. Peterson and H. Kwon, An optimization based domain decomposition method for partial differential equations , *Comput. Math. Appl.* 37(1999), pp.77-93.

[29] J. G. Heywood and R. Rannacher, Finite element approximations of the nonstationary Navier-Stokes problem, I. Regularity solutions and second-order error estimates for spatial discretization, *SIAM J. Numer. Anal.* 19(1982), pp.275-311.

[30] L. S. Hou, Error estimates form semidiscrete finite element approximations of the Stokes equations under minimal regularity assumptions, *to appear*.

[31] G. Hsiao, R. Kleinman and G. Roach, Weak solutions of fluid-solid interaction problems. *Math. Nachr.* 218(2000), pp.139-163

[32] P. LeTallec and S. Mani, Numerical analysis of a linearised fluid-structure interaction problem, *Numer. Math.* 87(2000), no.2, pp.317-354

[33] J.-L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications I*, Springer-Verlag, Berlin, 1972.

[34] S. Micu and E. Zuazua, Asymptotics for the spectrum of a fluid/structure hybrid system arising in the control of noise. *SIAM J. Math. Anal.* 29(1998), no.4, pp.967-1001.

[35] R. Rodríguez and J. Solomin, The order of convergence of eigenfrequences in finite element approximations of fluid-structure interaction problems. *Math. Comp.* 65(1996), no.216, pp1463-1475.

[36] M.Rumpf, On equilibria in the interaction of fluids and elastic solids. Theory of the Navier-Stokes equations, *Ser. Adv. Math. Appl. Sci.* 47(1998), pp.136-158.

[37] R. Schulkes, Interations of an elastic solid with a viscous fluid: eigenmode analysis. *J. Comput. Phys.* 100(1992), no.2, pp.270-283.

[38] R. Temam, *Navier-Sotkes equations*, North-Holland, New York, 1977.

[39] V. Thomee, *Galerkin finite element methods for parabolic equations*, Springer, Berlin, 1997.

[40] M. J. Vivsik and A.V.Fursikov, Mathematical problems of statistical hydromechanics, *Mathematics and its applications in physics and technology* 41, 1986.