

**Exact boundary controllability results for sandwich beam systems**

by

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## DEDICATION

I dedicate this thesis to my late high school friend Suresh.

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## CHAPTER 1. Introduction

A *distributed parameter system* is generally understood to be a control system governed by a partial differential equation. Often the control function is distributed over a portion of the domain of the PDE or on its boundary. If the control function is a scalar function of time, it is referred to as a *scalar control system*. Linear scalar distributed systems can be represented as follows:

$$\dot{x}(t) = \mathcal{A}x(t) + bu(t) \tag{1.1}$$

$$x(0) = x_0, \tag{1.2}$$

where  $x(t)$  is called the *state* of the system,  $\mathcal{A}, b$  are linear operators and  $u(t)$  is the control input. In this thesis, we consider control systems of this type that arise from the study of composite beams.

The fundamental controllability problem is the following: given an initial state  $x(0)$  and a terminal state  $x(T)$ , find a control that "steers" the solution of (1.1-1.2) from  $x(0)$  to  $x(T)$  in time  $T$ . Implicit in solving this problem is describing an appropriate function space  $X$  for which (1.1-1.2) has a unique solution which evolves continuously in time (so that it makes sense to speak of initial and terminal values). If such a control function can be found for all possible  $x(0), x(T) \in X$ , the pair  $(\mathcal{A}, b)$  is said to *exactly controllable on  $X$  in time  $T$* .

There is a dual concept to controllability known as observability which is associated with the following system:

$$\dot{x}(t) = \mathcal{A}x(t), x(0) = x_0 \tag{1.3}$$

$$y(t) = \mathcal{C}x(t), \tag{1.4}$$

where  $\mathcal{A}, \mathcal{C}$  are known operators and  $x(0)$  is the unknown initial condition. The fundamental

observability problem involves finding the initial value  $x(0)$  (and hence the solution  $x(t) \forall t > 0$ ), given the observations  $y(t)$  in an interval  $[0, T]$ . If this is possible, then the pair  $(\mathcal{A}, \mathcal{C})$  is said to be observable in time  $T$ .

Let us briefly consider the finite dimensional situation with a single control input  $u(t)$ . In this case,  $x(t) \in \mathbb{R}^n$ ,  $\mathcal{A} \in \mathbb{R}^{n \times n}$  and  $b \in \mathbb{R}^{n \times 1}$ . The solvability of (1.1)-(1.2) follows from the existence and uniqueness theory of ordinary differential equations. The controllability of (1.1)-(1.2) is equivalent to the Kalman's rank condition i.e the matrix  $\mathcal{K} = [b \ \mathcal{A}b \ \mathcal{A}^2b \ \dots \ \mathcal{A}^{n-1}b]$  should have full row rank. Observability of (1.3)-(1.4) is equivalent to the condition that the matrix  $[\mathcal{C}, \ \mathcal{C}\mathcal{A}, \ \mathcal{C}\mathcal{A}^2, \ \dots, \ \mathcal{C}\mathcal{A}^{n-1}]^T$  have full column rank. Furthermore, controllability/observability in a specific time  $T_0$  implies the same in an arbitrary time  $t > 0$ . We have the following theorem which illustrates the duality between the two concepts:

**Theorem 1.** *The pair  $(\mathcal{A}, b)$  is controllable  $\Leftrightarrow (\mathcal{A}^*, b^*)$  is observable.*

However, in the case of infinite dimensional systems, where  $\mathcal{A}$  is an operator, the situation is more complex and its analysis involves machinery from functional and harmonic analysis. For example, Theorem 1 still holds, where the duality exists in a more general function space setting. In addition, controllability/observability in general depends on the time interval  $[0, T]$ .

On one hand, the problem of driving a solution of Equation (1.1) from a specified initial state to a specified terminal state in time  $T$  can often be equivalently stated as a moment problem of the following form:

$$\int_0^T e^{\lambda_i t} u(t) dt = c_i, \quad i = 1, 2, 3, \dots \quad (1.5)$$

where  $\{c_i\}$  and  $\{\lambda_i\}$  are given sequences of complex numbers and  $u(t)$  is the control function that we seek, usually belonging to  $L^2[0, T]$ , where  $T$  is the control time. The methods used to solve (1.5) depend on the type of PDE that describes (1.1) and involves techniques from functional analysis. In particular, the amount of damping in (1.1) greatly influences the form of the sequence  $\{e^{\lambda_i t}\}$  in (1.5) and hence also the techniques involved in solving the control problem. On the other hand, one can use techniques from partial differential equations to prove observability of a dual system of the form (1.3)-(1.4) which in turn implies exact controllability

of the original system. In particular one needs to show an observability estimate of the form

$$\int_0^T \|\mathcal{C}e^{At}x(0)\|^2 dt \geq C\|x(0)\|^2, \quad (1.6)$$

where  $C, T > 0$ . The estimate (1.6) implies that if the observation  $y(t) = \mathcal{C}e^{At}x(0)$  is zero for the time interval  $[0, T]$ , then the initial condition  $x(0)$  is zero. The technique of multipliers is used to show (1.6) after which the HUM principle (Hilbert's Uniqueness method, see for e.g [22],[18],[19]) is used to conclude exact controllability in a suitable function space.

A physical example of a scalar control system is a beam subject to boundary forces, which may be viewed as controls, e.g., an Euler-Bernoulli beam

$$w_{tt} + w_{xxxx} = 0, \quad 0 < x < 1, t > 0 \quad (1.7)$$

$$w(0, t) = w(1, t) = w_{xx}(1, t) = 0, w_{xx}(0, t) = u(t), \quad t > 0. \quad (1.8)$$

In this example the bending moment at the left end is viewed as the control. The state of the system is  $(w, w_t)$ , and an appropriate choice for the state space  $X$  is  $H_0^1(0, 1) \times H^{-1}(0, 1)$ . Using semigroup theory, the system can be represented in the form (1.1)-(1.2). For this particular example, it is known that exact controllability holds in the space  $H_0^1(0, 1) \times H^{-1}(0, 1)$  in any arbitrary time  $T > 0$ . This is in contrast to Dirichlet boundary control for the wave equation,

$$w_{tt} - w_{xx} = 0 \quad 0 < x < 1, t > 0 \quad (1.9)$$

$$w(0, t) = 0, w(1, t) = u(t), \quad t > 0, \quad (1.10)$$

where exact controllability in the space  $L^2(0, 1) \times H^{-1}(0, 1)$  holds only if  $T > 2$ . The difference is due to the fact that the wave equation has a finite wave speed, and there exist nontrivial solutions of the wave equation that do not touch a controlled boundary up to the time instant  $T = 2$ . (Thus observability as in (1.6) cannot hold if  $T \leq 2$ .)

In this thesis, we investigate exact boundary controllability results for two beam systems that arise in the theory of composite structures. Both of these models are based upon a three layer beam (or "sandwich beam") in which the central layer allows shear and the outer layers are rigid with respect to shear. The first of these, known as the Mead-Markus [25] model turns out to have controllability properties that are similar to the Euler Bernoulli system on

which the model is partly based (the outer layers). The second of these systems, known as the Rao-Nakra [30] model turns out to have controllability properties similar to the wave equation, due to the presence of finite wave speeds. To date, very little appears to have been published concerning boundary controllability of sandwich beam structures. For spectral studies we mention [8] and for stability studies we mention [13] and [9]. A huge body of literature exists for controllability of elastic systems. To mention just a few, we have [23],[21],[19],[20] and references therein.

## 1.1 Preliminaries

In this section we briefly describe a few elements of semigroup theory. A detailed treatment of the same can be found in [2],[27].

**Definition 1.** Let  $X$  be a Banach space. A one parameter family  $\mathcal{T}(t) : X \rightarrow X, 0 \leq t < \infty$ , of bounded linear operators is a semigroup on  $X$  if

1.  $\mathcal{T}(0) = I$  ( $I$  is the identity operator) on  $X$
2.  $\mathcal{T}(t + s) = \mathcal{T}(t)\mathcal{T}(s) \forall t, s \geq 0$ .

**Remark 1.** The family  $\{\mathcal{T}(t)\}_{t \in \mathbb{R}}$  is called a  $C_0$  group of operators if  $\mathcal{T}(t)$  is invertible  $\forall t \in \mathbb{R}$  and  $\mathcal{T}(t)^{-1} = \mathcal{T}(-t)$ .

**Definition 2.** A semigroup  $\mathcal{T}(t), 0 \leq t < \infty$ , of bounded linear operators is a strongly continuous semigroup if

$$\lim_{t \downarrow 0} \mathcal{T}(t)x = x, \forall x \in X. \quad (1.11)$$

A strongly continuous semigroup is sometimes called a  $C_0$  semigroup.

**Definition 3.**  $\mathcal{A}$  is the infinitesimal generator of a  $C_0$  semigroup  $\{\mathcal{T}(t)\}_{t \geq 0}$  if

$$\begin{aligned} \mathcal{A}x &= \lim_{t \downarrow 0} \frac{\mathcal{T}(t)x - x}{t} = \left. \frac{d^+ \mathcal{T}(t)x}{dt} \right|_{t=0} \\ D(\mathcal{A}) &= \{x \in X : \lim_{t \downarrow 0} \frac{\mathcal{T}(t)x - x}{t} \in X\}. \end{aligned} \quad (1.12)$$

We consider systems whose state evolves according to an equation given by the following:

$$\dot{x}(t) = \mathcal{A}x(t) + f(t) \quad (1.13)$$

$$x(0) = x_0, \quad (1.14)$$

where  $\mathcal{A}$  is the infinitesimal generator of a  $C_0$  semigroup on  $X$ ,  $f(t) \in L^2([0, T], X)$  and  $x_0 \in X$ . In Equation (1.13),  $X$  is referred to as the state space,  $x(t)$  is the state of the system and  $x_0$  is the initial condition.

There are conditions on the resolvent of the operator  $\mathcal{A}$  which are sufficient for  $\mathcal{A}$  to be the infinitesimal generator of a  $C_0$  semigroup  $\{\mathcal{T}(t)\}_{t \geq 0}$  of bounded operators (e.g Hille-Yoshida theorem (see [27])).

**Definition 4.** A mild solution in  $[0, T]$  for Equation (1.13) is defined as:

$$x(t) = \mathcal{T}(t)x_0 + \int_0^t \mathcal{T}(t-s)f(s)ds, \forall t \in [0, T], \quad (1.15)$$

where  $\{\mathcal{T}(t)\}_{t \geq 0}$  is the semigroup generated by  $\mathcal{A}$ . The condition  $f(t) \in L^1([0, T], X)$  guarantees the continuity of the mild solution defined above (i.e  $x(t) \in C([0, T], X)$ )(see [27]).

In this thesis we look at scalar control systems described as below.

$$\dot{x}(t) = \mathcal{A}x(t) + bu(t) \quad (1.16)$$

$$x(0) = x_0 \in X,$$

for which the mild solution is given by

$$x(t) = \mathcal{T}(t)x_0 + \int_0^t \mathcal{T}(t-s)bu(s)ds, \forall t \in [0, T]. \quad (1.17)$$

We define the control to state map as follows.

$$\Phi_t(u) = \int_0^t \mathcal{T}(t-s)bu(s)ds. \quad (1.18)$$

If  $b \in X$  then  $\Phi_t$  is a continuous map from  $L^2[0, T]$  to  $X$  and hence  $\|\Phi_t(u(\cdot))\|_X \leq C\|u(\cdot)\|_{L^2(0,t)}$ . However, for boundary control problems, the mapping  $\Phi_t$  may remain continuous even though  $b \notin X$ . If this is the case, we say that  $b$  is admissible. More precisely,

**Definition 5.**  $b$  is an admissible input for Equation (1.16) if  $\forall t > 0, \exists C > 0$  such that

$$\|\Phi_t(u(\cdot))\|_X \leq C\|u(\cdot)\|_{L^2(0,t)}. \quad (1.19)$$

Also Equation (1.17) remains valid for such an admissible input. Hence admissibility of  $b$  in (1.16) is equivalent to the continuity of the control to state map. Equation (1.19) guarantees that the mild solution  $x(t) \in C([0, T], X)$ .

If  $b \notin X$ , we might still be able to guarantee the (1.19). A necessary and sufficient condition for well-posedness is given by the Carleson measure criterion (see e.g [17],[31]).

**Definition 6.** The system of equations given by (1.16) with  $b$  admissible, is said to be exactly controllable in  $X$  in time  $T$ , if there exists a control input  $u(t) \in L^2[0, T]$  which can steer the system from any initial state  $x(0)$  to any final state  $x(T) \in X$  in the finite time interval  $[0, T]$ .

**Definition 7.** The system of equations given by (1.16) with  $b$  admissible, is said to be exactly null controllable in  $X$  in time  $T$ , if there exists a control input  $u(t) \in L^2[0, T]$  which can steer the system from any initial state  $x(0)$  to the origin in the finite time interval  $[0, T]$ .

In Definitions 6 and 7, *steering* means that the system evolves from any initial condition  $x_0 \in X$  to any final condition  $x_T \in X$  in a finite time interval  $[0, T]$  according to Equation (1.15), maintaining  $x(t) \in C([0, T], X)$ .

**Definition 8.** The set of all states that can be reached starting at the origin using a control  $u(t) \in L^2[0, T]$  in time  $T$  is called the reachable subspace  $\mathcal{R}$  i.e

$$\mathcal{R} = \{x_T \in X : x_T = \int_0^T \mathcal{T}(T-s)bu(s)ds\}. \quad (1.20)$$

It is clear that  $\mathcal{R}$  is a subspace since if  $x_1 = \int_0^T \mathcal{T}(T-s)bu_1(s)ds$  and  $x_2 = \int_0^T \mathcal{T}(T-s)bu_2(s)ds$ , then  $\alpha x_1 + \beta x_2 = \int_0^T \mathcal{T}(T-s)b(\alpha u_1(s) + \beta u_2(s))ds$  i.e the state  $\alpha x_1 + \beta x_2$  can be reached from the origin using the control  $\alpha u_1(s) + \beta u_2(s)$  in time  $T$ .

Next we prove a theorem connecting the above concepts.

**Theorem 2.** Let  $\{\mathcal{T}(t)\}_{t \in \mathbb{R}}$  be a  $C_0$  group of operators i.e they are invertible and  $(\mathcal{T}(t))^{-1} = \mathcal{T}(-t) \forall t \in \mathbb{R}$ . Then the following are equivalent:

1.  $\mathcal{R} = X$  in time  $T$ .
2. The system (1.16) is exactly null controllable in  $X$  in time  $T$ .
3. The system (1.16) is exactly controllable in  $X$  in time  $T$ .

**Proof :** (1)  $\Rightarrow$  (2) Since the entire space is reachable, the state  $-\mathcal{T}(T)x(0)$  is reachable.

Hence we have

$$-\mathcal{T}(T)x(0) = \int_0^T \mathcal{T}(T-s)bu(s)ds.$$

Hence, given any initial state  $x(0)$ , we have

$$x(T) = \mathcal{T}(T)x(0) + \int_0^T \mathcal{T}(T-s)bu(s)ds = -\int_0^T \mathcal{T}(T-s)bu(s)ds + \int_0^T \mathcal{T}(T-s)bu(s)ds = 0$$

(2)  $\Rightarrow$  (3) Let  $x(0), x(T)$  be arbitrary initial and final states in  $X$  respectively. We are given that there exist controls  $u_1(t)$  and  $u_2(t)$  in  $L^2[0, T]$  such that the states  $x(0)$  and  $-\mathcal{T}(-T)x(T)$  can be steered to zero i.e there exist controls  $u_1(t)$  and  $u_2(t)$  such that the following are true:

$$\begin{aligned} \mathcal{T}(T)x(0) + \int_0^T \mathcal{T}(T-s)bu_1(s)ds &= 0. \\ -x(T) + \int_0^T \mathcal{T}(T-s)bu_2(s)ds &= 0. \end{aligned}$$

Adding the above two equations and rearranging we get,

$$x(T) = \mathcal{T}(t)x(0) + \int_0^T \mathcal{T}(T-s)b(u_1(s) + u_2(s))ds,$$

or in other words,  $x(0)$  can be steered to  $x(T)$  in time  $T$  using the control  $u_1(s) + u_2(s)$ .

(3)  $\Rightarrow$  (1) We are given that the entire space  $X$  is exactly controllable. This means that in particular, we can reach any state  $x(T) \in X$  from the origin in time  $T$  using a control  $u(t)$ . This means that  $\mathcal{R} = X$ .

**Remark 2.** The group property is used only to show (2)  $\Rightarrow$  (3). Otherwise (1)  $\Leftrightarrow$  (3)  $\Rightarrow$  (2) even in the semigroup case. The collection of operators  $\{\mathcal{T}(t)\}_{t \in \mathbb{R}}$  forms a group if the system (1.16) is conservative i.e there exists a certain energy that is conserved for all time. An example of this type is considered in Chapter 3.

**Remark 3.** If  $\{\mathcal{T}(t)\}_{t \in \mathbb{R}}$  is not invertible, then 1,2 and 3 in Theorem 2 are not generally equivalent. For example, in the case of Dirichlet control on the boundary applied to the heat equation, 2 is known to hold, while  $\mathcal{R}$  is only dense in  $X$  ( $\mathcal{R} \neq X$ ). In this case, 3 also fails. However, a weaker concept called approximate controllability does hold i.e given an  $\epsilon > 0$ ,  $x(0), x(T) \in X$ , there exists a trajectory  $x(t)$  from  $B_\epsilon(x(0))$  to  $B_\epsilon(x(T))$  where  $B_\epsilon(x(0)), B_\epsilon(x(T))$  are balls of radii  $\epsilon$  centered at  $x(0), x(T)$  respectively.

This thesis is organized as follows. In Chapter 2, we consider the Mead-Markus model of a damped three layer sandwich beam (see e.g [25]). The moment method is used to conclude null controllability in a certain function space. In Chapter 3, we consider a conservative Rao-Nakra (see [30]) beam. In this case, the multiplier method is used to prove an exact controllability result. In Chapter 4, we summarize some controllability results that were proven using the moment method in [14],[15] and [16] for a multilayer Rao-Nakra sandwich beam. In Chapter 5, we conclude with a summary of both methods and topics of future research. In the appendices, we mention some open problems.

## CHAPTER 2. Null controllability of a damped Mead-Markus sandwich beam using the moment method

### 2.1 Introduction

In this chapter, boundary null-controllability of the Mead-Markus model (see [25]) of a damped sandwich beam with shear damping is considered. A null controllability result is proved using the moment method. The equations of motion as formulated in Fabiano and Hansen [3] are as given below :

$$mw_{tt} + \left(A + \frac{B^2}{C}\right)w_{xxxx} - \frac{B}{C}s_{xxx} = 0 \quad (2.1)$$

$$\beta s_t + \gamma s - \frac{1}{C}s_{xx} + \frac{B}{C}w_{xxx} = 0, \quad (2.2)$$

with homogenous boundary conditions

$$w(0, t) = w(1, t) = 0, s_x(1, t) = 0, w_{xx}(1, t) = 0, \quad (2.3)$$

with controlled moment at the left end (see Hansen [10])

$$w_{xx}(0, t) = u(t), s_x(0, t) = Bu(t), \quad (2.4)$$

and initial conditions :

$$w(x, 0) = w^0(x), w_t(x, 0) = w^1(x), s(x, 0) = s^0(x). \quad (2.5)$$

In the above,  $w$  denotes the transverse displacement of the beam,  $s$  is proportional to the shear of the middle layer,  $u(t)$  represents moment control,  $m$  is the mass of the beam,  $A, B$  and  $C$  are material constants,  $\gamma$  and  $\beta$  are the elastic and damping coefficients of the middle layer respectively. For simplicity we assume that the beam is of unit length.

In this chapter, we prove that (2.1)-(2.5) is exactly null controllable to a particular one-dimensional state in which transverse displacement vanishes. The chapter is organized as follows. In Section 2.2, the semigroup formulation of (2.1)-(2.5) is discussed. The spectral analysis of (2.1)-(2.5) is done in Section 2.3. The wellposedness of (2.1)-(2.5) is discussed in Section 2.4, and the moment problem and its solution is discussed in Section 2.5.

## 2.2 Semigroup formulation

Let

$$\begin{pmatrix} w_1(t, x) \\ w_2(t, x) \\ w_3(t, x) \end{pmatrix} = \begin{pmatrix} w(t, x) \\ w_t(t, x) \\ s(t, x) \end{pmatrix}.$$

The arguments  $(t, x)$  will be omitted from now on for simplicity. First we consider the problem (2.1)-(2.5) with  $u(t) = 0$  (i.e the case of homogenous boundary conditions) and obtain the following:

$$\frac{d}{dt} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \mathcal{A}(\beta) \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, \quad \begin{pmatrix} w_1(0) \\ w_2(0) \\ w_3(0) \end{pmatrix} = \begin{pmatrix} w^0(x) \\ w^1(x) \\ s^0(x) \end{pmatrix},$$

where (using  $D = \frac{\partial}{\partial x}$ )

$$\left\{ \begin{array}{l} \mathcal{A}(\beta) = \begin{pmatrix} 0 & I & 0 \\ -\frac{1}{m}(A + \frac{B^2}{C})D^4 & 0 & \frac{B}{Cm}D^3 \\ -\frac{B}{C\beta}D^3 & 0 & (-\frac{\gamma}{\beta} + \frac{1}{C\beta}D^2) \end{pmatrix}, \\ \mathcal{D}(\mathcal{A}(\beta)) = H_w^4 \times H_w^2 \times H_s^3 \end{array} \right. \quad (2.6)$$

and

$$H_w^4 = \{\phi \in H^4(0, 1) : \phi(0) = \phi(1) = \phi_{xx}(0) = \phi_{xx}(1) = 0\} \quad (2.7)$$

$$H_w^2 = \{\phi \in H^2(0, 1) : \phi(0) = \phi(1) = 0\}$$

$$H_s^3 = \{\phi \in H^3(0, 1) : \phi_x(0) = \phi_x(1) = 0\},$$

$\mathcal{A}(\beta) : \mathcal{D}(\mathcal{A}(\beta)) \rightarrow \mathcal{H} = H_w^2 \times L^2(0, 1) \times H^1(0, 1)$  where  $\mathcal{H}$  is a Hilbert space with the following energy inner product (see Fabiano, Hansen [3]) :

$$\left\langle \begin{pmatrix} f \\ g \\ h \end{pmatrix}, \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right\rangle_e = \frac{1}{2} \int_0^1 (g\bar{v} + \frac{A}{m} f_{xx} \bar{u}_{xx} + \frac{\gamma}{m} h \bar{w}) + \frac{1}{Cm} (Bf_{xx} - h_x)(B\bar{u}_{xx} - \bar{w}_x) dx.$$

**Theorem 3.**  $\mathcal{A}^*(\beta) = -\mathcal{A}(-\beta)$  and  $\mathcal{D}(\mathcal{A}^*(\beta)) = \mathcal{D}(\mathcal{A}(\beta))$

**Proof:** We use the definition of  $\mathcal{A}$  and  $\mathcal{D}(\mathcal{A})$  from Equations (2.6) and (2.7).

Let

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} \in \mathcal{D}(\mathcal{A}), \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \mathcal{H}.$$

We calculate the following:

$$\left\langle \mathcal{A}(\beta) \begin{pmatrix} f \\ g \\ h \end{pmatrix}, \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right\rangle_e.$$

We omit the factor of half for simplicity. We have several terms as follows:

$$\begin{aligned} \text{Term 1} &= \int_0^1 \left( -\frac{1}{m} \left( A + \frac{B^2}{C} \right) f_{xxx} + \frac{B}{C} h_{xxx} \right) \bar{v} + \frac{A}{m} g_{xx} \bar{u}_{xx}. \\ \text{Term 2} &= \int_0^1 \frac{\gamma}{m} \left( -\frac{B}{C\beta} f_{xxx} - \frac{\gamma}{\beta} h + \frac{1}{C\beta} h_{xx} \right) \bar{w} \\ \text{Term 3} &= \int_0^1 \frac{1}{Cm} \left( Bg_{xx} - \left( -\frac{B}{C\beta} f_{xxx} - \frac{\gamma}{\beta} h_x + \frac{1}{C\beta} \right) \right) (B\bar{u}_{xx} - \bar{w}_x). \end{aligned}$$

We look at terms involving  $g$  first. We have

$$\begin{aligned} \int_0^1 \frac{B}{Cm} g_{xx} (B\bar{u}_{xx} - \bar{w}_x) &= \int_0^1 \frac{B}{Cm} g (B\bar{u}_{xxxx} - \bar{w}_{xxx}) + \frac{B}{Cm} g_x (B\bar{u}_{xx} - \bar{w}_x) \Big|_{x=0}^{x=1}. \\ \int_0^1 \frac{A}{m} g_{xx} \bar{u}_{xx} &= \frac{A}{m} \int_0^1 g \bar{u}_{xxxx} + \frac{A}{m} g_x \bar{u}_{xx} \Big|_{x=0}^{x=1}. \end{aligned}$$

So collecting all  $g$  terms, we have

$$g \text{ terms} = \int_0^1 \frac{B}{Cm} g(B\bar{u}_{xxxx} - \bar{w}_{xxx}) + \frac{A}{m} \int_0^1 g\bar{u}_{xxxx} + \frac{B}{Cm} g_x(B\bar{u}_{xx} - \bar{w}_x)|_{x=0}^{x=1} + \frac{A}{m} (g_x\bar{u}_{xx})|_{x=0}^{x=1}.$$

Next we look at terms involving  $f_{xx}$ .

$$\begin{aligned} -\frac{\gamma B}{mC\beta} \int_0^1 f_{xxx}\bar{w} &= \frac{\gamma B}{mC\beta} \int_0^1 f_{xx}\bar{w}_x. \\ \frac{B^2}{C^2m\beta} \int_0^1 f_{xxxx}\bar{u}_{xx} &= -\frac{B}{C^2m\beta} \int_0^1 f_{xx}\bar{u}_{xxxx} + \frac{B^2}{C^2m\beta} (f_{xxx}\bar{u}_{xx})|_{x=0}^{x=1}. \\ -\frac{B}{C^2m\beta} \int_0^1 f_{xxxx}\bar{w}_x &= -\frac{B}{C^2m\beta} \int_0^1 f_{xx}\bar{w}_{xxx} - \frac{B}{C^2m\beta} (f_{xxx}\bar{w}_x)|_{x=0}^{x=1}. \end{aligned}$$

Collecting all terms involving  $f_{xx}$ , we have:

$$\begin{aligned} f_{xx} \text{ terms} &= \frac{\gamma B}{mC\beta} \int_0^1 f_{xx}\bar{w}_x - \frac{B}{C^2m\beta} \int_0^1 f_{xx}\bar{u}_{xxxx} - \frac{B}{C^2m\beta} \int_0^1 f_{xx}\bar{w}_{xxx} \\ &\quad + \frac{B^2}{C^2m\beta} (f_{xxx}\bar{u}_{xx})|_{x=0}^{x=1} - \frac{B}{C^2m\beta} (f_{xxx}\bar{w}_x)|_{x=0}^{x=1}. \end{aligned}$$

Next we look at terms involving  $h$ .

$$\begin{aligned} -\frac{\gamma^2}{m\beta} \int_0^1 h\bar{w} \\ \frac{\gamma B}{mC\beta} \int_0^1 h_x\bar{u}_{xx} &= -\frac{\gamma B}{mC\beta} \int_0^1 h\bar{u}_{xxx}. \\ \frac{\gamma}{mC\beta} \int_0^1 h_{xx}\bar{w} &= \frac{\gamma}{mC\beta} \int_0^1 h\bar{w}_{xx} - \frac{\gamma}{mC\beta} (h\bar{w}_x)|_{x=0}^{x=1}. \end{aligned}$$

Collecting all terms involving  $h$ , we get:

$$h \text{ terms} = -\frac{\gamma^2}{m\beta} \int_0^1 h\bar{w} - \frac{\gamma B}{mC\beta} \int_0^1 h\bar{u}_{xxx} + \frac{\gamma}{mC\beta} \int_0^1 h\bar{w}_{xx} - \frac{\gamma}{mC\beta} (h\bar{w}_x)|_{x=0}^{x=1}.$$

Finally we look at terms involving  $h_x$ .

$$\begin{aligned} \frac{B}{C} \int_0^1 h_{xxx}v &= \frac{B}{C} \int_0^1 h_x\bar{v}_{xx} + \frac{B}{C} (h_{xx}\bar{v})|_{x=0}^{x=1}. \\ -\frac{B}{C^2m\beta} \int_0^1 h_{xxx}\bar{u}_{xx} &= -\frac{B}{C^2m\beta} \int_0^1 h_x\bar{u}_{xxxx} - \frac{B}{C^2m\beta} (h_{xx}\bar{u}_{xx})|_{x=0}^{x=1}. \\ \frac{1}{C^2m\beta} \int_0^1 h_{xxx}\bar{w}_x &= \frac{1}{C^2m\beta} \int_0^1 h_x\bar{w}_{xxx} + \frac{1}{C^2m\beta} (h_{xx}\bar{w}_x)|_{x=0}^{x=1}. \end{aligned}$$

$$\frac{\gamma}{mC\beta} \int_0^1 h_{xx} \bar{w} = -\frac{\gamma}{mC\beta} \int_0^1 h_x \bar{w}_x.$$

Collecting terms involving  $h_x$ , we have

$$\begin{aligned} h_x \text{ terms} &= \frac{B}{C} \int_0^1 h_x \bar{v}_{xx} - \frac{B}{C^2 m \beta} \int_0^1 h_x \bar{u}_{xxxx} + \frac{1}{C^2 m \beta} \int_0^1 h_x \bar{w}_{xxx} - \frac{\gamma}{mC\beta} \int_0^1 h_x \bar{w}_x \\ &\quad + \frac{B}{C} (h_{xx} \bar{v})|_{x=0}^{x=1} - \frac{B}{C^2 m \beta} (h_{xx} \bar{u}_{xx})|_{x=0}^{x=1} + \frac{1}{C^2 m \beta} (h_{xx} \bar{w}_x)|_{x=0}^{x=1}. \end{aligned}$$

We have taken care of all terms. Adding terms involving  $g$ ,  $f_{xx}$ ,  $h$  and  $h_x$  we have the following:

$$\begin{aligned} \left\langle \mathcal{A}(\beta) \begin{pmatrix} f \\ g \\ h \end{pmatrix}, \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right\rangle_e &= \frac{1}{2} \int_0^1 g \left( \frac{1}{m} \bar{u}_{xxxx} - \frac{B^2}{Cm} \bar{w}_{xxx} \right) - \frac{A}{2m} \int_0^1 f_{xx} \bar{v}_{xx} \\ + \frac{\gamma}{2m} \int_0^1 h \left( -\frac{B}{C\beta} \bar{v}_{xxx} - \frac{\gamma}{\beta} \bar{w} + \frac{1}{C\beta} \bar{w}_{xx} \right) &+ \frac{1}{2} \int_0^1 (Bf_{xx} - h_x) \left( -B\bar{v}_{xx} - \frac{1}{C\beta} \bar{w}_{xxx} + \frac{B}{C\beta} \bar{u}_{xxxx} + \frac{\gamma}{\beta} \bar{w}_x \right) \\ + \frac{B}{Cm} g_x (B\bar{u}_{xx} - \bar{w}_x)|_{x=0}^{x=1} &+ \frac{A}{m} (g_x \bar{u}_{xx})|_{x=0}^{x=1} + \frac{A}{m} (g_x \bar{u}_{xx})|_{x=0}^{x=1} + \frac{B^2}{C^2 m \beta} (f_{xxx} \bar{u}_{xx})|_{x=0}^{x=1} \\ - \frac{B}{C^2 m \beta} (f_{xxx} \bar{w}_x)|_{x=0}^{x=1} &- \frac{\gamma}{mC\beta} (h\bar{w}_x)|_{x=0}^{x=1} + \frac{B}{C} (h_{xx} \bar{v})|_{x=0}^{x=1} - \frac{B}{C^2 m \beta} (h_{xx} \bar{u}_{xx})|_{x=0}^{x=1} \\ &\quad + \frac{1}{C^2 m \beta} (h_{xx} \bar{w}_x)|_{x=0}^{x=1}. \end{aligned}$$

If we choose

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \mathcal{D}(\mathcal{A}(\beta)),$$

then all the boundary terms above vanish. Finally, we see that

$$\begin{aligned} \left\langle \mathcal{A}(\beta) \begin{pmatrix} f \\ g \\ h \end{pmatrix}, \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right\rangle_e &= \frac{1}{2} \int_0^1 g \left( \frac{1}{m} \bar{u}_{xxxx} - \frac{B^2}{Cm} \bar{w}_{xxx} \right) - \frac{A}{2m} \int_0^1 f_{xx} \bar{v}_{xx} \\ + \frac{\gamma}{2m} \int_0^1 h \left( -\frac{B}{C\beta} \bar{v}_{xxx} - \frac{\gamma}{\beta} \bar{w} + \frac{1}{C\beta} \bar{w}_{xx} \right) &+ \frac{1}{2} \int_0^1 (Bf_{xx} - h_x) \left( -B\bar{v}_{xx} - \frac{1}{C\beta} \bar{w}_{xxx} + \frac{B}{C\beta} \bar{u}_{xxxx} + \frac{\gamma}{\beta} \bar{w}_x \right) \\ &= \left\langle \begin{pmatrix} f \\ g \\ h \end{pmatrix}, -\mathcal{A}(-\beta) \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right\rangle_e, \end{aligned}$$

where

$$-\mathcal{A}(-\beta) \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -v \\ \frac{1}{m}(A + \frac{B^2}{C})u_{xxxx} - \frac{B}{Cm} \\ -\frac{B}{C\beta}u_{xxx} - \frac{\gamma}{\beta}w + \frac{1}{c\beta}w_{xx} \end{pmatrix}.$$

Hence we have the proof of Theorem 3.

**Theorem 4.**  $\mathcal{A}(\beta)$  is the infinitesimal generator of a strongly continuous semigroup on  $\mathcal{H}$ . Furthermore  $\mathcal{A}(\beta)$  extends by duality to an isomorphic semigroup on  $\mathcal{H}_{-\frac{1}{2}} := H_0^1(0, 1) \times H^{-1}(0, 1) \times L^2(0, 1)$ .

**Proof:** We check that both  $\mathcal{A}$  and  $\mathcal{A}^*$  are dissipative on  $\mathcal{H}$ . Then the result will follow from the Lumer-Phillips theorem (see for e.g. Pazy [27]). We show the calculation for  $\mathcal{A}$ . Since from Theorem 3, we have that  $\mathcal{A}(\beta)^* = -(\mathcal{A}(-\beta))$ , same is true for  $\mathcal{A}(\beta)^*$ .

$$\begin{aligned} \left\langle \mathcal{A} \begin{pmatrix} f \\ g \\ h \end{pmatrix}, \begin{pmatrix} f \\ g \\ h \end{pmatrix} \right\rangle_e &= \frac{1}{2} \int_0^1 \frac{-1}{m} (A + \frac{B^2}{C}) f_{xxxx} \bar{g} + \frac{B}{Cm} h_{xxx} \bar{g} + \frac{A}{m} g_{xx} \bar{f}_{xx} \\ &\quad + \frac{\gamma}{m} (\frac{-B}{C\beta} f_{xxx} \bar{h} - \frac{\gamma}{\beta} h \bar{h} + \frac{1}{C\beta} h_{xx} \bar{h}) \\ &\quad + \frac{1}{Cm} (B g_{xx} + \frac{B}{C\beta} f_{xxxx} + \frac{\gamma}{\beta} h_x - \frac{1}{C\beta} h_{xxx}) (B \bar{f}_{xx} - \bar{h}_x) \\ &= -\frac{\gamma^2}{m\beta} |h|^2 - \frac{B^2}{mC^2\beta} |f_{xxx}|^2 - \frac{B\gamma}{mC\beta} f_{xxx} \bar{h} - \frac{B\gamma}{mC\beta} \bar{f}_{xx} h \\ &= \frac{-1}{2m\beta} |\gamma h + \frac{B}{C} f_{xxx}|^2 \leq 0. \end{aligned}$$

The semigroup is easily extended by duality to  $\mathcal{H}_{-1} = (\mathcal{D}(\mathcal{A}(\beta)^*))^*$  as in e.g. Weiss [31]. The interpolation to  $\mathcal{H}_{-\frac{1}{2}} := H_0^1(0, 1) \times H^{-1}(0, 1) \times L^2(0, 1)$  is easily justified once the Riesz basis property is proved. Hence pending the proof of Theorem 5, Theorem 4 is proved.

**Remark 4.** The space  $\mathcal{H}_{-\frac{1}{2}}$  is the completion of the set of finite linear combinations of eigenfunctions with respect to the norm given by  $\|\sum_{k \in J} c_k \phi_k\|_{\mathcal{H}_{-\frac{1}{2}}} = \|\sum_{k \in J} \frac{c_k}{\sqrt{|\lambda_k|}} \phi_k\|_e$ . The eigenfunctions of  $\mathcal{A}(\beta)$  are seen to be sinusoidal in Section 2.3. This can be used to show that  $\mathcal{H}_{-\frac{1}{2}} := H_0^1(0, 1) \times H^{-1}(0, 1) \times L^2(0, 1)$ .

Using a standard integration by parts against a function in  $\mathcal{D}(\mathcal{A}(\beta))$ , one can reformulate (2.1) - (2.5) as a problem with homogenous boundary conditions, but non-homogenous right hand side. Formally, one obtains the following system:

$$\frac{d}{dt} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \mathcal{A}(\beta) \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} + fu(t) \quad (2.8)$$

$$f = \begin{pmatrix} 0 \\ \frac{1}{m}(A + \frac{B^2}{C})\delta' \\ 0 \end{pmatrix}, \quad \begin{pmatrix} w_1(0) \\ w_2(0) \\ w_3(0) \end{pmatrix} = \begin{pmatrix} w^0(x) \\ w^1(x) \\ s^0(x) \end{pmatrix}, \quad (2.9)$$

where in (2.9),  $\delta'$  is the distributional derivative of the Dirac delta distribution. Henceforth we adopt the above formulation of (2.1) - (2.5). We will see later that (2.8) - (2.9) is well posed on a subspace of  $\mathcal{H}_{-\frac{1}{2}}$ .

**Remark 5.** We clarify the appearance of  $\delta'$  in (2.8)-(2.9) by the following argument. Let  $\phi(x)$  be a test function satisfying all the homogenous boundary conditions in Equation (2.7). We multiply (2.1) and (2.2) by  $\phi(x)$  and integrate by parts in space using (2.3) and (2.4). We get the following weak formulation:

$$\int_0^1 mw_{tt} + (A + \frac{B^2}{C}) \int_0^1 w\phi_{xxxx} + \int_0^1 s\phi_{xxx} = -\frac{1}{m}(A + \frac{B^2}{C})\phi_x(0)u(t)$$

$$\int_0^1 \beta s_t\phi + \int_0^1 s\phi - \frac{1}{C} \int_0^1 s\phi_{xx} - (w_x(1)\phi_x(1) - w_x(0)\phi_x(0)) - \int_0^1 w\phi_{xxx} = 0.$$

The term  $-\phi_x(0)$  justifies the term  $\delta'$  in (2.8) - (2.9). Furthermore (2.8) - (2.9) also yield the same weak problem as above.

### 2.3 Spectral Analysis of $\{\mathcal{A}(\beta)\}$

Under some parametric restrictions, it is shown in Lemma 1 that the spectrum of  $\mathcal{A}(\beta)$  (henceforth denoted by  $\Lambda$ ) consists of a negative real branch  $\{\lambda_{k,1}\}_{k=1}^\infty \cup \lambda_0$  and two complex conjugate branches  $\{\lambda_{k,2}\}_{k=1}^\infty, \{\lambda_{k,3}\}_{k=1}^\infty$ , where  $\lambda_0 = -\frac{\gamma}{\beta}$ . It is also shown that each of the

three branches grow asymptotically at a quadratic rate with  $\lim_{k \rightarrow \infty} \arg(-\lambda_{k,j}) = \theta$ , for some  $\theta \in (0, \frac{\pi}{2})$ ,  $j = 1, 2, 3$ . The eigenfunctions can be calculated and are given by the following:

$$\phi_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \{\phi_{k,j}\} = \begin{pmatrix} \frac{1}{\lambda_{k,j}} \sin(\alpha_k x) \\ \sin(\alpha_k x) \\ A_{k,j} \cos(\alpha_k x) \end{pmatrix}, \quad (2.10)$$

$$A_{k,j} = \frac{\lambda_{k,j} + \frac{1}{m}(A + \frac{B^2}{C})\frac{\alpha_k^4}{\lambda_{k,j}}}{\frac{B}{C}\alpha_k^3}, \quad (2.11)$$

where  $\alpha_k = k\pi$  and  $\lambda_{k,j}$ 's are the solutions of the following cubic equation [3]:

$$\begin{aligned} \beta m \lambda_k^3 + \left(\frac{\alpha_k^2}{C} + \gamma\right) m \lambda_k^2 + \beta \frac{AC + B^2}{C} \alpha_k^4 \lambda_k \\ + \frac{AC + B^2}{C} \alpha_k^4 \left(\gamma + \frac{A}{AC + B^2} \alpha_k^2\right) = 0. \end{aligned} \quad (2.12)$$

The following change of variables

$$x = \sqrt{\frac{m}{A}} \frac{1}{\alpha_k^2} \lambda_k, \quad \hat{\beta} = \sqrt{\frac{m}{A}} \frac{1}{C\beta}, \quad \kappa = 1 + \frac{B^2}{AC},$$

converts (2.12) to the following:

$$x^3 + \hat{\beta}x^2 + \kappa x + \hat{\beta} + \frac{\hat{\beta}C\gamma}{\alpha_k^2}(x^2 + \kappa) = 0. \quad (2.13)$$

**Lemma 1.** *If  $\kappa \in (1, 9]$ ,  $\hat{\beta} > \frac{\kappa}{\sqrt{2}}$ , and  $\gamma < \frac{1}{\kappa C}$  the eigenvalues are separated and the following estimates hold for  $j = 1, 2, 3$ :*

$$\exists \theta \in (0, \frac{\pi}{2}) \text{ such that } |\arg(-\lambda_{k,j})| \leq \theta, \quad \forall k \in \mathbb{N}, \quad (2.14)$$

$$\exists C_1, C_2 > 0 \text{ such that } C_1 k^2 \leq |\lambda_{k,j}| \leq C_2 k^2, \quad \forall k \in \mathbb{N}, \quad (2.15)$$

$$\exists \bar{\sigma} > 0 \text{ such that } |\lambda_{m,j} - \lambda_{n,j}| \geq \bar{\sigma} |m^2 - n^2|, \quad \forall m, n \in \mathbb{N}. \quad (2.16)$$

**Proof:** We observe that as  $\beta$  takes all values from 0 to  $\infty$ , so does  $\hat{\beta}$  and vice-versa. We also observe that as  $k \rightarrow \infty$ , Rouché's theorem implies that the roots of (2.13) with  $\gamma \neq 0$  will be close to the roots of (2.13) with  $\gamma = 0$ . Thus, let us consider the case  $\gamma = 0$  first. Then we are interested in the roots of

$$f(x) = x^3 + \hat{\beta}x^2 + \kappa x + \hat{\beta} = 0. \quad (2.17)$$

It can be shown [3] that if  $1 < \kappa \leq 9$  then (2.17) has one negative real root (say  $a < 0$ ) and a pair of complex conjugate roots ( $b \pm ci$ ,  $b < 0$ ). Each root of (2.12) with  $\gamma = 0$  is a real multiple of one of the three roots of (2.17). The roots of (2.12) with  $\gamma = 0$  are given by

$$s_{k,1} = a(k\pi)^2 \sqrt{\frac{A}{m}}, \quad s_{k,2} = (b + ci)(k\pi)^2 \sqrt{\frac{A}{m}}, \quad s_{k,3} = (b - ci)(k\pi)^2 \sqrt{\frac{A}{m}}.$$

The argument of the branches is given by the argument of the roots of (2.17). Hence the roots lie within a sector in the left half plane; i.e.  $\exists \theta \in (0, \frac{\pi}{2})$  such that  $|\arg(-s_{k,j})| \leq \theta \forall k \in \mathbb{N}, j = 1, 2, 3$ . The  $s_{k,j}$ 's satisfy  $|s_{m,j} - s_{n,j}| \geq \sigma|(m^2 - n^2)|, \forall m, n \in \mathbb{N}$  for some  $\sigma > 0$ . Now, we return to the roots of (2.12) with  $\gamma \neq 0$ . The following change of variables

$$y = x \sqrt{\frac{1 + \frac{C\gamma}{\alpha_k^2}}{1 + \frac{C\gamma\kappa}{\alpha_k^2}}}$$

converts (2.13) to the following form:

$$y^3 + \hat{\beta}' y^2 + \kappa' y + \hat{\beta}' = 0, \quad (2.18)$$

where

$$\hat{\beta}' = \hat{\beta} \frac{(1 + \frac{C\gamma}{\alpha_k^2})^{\frac{3}{2}}}{\sqrt{1 + \frac{C\gamma\kappa}{\alpha_k^2}}}, \quad \kappa' = \kappa \sqrt{\frac{1 + \frac{C\gamma}{\alpha_k^2}}{1 + \frac{C\gamma\kappa}{\alpha_k^2}}}.$$

Hence (2.18) can be handled in a way similar to (2.17) to conclude that if  $1 < \kappa \leq 9$ , then (2.13) has one negative real root ( $a < 0$ ) and a pair of complex conjugate roots ( $b \pm ci$ ,  $b < 0$ ) with negative real part  $\forall k \in \mathbb{N}$  and  $\exists C_1, C_2 > 0$  such that  $C_1 k^2 \leq |\lambda_{k,j}| \leq C_2 k^2, \forall j = 1, 2, 3, k \in \mathbb{N}$ . Hence (2.15) holds.

Next we show that  $\forall \hat{\beta} > \frac{\kappa}{\sqrt{2}}, \lambda_{k_1,j} \neq \lambda_{k_2,j}$  if  $k_1 \neq k_2$ . Let  $\rho = \sqrt{\frac{A}{m}}, \epsilon = \frac{C\gamma}{k^2\pi^2}$ . Let  $\lambda$  denote an eigenvalue which is common to  $k = k_1, k_2$ . Then rewriting (2.12) in terms of  $\lambda$  we get the following:

$$\lambda^3 + \rho k^2 \hat{\beta} (1 + \epsilon) \lambda^2 + \rho^2 k^4 \kappa \lambda + \hat{\beta} (1 + \epsilon \kappa) \rho^3 k^6 = 0. \quad (2.19)$$

Subtracting the equations corresponding to  $k = k_1$  and  $k = k_2$  in (2.19), we get

$$\lambda^2 \rho \hat{\beta} (k_1^2 - k_2^2) + \rho^2 \kappa (k_1^4 - k_2^4) \lambda + \hat{\beta} \rho^3 (k_1^6 - k_2^6) + \hat{\beta} \frac{C\gamma\kappa}{\pi^2} \rho^3 (k_1^4 - k_2^4) = 0 \quad (2.20)$$

$$(k_1^2 - k_2^2) \left[ \lambda^2 \hat{\beta} + \rho \kappa (k_1^2 + k_2^2) \lambda + \kappa \rho^2 (k_1^4 + k_2^4 + k_1^2 k_2^2) + \hat{\beta} \frac{C\gamma\kappa}{\pi^2} \rho^2 (k_1^2 + k_2^2) \right] = 0. \quad (2.21)$$

It can be checked that if  $\hat{\beta} > \frac{\kappa}{\sqrt{2}}$  then the quadratic in (2.21) has no real roots and hence there is no common real root between  $k = k_1$  and  $k = k_2$ . Using the fact that the real roots of (2.19) are distinct, we show the same for the complex conjugate roots.

Hence the eigenvalues  $\{\lambda_{k,j}\}$  are distinct for all  $k \in \mathbb{N}$  and  $j = 1, 2, 3$ .

Next we obtain estimates on  $\lambda_{k,j}$ 's. We choose  $k \geq K_0$  such that  $|\lambda_{k,j} - s_{k,j}| < \epsilon_0 = \frac{\sigma}{4}$ ,  $\forall j = 1, 2, 3$ . Then  $\forall m, n \in \mathbb{N}, m, n \geq K_0$  and  $j = 1, 2, 3$ , we have

$$\begin{aligned} |\lambda_{m,j} - \lambda_{n,j}| &\geq |s_{m,j} - s_{n,j}| - |\lambda_{m,j} - s_{m,j}| - |\lambda_{n,j} - s_{n,j}| \\ &\geq \sigma|m^2 - n^2| - 2\epsilon_0 \\ &= \sigma|m^2 - n^2| - \frac{\sigma}{2} \\ &\geq \frac{\sigma}{2}|m^2 - n^2|. \end{aligned}$$

Also  $\forall m, n < K_0$  we have finitely many roots. Let  $\hat{\sigma} = \min(|\lambda_{m,j} - \lambda_{n,j}| : m, n = 1, \dots, K_0)$ .

Then

$$|\lambda_{m,j} - \lambda_{n,j}| \geq \hat{\sigma} \geq \frac{\hat{\sigma}}{K_0^2 - 1} |m^2 - n^2| \quad \forall m, n < K_0.$$

Now let  $\bar{\sigma} = \min(\sigma, \frac{\hat{\sigma}}{K_0^2 - 1})$ . Then we have

$$|\lambda_{m,j} - \lambda_{n,j}| \geq \bar{\sigma} |m^2 - n^2|, \quad \forall m, n \in \mathbb{N}, j = 1, 2, 3.$$

Hence (2.16) holds. Also by a similar argument we can show that  $\exists \theta \in (0, \frac{\pi}{2})$  such that  $|\arg(-\lambda_{k,j})| \leq \theta$ ,  $\forall k \in \mathbb{N}, j = 1, 2, 3$ . Hence (2.14) holds. Finally, to ensure that  $\lambda_0 \neq \lambda_{k,j}$ ,  $\forall k \in \mathbb{N}, j = 1, 2, 3$ , we look at Equation (2.13) again:

$$f(x) = x^3 + \hat{\beta}x^2 + \kappa x + \hat{\beta} + \frac{\hat{\beta}C\gamma}{\alpha_k^2}(x^2 + \kappa). \quad (2.22)$$

We note that  $f(-\hat{\beta}(1 + \frac{C\gamma}{k^2\pi^2})) < 0$  and  $f(-\frac{\hat{\beta}}{\kappa}(1 + \frac{C\gamma}{k^2\pi^2})) > 0$ . Hence we have that the real root of (2.22) satisfies

$$a \in \left(-\hat{\beta}\left(1 + \frac{C\gamma}{k^2\pi^2}\right), -\frac{\hat{\beta}s}{\kappa}\left(1 + \frac{C\gamma}{k^2\pi^2}\right)\right).$$

Hence the real root of Equation (2.12) (denoted by  $\tilde{a}$ ) satisfies

$$\tilde{a} \in \left( -\left(1 + \frac{C\gamma}{k^2\pi^2}\right) \frac{1}{C\beta} k^2\pi^2, -\left(1 + \frac{C\gamma}{k^2\pi^2}\right) \frac{1}{C\beta} \frac{k^2\pi^2}{\kappa} \right).$$

Hence we have that if  $\gamma < \frac{1}{\kappa C}$  then

$$\lambda_0 = -\frac{\gamma}{\beta} > -\frac{1}{\kappa\beta C} > -\left(1 + \frac{C\gamma}{k^2\pi^2}\right) \left(\frac{1}{C\beta} \frac{k^2\pi^2}{\kappa}\right)$$

and hence  $\lambda_0 \neq \lambda_{k,j}$ ,  $\forall k \in \mathbb{N}, j = 1, 2, 3$ . This completes the proof of Lemma 1.

**Definition 9.** A basis for a Hilbert space is a *Riesz basis* if it is equivalent to an orthonormal basis, that is if it is obtained from an orthonormal basis by means of a bounded invertible operator.

**Theorem 5.**  $\{\phi_{k,j}\} \cup \{\phi_0\}$  forms a Riesz basis for  $(\mathcal{H}, \langle \cdot, \cdot \rangle_e)$ .

**Proof:** Let

$$\begin{aligned} \theta_0 &= (0, 0, 1)^T, \theta_{k,1} = \left(\frac{\sqrt{2}}{k^2\pi^2} \sin(k\pi x), 0, 0\right)^T \\ \theta_{k,2} &= (0, \sqrt{2} \sin(k\pi x), 0)^T, \theta_{k,3} = (0, 0, \frac{\sqrt{2}}{k\pi} \cos(k\pi))^T. \end{aligned}$$

Then  $\theta_k$ 's are related to  $\phi_k$ 's in the following way:

$$\begin{aligned} \phi_0 &= \theta_0 \\ \phi_{k,j} &= \frac{k^2\pi^2}{\sqrt{2}\lambda_{k,j}} \theta_{k,1} + \frac{1}{\sqrt{2}} \theta_{k,2} + \frac{k\pi A_{k,j}}{\sqrt{2}} \theta_{k,3}, \quad \forall k \in \mathbb{N}, j = 1, 2, 3. \end{aligned}$$

It can be checked that  $\{\theta_0\} \cup \{\theta_{k,j}\}$  is an orthogonal basis in  $(\mathcal{H}, \langle \cdot, \cdot \rangle_e)$ . Also  $\exists M > 0$  such that  $\frac{1}{M} < \|\theta_{k,j}\|_e < M$ ,  $\forall k \in \mathbb{N}, j = 1, 2, 3$ . Hence  $\{\theta_0\} \cup \{\theta_{k,j}\}$  is equivalent to an orthonormal basis in the energy inner product. We define a mapping  $L : \mathcal{H} \rightarrow \mathcal{H}$  as follows:

$$L \begin{bmatrix} \theta_0 \\ \theta_{k,1} \\ \theta_{k,2} \\ \theta_{k,3} \end{bmatrix} = \begin{bmatrix} \phi_0 \\ \phi_{k,1} \\ \phi_{k,2} \\ \phi_{k,3} \end{bmatrix}.$$

Let

$$L_k = \begin{pmatrix} \frac{k^2\pi^2}{\lambda_{k,1}} & \frac{1}{\sqrt{2}} & \frac{k\pi A_{k,1}}{\sqrt{2}} \\ \frac{k^2\pi^2}{\lambda_{k,2}} & \frac{1}{\sqrt{2}} & \frac{k\pi A_{k,2}}{\sqrt{2}} \\ \frac{k^2\pi^2}{\lambda_{k,3}} & \frac{1}{\sqrt{2}} & \frac{k\pi A_{k,3}}{\sqrt{2}} \end{pmatrix}$$

denote the matrix of transformation of  $T$  between blocks. The set  $\{\phi_{k,j}\}$  is block orthogonal with block size 3 corresponding to  $j = 1, 2, 3$ . Hence  $\{\phi_{k,j}\} \cup \{\phi_0\}$  will be a Riesz basis if each of the block matrices  $L_k$  are uniformly bounded and invertible [4]. Using (2.15) in Lemma 1, it can be easily checked that  $A_{k,j} \sim \mathcal{O}(\frac{1}{k})$ , where  $A_{k,j}$ 's are described in (2.11). Hence  $L_k$  has an invertible limiting form for  $k$  large enough. Also  $L_k$  is invertible for small  $k$  due to the separation of the eigenvalues. Hence we also have that  $|\det(L_k)| \leq C$  for some  $C > 0$ . Thus,  $\{\phi_{k,j}\} \cup \{\phi_0\}$  forms a Riesz basis for  $(\mathcal{H}, \langle \cdot, \cdot \rangle_e)$ . This proves Theorem 5.

**Remark 6.** We show that if  $\lambda \neq \lambda_{k,j}, k \in \mathbb{N}, j = 1, 2, 3$ , then  $\lambda \in \rho(\mathcal{A})$ . Let  $J$  denote the set of indices for the eigenvalues of  $\mathcal{A}(\beta)$ . Let  $f = \sum_{k \in J} d_k \phi_k$ . Then it is clear that the solution to  $(\mathcal{A} - \lambda I)u = f$  is given by  $u = \frac{d_k}{(\lambda_k - \lambda)} \phi_k$ , and  $u \in \mathcal{H}$  since the sequence  $\{\frac{1}{(\lambda_k - \lambda)}\}_{k \in J}$  is bounded. This means that the operator  $(\mathcal{A} - \lambda I)^{-1}$  is bounded  $\forall \lambda \neq \lambda_k, k \in J$ . Hence the spectrum of  $\mathcal{A}$  consists only of point spectrum.

## 2.4 Admissibility of input element

Let  $\{\psi_k\}$  be a sequence of eigenvectors of  $\mathcal{A}(\beta)^*$  which are also biorthogonal to  $\{\phi_k\}$  and

$$\tilde{w}(t, x) = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \sum_{k \in J} w_k(t) \phi_k(x) \quad (2.23)$$

be the solution of (2.8) where  $\phi_k$ 's are the eigenvectors of  $\mathcal{A}$  found above and  $w_k(t)$ 's are scalar functions of time. Also let

$$\tilde{w}(0, x) = \begin{pmatrix} w^0(x) \\ w^1(x) \\ s^0(x) \end{pmatrix} = \sum_{k \in J} w_k(0) \phi_k(x) \quad (2.24)$$

Substituting (2.23) in (2.8) , we get

$$\begin{aligned}
\tilde{w}_t &= \mathcal{A}\tilde{w} + \begin{pmatrix} 0 \\ \frac{1}{m}(A + \frac{B^2}{C})\delta' \\ 0 \end{pmatrix} u(t), \\
\Rightarrow \langle \tilde{w}_t, \psi_k(x) \rangle_e &= \langle \mathcal{A}\tilde{w}, \psi_k(x) \rangle_e + \left\langle \begin{pmatrix} 0 \\ \frac{1}{m}(A + \frac{B^2}{C})\delta' \\ 0 \end{pmatrix} u(t), \psi_k(x) \right\rangle_e, \\
\Rightarrow \left\langle \sum_{k \in J} w'_k(t) \phi_k(x), \psi_k(x) \right\rangle_e &= \left\langle \sum_{k \in J} w_k(t) (\mathcal{A}\phi_k(x)), \psi_k(x) \right\rangle_e \\
&\quad + \left\langle \sum_{k \in J} f_k \phi_k(x), \psi_k(x) \right\rangle_e u(t), \\
\Rightarrow w'_k(t) &= \lambda_k w_k(t) + f_k u(t), \\
\Rightarrow w_k(t) &= e^{\lambda_k t} w_k(0) + \int_0^t e^{\lambda_k(t-s)} f_k u(s) ds, \tag{2.25}
\end{aligned}$$

where

$$\begin{pmatrix} 0 \\ \frac{1}{m}(A + \frac{B^2}{C})\delta' \\ 0 \end{pmatrix} = \sum_{k \in J} f_k \phi_k(x), \quad f_k = \left\langle \begin{pmatrix} 0 \\ \frac{1}{m}(A + \frac{B^2}{C})\delta' \\ 0 \end{pmatrix}, \psi_k(x) \right\rangle_e. \tag{2.26}$$

The  $\psi_k$ 's can be computed explicitly and are as follows:

$$\psi_{k,j} = \begin{pmatrix} \frac{-1}{\hat{\lambda}_{k,j}} \sin(k\pi x) \\ \sin(k\pi x) \\ \hat{A}_{k,j} \cos(k\pi x) \end{pmatrix}, \quad \psi_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

where

$$\begin{aligned}
\hat{\lambda}_{k,j} &= \bar{\lambda}_{k,j}, \\
\hat{A}_{k,j} &= \frac{-\hat{\lambda}_{k,j} + \frac{1}{m}(A + \frac{B^2}{C}) \frac{(k\pi)^4}{\hat{\lambda}_{k,j}}}{\frac{B}{Cm} (k\pi)^3}, \quad k \in \mathbb{N}, j = 1, 2, 3. \tag{2.27}
\end{aligned}$$

The  $f_k$ 's can also be computed and are given by the following:

$$f_0 = 0$$

$$f_{k,j} = -\frac{1}{m}\left(A + \frac{B^2}{C}\right)k\pi, \quad \forall k \in \mathbb{N}, j = 1, 2, 3. \quad (2.28)$$

Next we show that  $\{f_{k,j}\}$  defined in (2.28) forms an admissible input for (2.25) in the Hilbert space  $\mathcal{H}_{-\frac{1}{4}} = (\{d_k\} : \{\frac{d_k}{\sqrt[4]{|\lambda_k|}}\} \in l^2(J))$ , where the norm is given by  $\|\{d_k\}\|_{\mathcal{H}_{-\frac{1}{4}}} = \|\frac{d_k}{\sqrt[4]{|\lambda_k|}}\|_{l^2(J)}$ . We define  $\mathcal{A}^{-\frac{1}{4}} : \mathcal{H}_{-\frac{1}{4}} \rightarrow l^2(J)$  in the following way:

$$\mathcal{A}^{-\frac{1}{4}}\{c_k\} = \left\{\frac{c_k}{\sqrt[4]{|\lambda_k|}}\right\}.$$

Then  $\mathcal{A}^{-\frac{1}{4}}$  is an isomorphism from  $\mathcal{H}_{-\frac{1}{4}}$  into  $l^2(J)$ .

**Remark 7. Carleson's measure criterion:** We recall the Carleson's measure criterion for  $\{f_k\}_{k \in J}$  to be admissible in  $l^2(J)$ . We first define the Carleson rectangle as below:

$$R(h, \omega) = \{z \in \mathbb{C} : 0 \leq \operatorname{Re}(z) \leq h, |\operatorname{Im}(z) - \omega| \leq h\}.$$

Then a necessary and sufficient condition for  $\{f_k\}_{k \in J}$  to be admissible in  $l^2(J)$  is given by (see e.g. [11, 17]):

$$\sum_{-\lambda_k \in R(h, \omega)} |f_k|^2 \leq Mh.$$

Using the Carleson's measure criterion, we have the following theorem:

**Theorem 6.**  $\{f_k\}_{k \in J}$  given by (2.28) forms an admissible input for (2.25) in  $\mathcal{H}_{-\frac{1}{4}}$ .

**Remark 8.** Equivalently, if  $\{w_k(0)\} \in \mathcal{H}_{-\frac{1}{4}}$ , then  $\{w_k(t)\}$  in (2.25) satisfies the following:

$$\|\{w_k(t)\}\|_{\mathcal{H}_{-\frac{1}{4}}} \leq C_T(\|\{w_k(0)\}\|_{\mathcal{H}_{-\frac{1}{4}}} + \|u\|_{L^2[0,T]}).$$

**Proof:** Equivalently, we prove that  $\mathcal{A}^{-\frac{1}{4}}\{f_k\}$  forms an admissible input for  $\mathcal{H}$ . This will prove the theorem since  $\mathcal{A}^{-\frac{1}{4}} : \mathcal{H}_{-\frac{1}{4}} \rightarrow l^2(J)$  is an isomorphism. We verify that  $\{\mathcal{A}^{-\frac{1}{4}}f_k\}_{k \in J}$  satisfies the Carleson measure criterion [11, 17]. Recall from (2.28) and Lemma 1 that  $f_k \sim$

$\mathcal{O}(k)$  and  $\lambda_k \sim \mathcal{O}(k^2)$ . Let  $g_k = \mathcal{A}^{-\frac{1}{4}} f_k \sim \mathcal{O}(\sqrt{k})$ . Then (2.25) can be rewritten in the following way.

$$\mathcal{A}^{-\frac{1}{4}} w_k(t) = \mathcal{A}^{-\frac{1}{4}} e^{\lambda_k t} w_k(0) + \int_0^t e^{\lambda_k(t-s)} g_k u(s) ds, \quad \forall k \in J. \quad (2.29)$$

If we choose  $\{w_k\} \in \mathcal{H}_{-\frac{1}{4}}$ , then  $\mathcal{A}^{-\frac{1}{4}} \{w_k\} \in l^2(J)$ . We define the following rectangle in the complex plane.

$$R(h, \omega) = \{z \in C : 0 \leq \text{Re}(z) \leq h, |\text{Im}(z) - \omega| \leq h\}.$$

Then we have,

$$\sum_{-\lambda_k \in R(k^2, \omega)} |g_k|^2 \leq \sum_{-\lambda_k \in R(k^2, 0)} |g_k|^2 \leq \mathcal{O}(k^2), \quad \forall h > 0, \quad (2.30)$$

where in (2.30) we have made use of the fact that the number of eigenvalues in  $R(k^2, 0)$  is  $\mathcal{O}(k)$  and that the worst case scenario happens when the rectangle is centered at the origin. Hence it follows that  $\exists M > 0$  such that

$$\sum_{-\lambda_k \in R(h, \omega)} |g_k|^2 \leq Mh$$

and this proves Theorem 6.

**Remark 9.** As a consequence, if the initial data in (2.9) belong to  $\mathcal{H}_{-\frac{1}{4}}$  and  $u(t) \in L^2[0, T]$  then there exists a unique solution  $\{w, w_t, s\}$  to (2.8) - (2.9) defined by (2.23) and (2.25). Furthermore for some  $C > 0$  we have the following:

$$\|\{w, w_t, s\}(\cdot, T)\|_{\mathcal{H}_{-\frac{1}{4}}} \leq C(\|\{w^0, w^1, s^0\}\|_{\mathcal{H}_{-\frac{1}{4}}} + \|u\|_{L^2[0, T]}).$$

## 2.5 The moment problem and its solution

For controllability to the zero state we seek  $u(t) \in L^2[0, T]$  which solves (2.29) with  $\{w_k(T)\} = 0, \forall k \in J$ . Let  $d_k = \mathcal{A}^{-\frac{1}{4}} w_k(0)$ . Then we have  $d_k \in l^2(J)$ , and (2.29) can be rewritten as:

$$-d_0 = \int_0^T e^{\lambda_0(t-s)} g_0 u(s) ds, \quad (2.31)$$

$$-d_{k,j} = \int_0^T e^{\lambda_{k,j}(t-s)} g_{k,j} u(s) ds, \quad \forall k \in \mathbb{N}, j = 1, 2, 3. \quad (2.32)$$

**Remark 10.** In (2.31),  $g_0 = \mathcal{A}^{-\frac{1}{4}}f_0 = 0$  and hence the eigenspace  $\text{span}(\{0, 0, 1\}^T)$  corresponding to the eigenvalue  $\lambda_0 = -\frac{\gamma}{\beta}$  cannot be controlled to the zero state. Hence we consider the solvability of (2.32).

From (2.28) we have that  $\{f_{k,j}; k \in \mathbb{N}, j = 1, 2, 3\}$  is bounded away from zero and hence we can rewrite (2.29) as follows:

$$\int_0^T e^{\lambda_{k,j}\tau} \tilde{u}(\tau) d\tau = c_{k,j}, \quad \forall k \in \mathbb{N}, j = 1, 2, 3, \quad (2.33)$$

where

$$c_{k,j} = \frac{-\mathcal{A}^{-\frac{1}{4}}\{e^{\lambda_{k,j}T} w_{k,j}(0)\}}{\mathcal{A}^{-\frac{1}{4}}f_{k,j}}, \quad \forall k \in \mathbb{N}, j = 1, 2, 3,$$

where  $T$  is the final time instant and  $\tilde{u}(t) = u(T - t)$ . Using Lemma 1 and (2.28) it can be shown that  $\exists \alpha, B > 0$  such that  $|c_{k,j}| \leq B e^{-\alpha k^2}$ ,  $\forall k \in \mathbb{N}, j = 1, 2, 3$ . If we are looking for a control in  $L^2[0, T]$ , then equation (2.33) can be rewritten as

$$\left\langle \tilde{u}(t), e^{\lambda_{k,j}t} \right\rangle_{L^2[0,T]} = c_{k,j}, \quad \forall k \in \mathbb{N}, j = 1, 2, 3, \quad (2.34)$$

where  $\{c_{k,j}\} \in l^2(J - \{0\})$ . Hence the original problem has been transformed in to the moment problem given by (2.34).

In order to solve the moment problem given by (2.34) we need the following theorem from Hansen [5] :

**Theorem 7.** Let  $\Lambda_0 := \{\lambda_k\}_{k=1}^\infty$  be a sequence of distinct complex numbers lying in  $\Delta_\theta := \{\lambda \in \mathbf{C} : |\arg(\lambda)| \leq \theta\}$ , which satisfy

$$|\lambda_k - \lambda_j| \geq \rho |k^\beta - j^\beta|, (\beta > 1, \rho \geq 0), \quad \epsilon(A + Bk^\beta) \leq \lambda_k < A + Bk^\beta.$$

Then there exist a sequence of functions  $q_k(T, t)$  which are biorthogonal to  $\{e^{\lambda_k t}\}$  in  $L^2[0, T]$  and satisfy

$$e^{mk} < \|q_k(T, t)\|_{L^2[0,T]} \leq K_T e^{kM}, (m, M > 0).$$

After reindexing the eigenvalues, it is clear from Lemma 1 that the eigenvalues  $\{\lambda_k\}_{k \in J - \{0\}}$  satisfy the estimates needed by Theorem 7. Hence a solution to the moment problem given by (2.34) is given by

$$\tilde{u}(t) = \sum_{k \in J - \{0\}} c_k q_k(T, t).$$

We also have the following:

$$\|\tilde{u}\|_{L^2[0, T]} \leq \sum_{k \in J - \{0\}} |c_k| \|q_k\|_{L^2[0, T]} \leq \sum_{k \in J - \{0\}} B K_T e^{-\alpha k^2} e^{kM} < \infty.$$

We conclude by stating the main theorem of the chapter.

**Theorem 8.** *Assume  $B < 2\sqrt{2AC}$ ,  $\beta < \sqrt{\frac{2m}{A}} \frac{1}{B^2 + AC}$ , and  $\gamma < \frac{1}{\kappa C}$ . Given any initial state  $\{w^0, w^1, s^0\} \in \mathcal{H}_{-\frac{1}{4}}$  and  $T > 0$ ,  $\exists u \in L^2(0, \infty)$  supported on  $[0, T]$  such that  $\forall t \geq T$ ,  $\{w, w_t, s\}(\cdot, T) = \{(0, 0, e^{-\frac{\gamma}{\beta} T} K_0)\}$ , where  $K_0$  is a constant determined by the initial data.*

**Remark 11.** The restriction on  $\beta$  prevents the eigenvalues from being overdamped (asymptotically). Without the other parametric restrictions there is at most a finite number of repeated eigenvalues which could result in a lack of controllability.

**Remark 12.** The undamped model has been proved to be exactly controllable in  $H_0^1(0, 1) \times H^{-1}(0, 1)$  in [28]. Here in the damped case, we obtain greater regularity due to analytic smoothing (see Hansen and Lasiecka [13]). Thus, the space of exact null controllability is smaller. On the other hand, the homogenous problem is well posed in  $\mathcal{H}_{-\frac{1}{2}} = H_0^1(0, 1) \times H^{-1}(0, 1) \times L^2(0, 1)$ . If we are given initial data in  $H_0^1(0, 1) \times H^{-1}(0, 1) \times L^2(0, 1)$  the zero control can be applied for a short time  $\epsilon$  after which Theorem 8 applies.

**Remark 13.** It is also interesting to note that the uncontrollable subspace does not exist in the undamped case, whereas in the damped case, we have a one dimensional state that is not controllable.

**Remark 14.** As in Remark 3, Chapter 1, approximate controllability holds in  $\mathcal{H}_{-\frac{1}{4}}$  modulo a one-dimensional space for the damped Mead-Markus model since the operators  $\{\mathcal{T}(t)\}_{t \in \mathbb{R}}$  are not invertible due to the diffusive nature of the Equation (2.2).

## CHAPTER 3. Exact controllability of an undamped Rao-Nakra sandwich beam using the multiplier method

### 3.1 Introduction

In this chapter, exact controllability of an undamped Rao-Nakra sandwich beam (see e.g [6],[30]) is considered. The classical Rao-Nakra [30] sandwich beam model consists of two outer “face plate” layers (which are assumed to be relatively stiff) which “sandwich” a much more compliant “core layer”. The Rao-Nakra model is derived using Euler-Bernoulli beam assumptions for the face plate layers, Timoshenko beam assumptions for the core layer and a “no-slip” assumption for the displacements along the interface. We consider the undamped case and a beam of unit length for simplicity. We first consider the problem with the omission of the moment of inertia term i.e  $-\alpha w_{xxtt}$ . The equations of motion are as follows:

$$mw_{tt} + Kw_{xxxx} - N^2h_2G_2w_{xx} - NG_2(-v_x^1 + v_x^3) = 0 \quad (3.1)$$

$$h_1\rho_1v_{tt}^1 - h_1E_1v_{xx}^1 - \frac{G_2}{h_2}(-v^1 + v^3) - NG_2w_x = 0 \quad (3.2)$$

$$h_3\rho_3v_{tt}^3 - h_3E_3v_{xx}^3 + \frac{G_2}{h_2}(-v^1 + v^3) + NG_2w_x = 0. \quad (3.3)$$

The boundary conditions are as follows:

$$w(0, t) = w(1, t) = w_x(0, t) = v^1(0, t) = v^3(0, t) = 0 \quad (3.4)$$

$$w_x(1, t) = M(t), v^1(1, t) = g_1(t), v^3(1, t) = g_3(t), \quad (3.5)$$

where  $M(t)$ ,  $g_1(t)$  and  $g_3(t)$  are control inputs at the right end. The initial conditions are as follows:

$$w(x, 0) = w^0(x), w_t(x, 0) = w^1(x) \quad (3.6)$$

$$v^1(x, 0) = v^{10}(x), v_t^1(x, 0) = v^{11}(x), v^3(x, 0) = v^{30}(x), v_t^3(x, 0) = v^{31}(x). \quad (3.7)$$

In the above,  $w$  denotes the transverse displacement of the beam,  $\varphi = h_2^{-1}(-v^1 + v^3) + Nw_x$  denotes the shear angle of the core layer,  $\mathbf{v}_\mathcal{O} = (v^1, v^3)^T$  is the vector of longitudinal displacement along the neutral axis of the outer layers. ( $i = 1, 3$  is for the outer layers,  $i = 2$  is for the core layer.) The density of the  $i$ th layer is denoted  $\rho_i$ , the thickness  $h_i$ , the Young's modulus  $E_i$ , the shear modulus of the core layer is  $G_2$ . We let  $m = \sum h_i \rho_i$  denote the mass density per length,  $\alpha = \rho_1 h_1^3/12 + \rho_3 h_3^3/12$  is a moment of inertia parameter,  $K = E_1 h_1^3/12 + E_3 h_3^3/12$  is the bending stiffness.

We use the multiplier method in conjunction with the HUM (Hilbert's Uniqueness Method) to prove exact controllability results. A classical reference for the HUM method is [22]. The use of multipliers can be found in [26],[21],[19],[20],[18] and references therein. In particular, Ho seems to be the first to use multipliers in a control application. The HUM method was formally introduced by J.L.Lions in a set of lectures at the Collège de France, 1986/1987.

### 3.2 Semigroup formulation

We consider the homogenous problem first. We recall the homogenous problem once again.

$$mw_{tt} + Kw_{xxxx} - N^2 h_2 G_2 w_{xx} - NG_2(-v_x^1 + v_x^3) = 0 \quad (3.8)$$

$$h_1 \rho_1 v_{tt}^1 - h_1 E_1 v_{xx}^1 - \frac{G_2}{h_2}(-v^1 + v^3) - NG_2 w_x = 0 \quad (3.9)$$

$$h_3 \rho_3 v_{tt}^3 - h_3 E_3 v_{xx}^3 + \frac{G_2}{h_2}(-v^1 + v^3) + NG_2 w_x = 0. \quad (3.10)$$

The boundary conditions are as follows:

$$w(0, t) = w_x(0, t) = w_x(1, t) = v^1(0, t) = v^3(0, t) = 0 \quad (3.11)$$

$$w_x(1, t) = 0, v^1(1, t) = 0, v^3(1, t) = 0. \quad (3.12)$$

The initial conditions are as follows:

$$w(x, 0) = w^0(x), w_t(x, 0) = w^1(x) \quad (3.13)$$

$$v^1(x, 0) = v^{10}(x), v_t^1(x, 0) = v^{11}(x), v^3(x, 0) = v^{30}(x), v_t^3(x, 0) = v^{31}(x). \quad (3.14)$$

Let  $Y = (w_1, w_2, w_3, w_4, w_5, w_6)^T = (w, v^1, v^3, w_t, v_t^1, v_t^3)^T = (U, V)^T$ . Then Equation (3.1 - 3.3) can be rewritten as follows.

$$\frac{dY}{dt} = \mathcal{A}Y, \quad Y(0) = (w^0, v^{10}, v^{30}, w^1, v^{11}, v^{31})^T, \quad (3.15)$$

where (using  $D = \frac{\partial}{\partial x}$ )

$$\tilde{\mathcal{A}} = \begin{pmatrix} -KD^4 + N^2h_2G_2D^2 & -NG_2D & NG_2D \\ \frac{NG_2}{h_1\rho_1}D & \frac{E_1}{\rho_1}D^2 - \frac{G_2}{h_2h_1\rho_1} & \frac{G_2}{h_2h_1\rho_1} \\ -\frac{NG_2}{h_3\rho_3}D & \frac{G_2}{h_2h_3\rho_3} & \frac{E_3}{\rho_3}D^2 - \frac{G_2}{h_2h_3\rho_3} \end{pmatrix},$$

$$\mathcal{A} = \begin{pmatrix} 0_{3 \times 3} & I_{3 \times 3} \\ \tilde{\mathcal{A}} & 0_{3 \times 3} \end{pmatrix}, \quad (3.16)$$

$$\mathcal{D}(\mathcal{A}) = H_w^4 \times H_{v^1}^2 \times H_{v^3}^2 \times H_w^2 \times H_{v^1}^1 \times H_{v^3}^1,$$

and

$$H_w^4 = \{\phi \in H^4(0, 1) : \phi(0) = \phi(1) = \phi_x(0) = \phi_x(1) = 0\}$$

$$H_{v^1}^2 = H_{v^3}^2 = \{\phi \in H^2(0, 1) : \phi(0) = \phi(1) = 0\}$$

$$H_w^2 = \{\phi \in H^2(0, 1) : \phi(0) = \phi(1) = \phi_x(0) = \phi_x(1) = 0\},$$

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{H} = H_w^2 \times H_0^1(0, 1) \times H_0^1(0, 1) \times L^2(0, 1) \times L^2(0, 1) \times L^2(0, 1).$$

We will refer to the Hilbert space  $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$  as the energy space, where the energy inner product is given by the following:

$$\left\langle \begin{pmatrix} w \\ v^1 \\ v^3 \\ w_t \\ v_t^1 \\ v_t^3 \end{pmatrix}, \begin{pmatrix} \hat{w} \\ \hat{v}^1 \\ \hat{v}^3 \\ \hat{w}_t \\ \hat{v}_t^1 \\ \hat{v}_t^3 \end{pmatrix} \right\rangle_e = \int_0^1 m w_t \bar{\hat{w}}_t + h_1 \rho_1 v_t^1 \bar{\hat{v}}_t^1 + h_3 \rho_3 v_t^3 \bar{\hat{v}}_t^3 + K w_{xx} \bar{\hat{w}}_{xx} + h_1 E_1 v_x^1 \bar{\hat{v}}_x^1$$

$$+h_3E_3v_x^3\bar{v}_x^3 + G_2h_2(h_2^{-1}(-v^1 + v^3) + Nw_x)(h_2^{-1}(-\bar{v}_1 + \bar{v}_3) + N\bar{w}_x)dx.$$

Consequently we have the following expression for the energy norm:

$$\begin{aligned} E(t) = \|(w, v^1, v^3, w_t, v_t^1, v_t^3)\|_{\mathcal{H}}^2 &= \int_0^1 mw_t^2 + h_1\rho_1(v_t^1)^2 + h_3\rho_3(v_t^3)^2 \\ &+ \int_0^1 Kw_{xx}^2 + h_1E_1(v_x^1)^2 + h_3E_3(v_x^3)^2 + G_2h_2(h_2^{-1}(-v^1 + v^3) + Nw_x)^2 dx. \end{aligned} \quad (3.17)$$

We define a new norm  $\tilde{E}(t)$  in the following way:

$$\tilde{E}(t) = \|w_{xx}\|_{L^2(0,1)}^2 + \|w_t\|_{L^2(0,1)}^2 + \|v_t^1\|_{L^2(0,1)}^2 + \|v_x^1\|_{L^2(0,1)}^2 + \|v_t^3\|_{L^2(0,1)}^2 + \|v_x^3\|_{L^2(0,1)}^2 \quad (3.18)$$

We show that that  $\tilde{E}(t)$  is equivalent to  $E(t)$ .

**Theorem 9.** *The norm  $\tilde{E}(t)$  and  $E(t)$  are equivalent i.e there exist constants  $F_1 > 0$  and  $F_2 > 0$  such that*

$$F_1\tilde{E}(t) \leq E(t) \leq F_2\tilde{E}(t), \quad (3.19)$$

where

$$\begin{aligned} F_1 &= \min(h_1\rho_1, h_3\rho_3, h_1E_1, h_3E_3, m, K) \\ F_2 &= \max(m, h_1\rho_1, h_3\rho_3, h_1E_1 + 2G_2\pi^2(N + \frac{1}{h_2}), h_3E_3 + 2G_2\pi^2(N + \frac{1}{h_2}), \\ &\quad K + NG_2\pi^2(Nh_2 + 1)). \end{aligned} \quad (3.20)$$

**Proof:** We first show the lower bound. We note that

$$\int_0^1 G_2h_2(h_2^{-1}(-v^1 + v^3) + Nw_x)^2 dx \geq 0.$$

The remaining terms in  $E(t)$  resemble those of  $\tilde{E}(t)$  except for multiplicative constants. We call the lowest of all those constants  $F_1$  and we have  $E(t) \geq F_1\tilde{E}(t)$ . Next we show the upper bound. We need the following estimate:

$$\int_0^1 G_2h_2(h_2^{-1}(-v^1 + v^3) + Nw_x)^2 = \int_0^1 \frac{G_2}{h_2}(-v^1 + v^3)^2 - 2NG_2(-v^1 + v^3)w_x + N^2G_2h_2w_x^2.$$

Next we use Cauchy-Schwartz and Poincare's inequality to obtain the following:

$$\int_0^1 \frac{G_2}{h_2}(-v^1 + v^3)^2 - 2NG_2(-v^1 + v^3)w_x + N^2G_2h_2w_x^2$$

$$\begin{aligned}
&\leq \int_0^1 \frac{2G_2}{h_2} ((v^1)^2 + (v^3)^2) + NG_2(\pi^2 w_{xx}^2 + 2((v^1)^2 + (v^3)^2) + N^2 G_2 h_2 \pi^2 w_{xx}^2 \\
&= (NG_2 \pi^2)(Nh_2 + 1) \int_0^1 w_{xx}^2 + 2G_2(N + \frac{1}{h_2}) \int_0^1 ((v^1)^2 + (v^3)^2) \\
&\leq (NG_2 \pi^2)(Nh_2 + 1) \int_0^1 w_{xx}^2 + 2G_2 \pi^2(N + \frac{1}{h_2}) \int_0^1 ((v_x^1)^2 + (v_x^3)^2)
\end{aligned}$$

Once again, the rest of the terms in  $E(t)$  along with the right hand side in the above estimate resembles  $\tilde{E}(t)$  except for multiplicative constants. We collect the constants for similar terms and call the greatest as  $F_2$ . This proves the theorem.

**Remark 15.** It turns out that  $\tilde{E}(t)$  is an easier norm to work with. The fact that  $F_1$  is independent of the coupling constant  $G_2$  will be needed later on while proving estimates. Furthermore, constants  $m, h_1 \rho_1, h_3 \rho_3, h_1 E_1, h_3 E_3, K$  can be chosen large enough so that  $F_2$  is also independent of  $G_2$ .

**Theorem 10.**  $\mathcal{A}$  is the infinitesimal generator of a strongly continuous semigroup on  $\mathcal{H}$ .

**Proof:** We can easily check that  $\mathcal{A}^* = -\mathcal{A}$  and  $D(\mathcal{A}^*) = D(\mathcal{A})$  by direct calculation. Next we check that both  $\mathcal{A}$  and  $\mathcal{A}^*$  are dissipative on  $\mathcal{H}$ . Then the result will follow from the Lumer-Phillips theorem (see e.g. Pazy [27]). We show the calculation for  $\mathcal{A}$ . The calculation for  $\mathcal{A}^*$  is similar. Let  $Y \in \mathcal{D}(\mathcal{A})$  be given by  $Y = (x_1, x_2, x_3, y_1, y_2, y_3)$ .  $\langle \mathcal{A}Y, Y \rangle_e$  consists of various terms. We omit writing  $dx$  inside the integrals in what follows, as it is understood from the limits of integration.

$$\begin{aligned}
\text{Term 1} &= \int_0^1 m(-KD^4 x_1 + N^2 h_2 G_2 D^2 x_1 + NG_2(-x_2 + x_3)) \bar{y}_1. \\
\text{Term 2} &= \int_0^1 h_1 \rho_1 \left( \frac{1}{h_1 \rho_1} (h_1 E_1 D^2 x_2 + \frac{G_2}{h_2} (-x_2 + x_3) + NG_2 D x_1) \right) \bar{y}_2 \\
&= \int_0^1 -h_1 E_1 D x_2 \bar{D} y_2 + \frac{G_2}{h_2} (-x_2 + x_3) \bar{y}_2 - NG_2 x_1 \bar{D} x_2. \\
\text{Term 3} &= \int_0^1 h_3 \rho_3 \left( \frac{1}{h_3 \rho_3} (h_3 E_3 D^2 x_3 - \frac{G_2}{h_2} (-x_2 + x_3) - NG_2 D x_1) \right) \bar{y}_3 \\
&= \int_0^1 (-h_3 E_3 D x_3 \bar{D} y_3 - \frac{G_2}{h_2} (-x_2 + x_3) \bar{y}_3 + NG_2 x_1 \bar{D} y_3). \\
\text{Term 4} &= \int_0^1 K D^2 y_1 \bar{D}^2 x_1 = \int_0^1 -K D y_1 \bar{D}^3 x_1 = \int_0^1 K y_1 \bar{D}^4 x_1.
\end{aligned}$$

$$\text{Term 5} = \int_0^1 h_1 E_1 D y_2 \overline{D x_2}.$$

$$\text{Term 6} = \int_0^1 h_3 E_3 D y_3 \overline{D x_3}.$$

$$\begin{aligned} \text{Term 7} &= \int_0^1 G_2 h_2 (h_2^{-1}(-y_2 + y_3) + N D y_1) (h_2^{-1}(-\bar{x}_2 + \bar{x}_3) + N D \bar{x}_1) \\ &= \int_0^1 \left( \frac{G_2}{h_2} (-y_2 + y_3) (-\bar{x}_2 + \bar{x}_3) - N^2 G_2 h_2 y_1 \overline{D^2 x_1} - N G_2 D(-y_2 + y_3) \bar{x}_1 - N G_2 y_1 \overline{D(-x_2 + x_3)} \right). \end{aligned}$$

Adding Terms 1 through 7 and rearranging, we get

$$\begin{aligned} \langle \mathcal{A}Y, Y \rangle_{\mathcal{H}} &= \int_0^1 K (\overline{D^4 x_1 y_1} - D^4 x_1 \bar{y}_1) + N^2 G_2 h_2 (D^2 x_1 \bar{y}_1 - \overline{D^2 x_1 y_1}) \\ &+ N G_2 (D(-x_2 + x_3) \bar{y}_1 - \overline{D(-x_2 + x_3) y_1}) + h_1 E_1 (D y_2 \overline{D x_2} - \overline{D y_2} D x_2) + h_3 E_3 (D y_3 \overline{D x_3} - \overline{D y_3} D x_3) \\ &+ N G_2 (x_1 \overline{D(-y_2 + y_3)} - D(-y_2 + y_3) \bar{x}_1) + \frac{G_2}{h_2} ((-y_2 + y_3) \overline{(-x_2 + x_3)} - (-x_2 + x_3) \overline{(-y_2 + y_3)}). \end{aligned}$$

Hence we have  $\text{Re} \langle \mathcal{A}Y, Y \rangle_{\mathcal{H}} = 0$  and hence  $\mathcal{A}$  is a dissipative operator and the proof is complete.

**Remark 16.** The fact that  $\mathcal{A}$  generates a semigroup of contractions (call it  $\mathcal{T}(t)$ ) means that  $(0, \infty) \in \rho(\mathcal{A})$  where  $\rho(\mathcal{A})$  denotes the resolvent of  $\mathcal{A}$  (see [2], page 124). Also, there is a doubly infinite array of Hilbert spaces  $\{\mathcal{X}_k, \langle \cdot, \cdot \rangle_k\}_{k \in \mathbb{Z}}$  satisfying the following properties:

1.  $\mathcal{X}_0 = \mathcal{H}, \mathcal{X}_1 = \mathcal{D}(\mathcal{A}), \mathcal{X}_k = (D(\mathcal{A}^k))$ , and  $\mathcal{X}_k$  is densely embedded in  $\mathcal{X}_j$  if  $k > j$ .
2.  $\langle x, y \rangle_k = \langle \mathcal{A}^k x, \mathcal{A}^k y \rangle_{\mathcal{H}} \quad \forall x \in \mathcal{X}_k$ .
3. If we define  $\mathcal{A}_k x = \mathcal{A}x, \forall x \in \mathcal{D}(\mathcal{A}_k)$  where  $\mathcal{D}(\mathcal{A}_k) = \{x \in \mathcal{X}_k : \mathcal{A}x \in \mathcal{X}_k\} = \mathcal{D}(\mathcal{A}^{k+1}) = \mathcal{X}_{k+1}$ , then  $\mathcal{A}_k$  generates a semigroup of contractions  $\mathcal{T}_k(t) = \mathcal{T}(t)|_{\mathcal{X}_k}$  on  $\mathcal{X}_k$  which is a restriction of  $\mathcal{T}(t)$  to  $\mathcal{X}_k$ . Each of the semigroups  $\mathcal{T}_k(t)$  are invariant on  $\mathcal{X}_k$ .

**Remark 17.** The point of Remark 16 is that if we choose initial conditions in a smoother space, i.e  $Y(0) \in \mathcal{X}_k$  for  $k$  large enough, then the invariance of the semigroup in  $\mathcal{X}_k$  implies that a semigroup solution exists in  $\mathcal{X}_k$ , i.e the smoother the initial data, the smoother are the solutions. Furthermore, the solution is given by  $Y(t) = \mathcal{T}(t)Y(0), t > 0$ . The existence of smooth solutions to the homogenous problem (3.15) justifies the integration by parts formulae

that we show in what follows. Furthermore,  $\mathcal{X}_k$  is a dense subspace of  $\mathcal{X} \forall k > 0$  and hence any inequality involving elements in  $\mathcal{X}_k, \forall k > 0$  is true for elements in  $\mathcal{X}$  using a standard density argument.

**Remark 18.** It can be shown that  $\frac{dE(t)}{dt} = 0$  along the solutions of (3.8)-(3.10) i.e energy is conserved. Therefore we have that  $E(t) = E(0), \forall t \in \mathbb{R}$ . Conservation of energy implies that the system (3.8)-(3.10) is time reversible and hence the semigroup  $\{\mathcal{T}(t)\}_{t>0}$  generated by  $\mathcal{A}$  is in fact a group  $\{\mathcal{T}(t)\}_{t \in \mathbb{R}}$ .

### 3.3 An important identity

In this section, we establish an identity which is true for smooth solutions which result from smooth initial conditions i.e  $Y(0) \in \mathcal{X}_k$  for  $k$  large enough. We omit writing  $dx$  and  $dt$  inside the integrals as it is understood from the limits of integration.

**Lemma 2.** *Let  $T > 0$ . Then every smooth solution of (3.8-3.14) satisfies the following identity:*

$$\begin{aligned} & \frac{h_1 E_1}{2} \int_0^T (v_x^1(1, t))^2 + \frac{h_3 E_3}{2} \int_0^T (v_x^3(1, t))^2 + \frac{K}{2} \int_0^T (w_{xx}(1, t))^2 = \\ & \frac{m}{2} \int_0^T \int_0^1 w_t^2 + \frac{3K}{2} \int_0^T \int_0^1 w_{xx}^2 + \frac{N^2 h_2 G_2}{2} \int_0^T \int_0^1 w_x^2 + \frac{h_3 \rho_3}{2} \int_0^T \int_0^1 (v_t^3)^2 \\ & + \frac{h_3 E_3}{2} \int_0^T \int_0^1 (v_x^3)^2 + \frac{h_1 \rho_1}{2} \int_0^T \int_0^1 (v_t^1)^2 + \frac{h_1 E_1}{2} \int_0^T \int_0^1 (v_x^1)^2 \\ & - \frac{G_2}{2h_2} \int_0^T \int_0^1 (-v^1 + v^3)^2 + X, \end{aligned} \quad (3.21)$$

where  $X = \int_0^1 m w_t x w_x + h_1 \rho_1 v_t^1 x v_x^1 + h_3 \rho_3 v_t^3 x v_x^3 \Big|_{t=0}^{t=T}$ .

**Proof:** We first multiply (3.8) by  $x w_x$  and integrate by parts. We get the following:

$$\begin{aligned} & \int_0^T \int_0^1 m w_t t x w_x = m \int_0^1 w_t x w_x \Big|_{t=0}^{t=T} - m \int_0^T \int_0^1 w_t x w_{xt}. \\ & m \int_0^T \int_0^1 w_t x w_{xt} = m \int_0^T \int_0^1 x \left( \frac{1}{2} w_t^2 \right)_x = -\frac{m}{2} \int_0^T \int_0^1 w_t^2. \end{aligned}$$

Hence we get

$$\int_0^T \int_0^1 m w_t t x w_x = m \int_0^1 w_t x w_x \Big|_{t=0}^{t=T} + \frac{m}{2} \int_0^T \int_0^1 w_t^2.$$

Next we have,

$$\begin{aligned} K \int_0^T \int_0^1 w_{xxxx} x w_x &= -K \int_0^T \int_0^1 w_{xxx} (x w_x)_x = -K \int_0^T \int_0^1 w_{xxx} w_x - K \int_0^T \int_0^1 x w_{xx} w_{xxx}. \\ K \int_0^T \int_0^1 w_{xxx} w_x &= -K \int_0^T \int_0^1 w_{xx}^2. \\ K \int_0^T \int_0^1 x w_{xx} w_{xxx} &= K \int_0^T \int_0^1 x \left( \frac{1}{2} w_{xx}^2 \right)_x = \frac{K}{2} \int_0^T w_{xx}^2(1) - \frac{K}{2} \int_0^T \int_0^1 x w_{xx} w_{xxx}. \end{aligned}$$

Hence we get

$$K \int_0^T \int_0^1 w_{xxxx} x w_x = \frac{3K}{2} \int_0^T \int_0^1 w_{xx}^2 - \int_0^T w_{xx}^2(1).$$

Next we have

$$\begin{aligned} -N^2 h_2 G_2 \int_0^T \int_0^1 w_{xx} x w_x &= N^2 h_2 G_2 \int_0^T \int_0^1 w_x (x w_x)_x \\ &= N^2 h_2 G_2 \int_0^T \int_0^1 w_x^2 + N^2 h_2 G_2 \int_0^T \int_0^1 x \left( \frac{1}{2} w_x^2 \right)_x = N^2 h_2 G_2 \int_0^T \int_0^1 w_x^2. \end{aligned}$$

We leave the coupled terms unchanged i.e

$$-N G_2 \int_0^T \int_0^1 (-v_x^1 + v_x^3) x w_x.$$

Putting everything together, from (3.8), we get

$$\begin{aligned} \frac{K}{2} \int_0^T w_{xx}^2(1) &= \int_0^T \int_0^1 \frac{m}{2} w_t^2 + \frac{3K}{2} w_{xx}^2 + \frac{N^2 h_2 G_2}{2} w_x^2 - N G_2 (-v_x^1 + v_x^3) x w_x \\ &\quad + m \int_0^1 w_t x w_x \Big|_{t=0}^{t=T}. \end{aligned} \quad (3.22)$$

Next we multiply (3.9) by  $x v_x^1$  and integrate by parts.

$$\begin{aligned} h_1 \rho_1 \int_0^T \int_0^1 v_{tt}^1 x v_x^1 &= h_1 \rho_1 \int_0^1 v_t^1 x v_x^1 \Big|_{t=0}^{t=T} - h_1 \rho_1 \int_0^T \int_0^1 v_t^1 x v_{xt}^1. \\ h_1 \rho_1 \int_0^T \int_0^1 x \left( \frac{1}{2} (v_t^1)^2 \right)_x &= -\frac{h_1 \rho_1}{2} \int_0^T \int_0^1 (v_t^1)^2. \end{aligned}$$

Hence we get,

$$h_1 \rho_1 \int_0^T \int_0^1 v_{tt}^1 x v_x^1 = h_1 \rho_1 \int_0^1 v_t^1 x v_x^1 + \frac{h_1 \rho_1}{2} \int_0^T \int_0^1 (v_t^1)^2.$$

Next we have

$$-h_1 E_1 \int_0^T \int_0^1 v_{xx}^1 x v_x^1 = -\frac{h_1 E_1}{2} \int_0^T (v_x^1(1, t))^2 + \frac{h_1 E_1}{2} \int_0^T \int_0^1 (v_x^1)^2.$$

We leave the coupled terms unchanged i.e

$$-\frac{G_2}{h_2} \int_0^T \int_0^1 (-v^1 + v^3) x v_x^1 - N G_2 \int_0^T \int_0^1 w_x x v_x^1.$$

In all, from (3.9) we get

$$\begin{aligned} \frac{h_1 E_1}{2} \int_0^T (v_x^1(1, t))^2 &= \int_0^T \int_0^1 \frac{h_1 \rho_1}{2} (v_t^1)^2 + \frac{h_1 E_1}{2} (v_x^1)^2 - \frac{G_2}{h_2} (-v^1 + v^3) x v_x^1 \\ &\quad - \int_0^T \int_0^1 N G_2 w_x x v_x^1 + h_1 \rho_1 \int_0^1 v_t^1 x v_x^1|_{t=0}^t. \end{aligned} \quad (3.23)$$

Similarly from (3.10), we get

$$\begin{aligned} \frac{h_3 E_3}{2} \int_0^T (v_x^3(1, t))^2 &= \int_0^T \int_0^1 \frac{h_3 \rho_3}{2} (v_t^3)^2 + \frac{h_3 E_3}{2} (v_x^3)^2 + \frac{G_2}{h_2} (-v^1 + v^3) x v_x^3 \\ &\quad + \int_0^T \int_0^1 N G_2 w_x x v_x^3 + h_3 \rho_3 \int_0^1 v_t^3 x v_x^3|_{t=0}^t. \end{aligned} \quad (3.24)$$

Adding Equations (3.22), (3.23) and (3.24) we get

$$\begin{aligned} &\frac{h_1 E_1}{2} \int_0^T (v_x^1(1, t))^2 + \frac{h_3 E_3}{2} \int_0^T (v_x^3(1, t))^2 + \frac{K}{2} \int_0^T (w_{xx}(1, t))^2 = \\ &\frac{m}{2} \int_0^T \int_0^1 w_t^2 + \frac{3K}{2} \int_0^T \int_0^1 w_{xx}^2 + \frac{N^2 h_2 G_2}{2} \int_0^T \int_0^1 w_x^2 + \frac{h_3 \rho_3}{2} \int_0^T \int_0^1 (v_t^3)^2 \\ &+ \frac{h_3 E_3}{2} \int_0^T \int_0^1 (v_x^3)^2 + \frac{h_1 \rho_1}{2} \int_0^T \int_0^1 (v_t^1)^2 + \frac{h_1 E_1}{2} \int_0^T \int_0^1 (v_x^1)^2 \\ &+ \frac{G_2}{2h_2} \int_0^T \int_0^1 x \left( \frac{1}{2} (-v^1 + v^3)^2 \right)_x + X, \end{aligned} \quad (3.25)$$

where  $X = \int_0^1 m w_t x w_x + h_1 \rho_1 v_t^1 x v_x^1 + h_3 \rho_3 v_t^3 x v_x^3|_{t=0}^t$ . We have one last calculation.

$$\frac{G_2}{2h_2} \int_0^T \int_0^1 x \left( \frac{1}{2} (-v^1 + v^3)^2 \right)_x = -\frac{G_2}{2h_2} \int_0^T \int_0^1 (-v^1 + v^3)^2.$$

This proves the Lemma.

### 3.4 Estimating $X$

We recall from the Lemma 2 that

$$X = \int_0^1 m w_t x w_x + h_1 \rho_1 v_t^1 x v_x^1 + h_3 \rho_3 v_t^3 x v_x^3|_{t=0}^t.$$

We first look at

$$X(t) = \int_0^1 m w_t x w_x + h_1 \rho_1 v_t^1 x v_x^1 + h_3 \rho_3 v_t^3 x v_x^3.$$

We have

$$\begin{aligned}
|X(t)| &= \left| \int_0^1 m w_t x w_x + h_1 \rho_1 v_t^1 x v_x^1 + h_3 \rho_3 v_t^3 x v_x^3 \right| \\
&\leq \int_0^1 m |w_t x w_x| + \int_0^1 h_1 \rho_1 |v_t^1 x v_x^1| + \int_0^1 h_3 \rho_3 |v_t^3 x v_x^3|. \\
&\quad \int_0^1 m |w_t x w_x| \leq \int_0^1 m (|w_t|^2 + |w_x|^2). \\
&\quad \int_0^1 h_1 \rho_1 |v_t^1 x v_x^1| \leq \int_0^1 h_1 \rho_1 (|v_t^1|^2 + |v_x^1|^2). \\
&\quad \int_0^1 h_3 \rho_3 |v_t^3 x v_x^3| \leq \int_0^1 h_3 \rho_3 (|v_t^3|^2 + |v_x^3|^2).
\end{aligned}$$

We also have by the Poincaré's inequality on  $(0, 1)$  that  $\|w_x\|_{L^2(0,1)}^2 \leq \pi^2 \|w_{xx}\|_{L^2(0,1)}^2$ . Using all of the above, we have that  $|X(t)| \leq D \tilde{E}(t)$ , where  $D = \max(m\pi^2, h_1\rho_1, h_3\rho_3)$ . Using Remark 18, and Equation (3.19), we have that  $|X| \leq \frac{2D}{F_1} E(0)$ , where

$$F_1 = \min(h_1\rho_1, h_3\rho_3, h_1E_1, h_3E_3, m, K).$$

### 3.5 Hidden Regularity Estimate

In this section, we derive a hidden regularity estimate for the homogenous problem (3.8)-(3.14). The regularity estimate will allow us to define a weak solution to the controlled equations (3.1-3.7) by the method of transposition.

Using the above estimate for  $X$  in the previous section, we can prove the following regularity estimate:

**Theorem 11.** *Let  $T > 0$ . Then every smooth solution of (3.8)-(3.14) satisfies the following "hidden" regularity estimate:*

$$\frac{h_1 E_1}{2} \int_0^T (v_x^1(1, t))^2 + \frac{h_3 E_3}{2} \int_0^T (v_x^3(1, t))^2 + \frac{K}{2} \int_0^T w_{xx}^2(1) \leq \left( \frac{TC + 2D}{F_1} \right) E(0), \quad (3.26)$$

where

$$D = \max(m\pi^2, h_1\rho_1, h_3\rho_3) \quad (3.27)$$

$$C = \max(m, 3K, N^2 h_2 G_2 \pi^2, h_1\rho_1, h_1E_1, h_3\rho_3, h_3E_3) \quad (3.28)$$

$$F_1 = \min(h_1\rho_1, h_3\rho_3, h_1E_1, h_3E_3, m, K). \quad (3.29)$$

As a consequence, the linear form  $(w^1, v^{11}, v^{31}, w^0, v^{10}, v^{30}) \rightarrow (w_{xx}(1, t), v_x^1(1, t), v_x^3(1, t))$  is continuous from  $\mathcal{H}$  to  $(L^2[0, T])^3$ .

**Remark 19.** Due to existence of finite energy solutions, we have that  $v_x^1(x, t), v_x^3(x, t)$  and  $w_{xx}(x, t)$  belong to  $L^2((0, 1) \times [0, T])$ . Theorem 11 implies that the point evaluations  $v_x^1(1, t), v_x^3(1, t)$  and  $w_{xx}(1, t)$  belong to  $L^2[0, T]$  which is a stronger result than the usual trace theorem, where we expect point evaluations to be less regular by half an order.

**Proof:** We begin by observing that in Lemma 2, Equation (3.21),

$$-\frac{G_2}{2h_2} \int_0^T \int_0^1 (-v^1 + v^3)^2 \leq 0,$$

and every other term except  $X$  looks like an energy term except for a multiplicative constant. We denote by  $C$  the greatest of all the constants multiplying "energy like" terms. Hence we have

$$\begin{aligned} & \frac{m}{2} \int_0^T \int_0^1 w_t^2 + \frac{3K}{2} \int_0^T \int_0^1 w_{xx}^2 + \frac{N^2 h_2 G_2}{2} \int_0^T \int_0^1 w_x^2 + \frac{h_3 \rho_3}{2} \int_0^T \int_0^1 (v_t^3)^2 \\ & + \frac{h_3 E_3}{2} \int_0^T \int_0^1 (v_x^3)^2 + \frac{h_1 \rho_1}{2} \int_0^T \int_0^1 (v_t^1)^2 + \frac{h_1 E_1}{2} \int_0^T \int_0^1 (v_x^1)^2 \leq TC\tilde{E}(t) \leq \frac{TC}{F_1} E(0). \end{aligned}$$

From Section 3.4, we also have that  $|X| \leq \frac{2D}{F_1} E(0)$ . Putting all the pieces above together, we have the statement of the theorem.

### 3.6 Solution of the non-homogenous problem

In this section, we use the regularity estimate proved in Theorem 11 to show the existence of weak solutions for the non-homogenous problem. To maintain consistency of notation, we rename the variables in Equations (3.1)-(3.7) as follows:

$$mz_{tt} + Kz_{xxxx} - N^2 h_2 G_2 z_{xx} - NG_2(-y_x^1 + y_x^3) = 0 \quad (3.30)$$

$$h_1 \rho_1 y_{tt}^1 - h_1 E_1 y_{xx}^1 - \frac{G_2}{h_2} (-y^1 + y^3) - NG_2 z_x = 0 \quad (3.31)$$

$$h_3 \rho_3 y_{tt}^3 - h_3 E_3 y_{xx}^3 + \frac{G_2}{h_2} (-y^1 + y^3) + NG_2 z_x = 0. \quad (3.32)$$

The boundary conditions are as follows:

$$z(0, t) = z(1, t) = z_x(0, t) = y^1(0, t) = y^3(0, t) = 0 \quad (3.33)$$

$$z_x(1, t) = M(t), y^1(1, t) = g_1(t), y^3(1, t) = g_3(t), \quad (3.34)$$

where  $M(t)$ ,  $g_1(t)$  and  $g_3(t)$  are control inputs at the right end. The initial conditions are as follows:

$$z(x, 0) = z^0(x), z_t(x, 0) = z^1(x) \quad (3.35)$$

$$y^1(x, 0) = y^{10}(x), y_t^1(x, 0) = y^{11}(x), y^3(x, 0) = y^{30}(x), y_t^3(x, 0) = y^{31}(x). \quad (3.36)$$

In order to define a solution for Equations (3.30)-(3.36), we first multiply Equation (3.1) by  $z$  and formally integrate by parts.

$$\begin{aligned} m \int_0^1 \int_0^T w_{tt}z &= \int_0^1 m(w_t z - w z_t)|_{t=0}^{t=T} + m \int_0^1 \int_0^T w z_{tt}. \\ K \int_0^1 \int_0^T w_{xxxx}z &= -K \int_0^1 \int_0^T w_{xxx}z_x = K \int_0^1 \int_0^T w_{xx}z_{xx} - K \int_0^T w_{xx}(1, t)M(t) \\ &= -K \int_0^T w_{xx}(1, t)M(t) - K \int_0^1 \int_0^T w_x z_{xxx} = -K \int_0^1 \int_0^T w_{xx}(1, t)M(t) + K \int_0^1 \int_0^T w z_{xxxx}. \\ -N^2 h_2 G_2 \int_0^1 \int_0^T w_{xx}z &= N^2 h_2 G_2 \int_0^1 \int_0^T w_x z_x = -N^2 h_2 G_2 \int_0^1 \int_0^T w z_{xx}. \\ -N G_2 \int_0^1 \int_0^T (-v^1 + v^3)_x z &= N G_2 \int_0^1 \int_0^T (-v^1 + v^3) z_x. \end{aligned}$$

Now we multiply Equation (3.2) with  $y^1$  and formally integrate by parts.

$$\begin{aligned} h_1 \rho_1 \int_0^1 \int_0^T v_{tt}^1 y^1 &= h_1 \rho_1 \int_0^1 (v_t^1 y^1 - v^1 y_t^1)|_{t=0}^{t=T} + h_1 \rho_1 \int_0^1 \int_0^T v^1 y_{tt}^1. \\ -h_1 E_1 \int_0^1 \int_0^T v_{xx}^1 y^1 &= -h_1 E_1 \int_0^T v_x^1(1, t)g_1(t) + h_1 E_1 \int_0^1 \int_0^T v_x^1 y_{xx}^1 \\ &= -h_1 E_1 \int_0^T v_x^1 g_1(t) - h_1 E_1 \int_0^1 \int_0^T v^1 y_{xx}^1. \end{aligned}$$

For the coupled terms, we leave  $-\frac{G_2}{h_2} \int_0^1 \int_0^T (-v^1 + v^3) y^1$  unchanged.

$$-N G_2 \int_0^1 \int_0^T w_x y^1 = N G_2 \int_0^1 \int_0^T w y_x^1.$$

Finally we multiply Equation (3.3) by  $y^3$  and formally integrate by parts.

$$\begin{aligned} h_3\rho_3 \int_0^1 \int_0^T v_{tt}^3 y^3 &= h_3\rho_3 \int_0^T (v_t^3 y^3 - v^3 y_t^3)|_{t=0}^{t=T} + h_3\rho_3 \int_0^1 \int_0^T v^3 y_{tt}^3. \\ -h_3E_3 \int_0^1 \int_0^T v_{xx}^3 y^3 &= -h_3E_3 \int_0^T v_x^3(1,t)g_3(t) + h_3E_3 \int_0^1 \int_0^T v_x^3 y_{xx}^3 \\ &= -h_3E_3 \int_0^T v_x^3 g_3(t) - h_3E_3 \int_0^1 \int_0^T v^3 y_{xx}^3. \end{aligned}$$

For the coupled terms, we leave  $\frac{G_2}{h_2} \int_0^1 \int_0^T (-v^1 + v^3)y^3$  unchanged.

$$NG_2 \int_0^1 \int_0^T w_x y^3 = -NG_2 \int_0^1 \int_0^T w y_x^3.$$

Adding everything from above and rearranging, we get the following:

$$\begin{aligned} \int_0^1 m(w_t(T)z(T) - w(T)z_t(T)) &+ h_1\rho_1(v_t^1(T)y^1(T) - v^1 y_t^1(T)) + h_3\rho_3(v_t^3(T)y^3(T) - v^3(T)y_t^3(T)) \\ &= \int_0^1 m(w^1 z^0 - w^0 z^1) + h_1\rho_1(v^{11}y^{10} - v^{10}y^{11}) + h_3\rho_3(v^{31}y^{30} - v^{30}y^{31}) \\ &\quad + \int_0^T K w_{xx}(1,t)M(t) + h_1E_1 v_x^1(1,t)g_1(t) + h_3E_3 v_x^3(1,t)g_3(t). \end{aligned}$$

We define the following linear form.

$$\begin{aligned} L_T((w^0, v^{10}, v^{30}, w^1, v^{11}, v^{31})) &= \tag{3.37} \\ \int_0^T K w_{xx}(1,t)M(t) &+ h_1E_1 v_x^1(1,t)g_1(t) + h_3E_3 v_x^3(1,t)g_3(t) \\ + \langle (-z^1, -y^{11}, -y^{31}, z^0, y^{10}, y^{30}), &(w^0, v^{10}, v^{30}, w^1, v^{11}, v^{31}) \rangle_{\mathcal{H}', \mathcal{H}} \end{aligned}$$

where

$$\begin{aligned} &\langle (-z^1, -y^{11}, -y^{31}, z^0, y^{10}, y^{30}), (w^0, v^{10}, v^{30}, w^1, v^{11}, v^{31}) \rangle_{\mathcal{H}', \mathcal{H}} \\ &= \int_0^1 m(w^1 z^0 - w^0 z^1) + h_1\rho_1(v^{11}y^{10} - v^{10}y^{11}) + h_3\rho_3(v^{31}y^{30} - v^{30}y^{31}). \end{aligned}$$

Using the above defined linear form, we can rewrite everything from above as follows:

$$\begin{aligned} L_T((w^0, v^{10}, v^{30}, w^1, v^{11}, v^{31})^T) &= \tag{3.38} \\ \langle (-z_t, -y_t^1, -y_t^3, z, y^1, y^3)|_{t=T}, &(w, v^1, v^3, w_t, v_t^1, v_t^3)|_{t=T} \rangle_{\mathcal{H}', \mathcal{H}} \end{aligned}$$

**Definition 10.** We say that  $(z, y^1, y^3, z_t, y_t^1, y_t^3)$  is a solution of (3.30)-(3.36) if

$$(z_t, y_t^1, y_t^3, z, y^1, y^3) \in \mathcal{H}'$$

and (3.38) is satisfied  $\forall t \in \mathbb{R}$  and  $\forall (w^0, v^{10}, v^{30}, w^1, v^{11}, v^{31}) \in \mathcal{H}$ .

**Theorem 12.** Given  $(z^1, y^{11}, y^{31}, z^0, y^{10}, y^{30}) \in \mathcal{H}'$  and  $M(t), g_1(t), g_3(t) \in L^2[0, T]$ , the problem (3.30)-(3.36) has a unique solution.

**Proof :** It is clear from Theorem 11 that the linear form defined in Equation (3.37) is bounded on  $\mathcal{H}$  for every  $t \in \mathbb{R}$ . Furthermore, due to conservation of energy, the linear map  $(w, v^1, v^3, w_t, v^1, v^3)|_{t=T} \rightarrow (w^0, v^{10}, v^{30}, w^1, v^{11}, v^{31})$  is an isomorphism from  $\mathcal{H}$  onto itself. Hence the linear form  $(w, v^1, v^3, w_t, v^1, v^3)|_{t=T} \rightarrow L_T((w^0, v^{10}, v^{30}, w^1, v^{11}, v^{31}))$  is also bounded on  $\mathcal{H}$ . Therefore, by the Riesz representation theorem, there exists a unique element  $(z_t, y_t^1, y_t^3, z, y^1, y^3)|_{t=T} \in \mathcal{H}'$  satisfying Equation (3.38). This proves the theorem.

### 3.7 Observability Estimate

In this section, we prove an observability estimate for solutions of the homogenous problem (3.8)-(3.14) for initial conditions in a smooth enough space as discussed in Remark 16. This estimate will aid us to solve the problem of controlling any initial state in  $\mathcal{H}'$  to zero terminal state.

**Theorem 13.** Let  $T > 0$ . Then every smooth solution of (3.8)-(3.14) satisfies the following observability estimate:

$$\int_0^T \frac{K}{2} w_{xx}^2(1) + \frac{h_1 E_1}{2} (v_x^1(1, t))^2 + \frac{h_3 E_3}{2} (v_x^3(1, t))^2 \geq (T(\frac{\tilde{C}}{F_2} - \frac{G_2}{F_1 h_2}) - \frac{2D}{F_1}) E(0), \quad (3.39)$$

where

$$\tilde{C} = \min(m, 3K, h_1 E_1, h_1 \rho_1, h_3 E_3, h_3 \rho_3)$$

$$D = \max(m\pi^2, h_1 \rho_1, h_3 \rho_3)$$

$$F_1 = \min(h_1 \rho_1, h_3 \rho_3, h_1 E_1, h_3 E_3, m, K)$$

$$F_2 = \max(m, h_1 \rho_1, h_3 \rho_3, h_1 E_1 + 2G_2 \pi^2 (N + \frac{1}{h_2}), h_3 E_3 + 2G_2 \pi^2 (N + \frac{1}{h_2}), \\ K + NG_2 \pi^2 (Nh_2 + 1)).$$

**Proof:** We first recall the identity proved in Lemma 2:

$$\begin{aligned}
& \frac{h_1 E_1}{2} \int_0^T (v_x^1(1, t))^2 + \frac{h_3 E_3}{2} \int_0^T (v_x^3(1, t))^2 + \frac{K}{2} \int_0^T (w_{xx}(1, t))^2 = \\
& \frac{m}{2} \int_0^T \int_0^1 w_t^2 + \frac{3K}{2} \int_0^T \int_0^1 w_{xx}^2 + \frac{N^2 h_2 G_2}{2} \int_0^T \int_0^1 w_x^2 + \frac{h_3 \rho_3}{2} \int_0^T \int_0^1 (v_t^3)^2 \\
& + \frac{h_3 E_3}{2} \int_0^T \int_0^1 (v_x^3)^2 \frac{h_1 \rho_1}{2} \int_0^T \int_0^1 (v_t^1)^2 + \frac{h_1 E_1}{2} \int_0^T \int_0^1 (v_x^1)^2 \\
& - \frac{G_2}{2h_2} \int_0^T \int_0^1 (-v^1 + v^3)^2 + X.
\end{aligned}$$

From above, we omit the term  $\frac{N^2 h_2 G_2}{2} \int_0^T \int_0^1 w_x^2$  as we are only looking for a lower bound for the right hand side. We also recall the estimate on the term  $X$  as derived in Section 3.4 i.e

$$|X| \leq \frac{2D}{F_1} E(0),$$

and hence

$$X \geq -\frac{2D}{F_1} E(0).$$

We use the above estimate along with similar estimating techniques as in Theorem 11. We first estimate as follows:

$$\left| \frac{G_2}{2h_2} \int_0^T \int_0^1 (-v^1 + v^3)^2 \right| \leq T \frac{G_2}{h_2} \tilde{E}(t) \leq T \frac{G_2}{F_1 h_2} E(0).$$

Hence

$$\frac{G_2}{2h_2} \int_0^T \int_0^1 (-v^1 + v^3)^2 \geq -T \frac{G_2}{F_1 h_2} E(0).$$

We also have

$$\begin{aligned}
& \frac{m}{2} \int_0^T \int_0^1 w_t^2 + \frac{3K}{2} \int_0^T \int_0^1 w_{xx}^2 + \frac{N^2 h_2 G_2}{2} \int_0^T \int_0^1 w_x^2 + \frac{h_3 \rho_3}{2} \int_0^T \int_0^1 (v_t^3)^2 \\
& + \frac{h_3 E_3}{2} \int_0^T \int_0^1 (v_x^3)^2 \frac{h_1 \rho_1}{2} \int_0^T \int_0^1 (v_t^1)^2 + \frac{h_1 E_1}{2} \int_0^T \int_0^1 (v_x^1)^2 \\
& \geq T \tilde{C} \tilde{E}(t) \geq \frac{T \tilde{C}}{F_2} E(0),
\end{aligned}$$

where

$$\tilde{C} = \min(m, 3K, h_1 E_1, h_1 \rho_1, h_3 E_3, h_3 \rho_3).$$

Putting all of the above together, we have the proof of the theorem.

Combining Theorems 11,13,Remark 15 and using a density argument as in Remark 17, we have the following theorem:

**Theorem 14.** *Assume that constants  $m, h_1\rho_1, h_3\rho_3, h_1E_1, h_3E_3, K$  are large enough so that  $F_2$  is independent of  $G_2$ . Assume that  $\frac{G_2}{F_1h_2} < \frac{\tilde{C}}{F_2}$  and  $T$  is large enough such that*

$$T > \left( \frac{\frac{2D}{F_1}}{\frac{\tilde{C}}{F_2} - \frac{G_2}{F_1h_2}} \right).$$

Then

$$\int_0^T \frac{K}{2} w_{xx}^2(1) + \frac{h_1E_1}{2} (v_x^1(1, t))^2 + \frac{h_3E_3}{2} (v_x^3(1, t))^2 \quad (3.40)$$

defines a norm equivalent to the energy norm  $\|\cdot\|_{\mathcal{H}}$ .

### 3.8 The HUM principle

In this section, we will apply Hilbert's Uniqueness Method (see e.g [19]) to derive an exact controllability result. First, we look at (3.30)-(3.36) with terminal state conditions and with controls given by  $M(t), g_1(t), g_3(t)$  i.e

$$mz_{tt} + Kz_{xxxx} - N^2h_2G_2z_{xx} - NG_2(-y_x^1 + y_x^3) = 0 \quad (3.41)$$

$$h_1\rho_1y_{tt}^1 - h_1E_1y_{xx}^1 - \frac{G_2}{h_2}(-y^1 + y^3) - NG_2z_x = 0 \quad (3.42)$$

$$h_3\rho_3y_{tt}^3 - h_3E_3y_{xx}^3 + \frac{G_2}{h_2}(-y^1 + y^3) + NG_2z_x = 0. \quad (3.43)$$

The boundary conditions are as follows:

$$z(0, t) = z(1, t) = z_x(0, t) = y^1(0, t) = y^3(0, t) = 0 \quad (3.44)$$

$$z_x(1, t) = M(t), y^1(1, t) = g_1(t), y^3(1, t) = g_3(t), \quad (3.45)$$

$$z(x, 0) = z^0(x), z_t(x, 0) = z^1(x) \quad (3.46)$$

$$y^1(x, 0) = y^{10}(x), y_t^1(x, 0) = y^{11}(x), y^3(x, 0) = y^{30}(x), y_t^3(x, 0) = y^{31}(x). \quad (3.47)$$

It follows from Theorem 12 that for initial conditions  $(z^1, y^{11}, y^{31}, z^0, y^{10}, y^{30}) \in \mathcal{H}'$ , the problem (3.41)-(3.47) has a unique solution  $(z_t, y_t^1, y_t^3, z, y^1, y^3) \in \mathcal{H}' \forall t \in \mathbb{R}$ .

**Definition 11.** The problem (3.41)-(3.47) is exactly controllable if for any given pair of initial and terminal states

$$((z^1, y^{11}, y^{31}, z^0, y^{10}, y^{30}), (z_T^1, y_T^{11}, y_T^{31}, z_T^0, y_T^{10}, y_T^{30})) \in \mathcal{H}',$$

there exist controls  $(M(t), g_1(t), g_3(t))$  in  $L^2[0, T]$  such that the solution of (3.41)-(3.47) satisfies

$$z(x, T) = z_T^0, z_t(x, T) = z_T^0 \quad (3.48)$$

$$y^1(x, T) = y_T^{10}, y_t^1(x, T) = y_T^{11}, y^3(x, T) = y_T^{30}, y_t^3(x, T) = y_T^{31}. \quad (3.49)$$

**Theorem 15.** Assume that constants  $m, h_1\rho_1, h_3\rho_3, h_1E_1, h_3E_3, K$  are large enough so that  $F_2$  is independent of  $G_2$ . Assume that  $\frac{G_2}{F_1h_2} < \frac{\tilde{C}}{F_2}$  and  $T$  is large enough such that

$$T > \left( \frac{\frac{2D}{F_1}}{\frac{\tilde{C}}{F_2} - \frac{G_2}{F_1h_2}} \right),$$

where

$$\tilde{C} = \min(m, 3K, h_1E_1, h_1\rho_1, h_3E_3, h_3\rho_3)$$

$$D = \max(m\pi^2, h_1\rho_1, h_3\rho_3)$$

$$F_1 = \min(h_1\rho_1, h_3\rho_3, h_1E_1, h_3E_3, m, K)$$

$$F_2 = \max\left(m, h_1\rho_1, h_3\rho_3, h_1E_1 + 2G_2\pi^2\left(N + \frac{1}{h_2}\right), h_3E_3 + 2G_2\pi^2\left(N + \frac{1}{h_2}\right), K + NG_2\pi^2(Nh_2 + 1)\right).$$

Then for any pair

$$((z^1, y^{11}, y^{31}, z^0, y^{10}, y^{30}), (z_T^1, y_T^{11}, y_T^{31}, z_T^0, y_T^{10}, y_T^{30})) \in \mathcal{H}',$$

there exist controls  $M(t), g_1(t), g_3(t)$  in  $L^2[0, T]$  such that the solution of (3.41)-(3.47) satisfies

$$z(x, T) = z_T^0, z_t(x, T) = z_T^0 \quad (3.50)$$

$$y^1(x, T) = y_T^{10}, y_t^1(x, T) = y_T^{11}, y^3(x, T) = y_T^{30}, y_t^3(x, T) = y_T^{31}. \quad (3.51)$$

**Remark 20.** Let us consider the solution of the problem

$$m\hat{z}_{tt} + K\hat{z}_{xxxx} - N^2h_2G_2\hat{z}_{xx} - NG_2(-\hat{y}_x^1 + \hat{y}_x^3) = 0 \quad (3.52)$$

$$h_1\rho_1\hat{y}_{tt}^1 - h_1E_1\hat{y}_{xx}^1 - \frac{G_2}{h_2}(-\hat{y}^1 + \hat{y}^3) - NG_2\hat{z}_x = 0 \quad (3.53)$$

$$h_3\rho_3\hat{y}_{tt}^3 - h_3E_3\hat{y}_{xx}^3 + \frac{G_2}{h_2}(-\hat{y}^1 + \hat{y}^3) + NG_2\hat{z}_x = 0, \quad (3.54)$$

$$\hat{z}(0, t) = \hat{z}(1, t) = \hat{z}_x(0, t) = \hat{y}^1(0, t) = \hat{y}^3(0, t) = 0 \quad (3.55)$$

$$\hat{z}_x(1, t) = 0, \hat{y}^1(1, t) = 0, \hat{y}^3(1, t) = 0, \quad (3.56)$$

$$\hat{z}(x, T) = z_T^0, \hat{z}_t(x, T) = z_T^1 \quad (3.57)$$

$$\hat{y}^1(x, T) = y_T^{10}, \hat{y}_t^1(x, T) = y_T^{11}, \hat{y}^3(x, T) = y_T^{30}, \hat{y}_t^3(x, T) = y_T^{31}. \quad (3.58)$$

and assume that there exists controls  $M(t), g_1(t), g_3(t) \in L^2[0, T]$  such that the solution of the problem:

$$mz_{tt} + Kz_{xxxx} - N^2h_2G_2z_{xx} - NG_2(-y_x^1 + y_x^3) = 0 \quad (3.59)$$

$$h_1\rho_1y_{tt}^1 - h_1E_1y_{xx}^1 - \frac{G_2}{h_2}(-y^1 + y^3) - NG_2z_x = 0 \quad (3.60)$$

$$h_3\rho_3y_{tt}^3 - h_3E_3y_{xx}^3 + \frac{G_2}{h_2}(-y^1 + y^3) + NG_2z_x = 0, \quad (3.61)$$

$$z(0, t) = z(1, t) = z_x(0, t) = y^1(0, t) = y^3(0, t) = 0 \quad (3.62)$$

$$z_x(1, t) = M(t), y^1(1, t) = g_1(t), y^3(1, t) = g_3(t), \quad (3.63)$$

$$z(x, 0) = z^0 - \hat{z}(x, 0), z_t(x, 0) = z^1 - \hat{z}_t(x, 0) \quad (3.64)$$

$$y^1(x, 0) = y^{10} - \hat{y}^1(x, 0), \quad (3.65)$$

$$y_t^1(x, 0) = y^{11} - \hat{y}_t^1(x, 0), y^3(x, 0) = y^{30} - \hat{y}^3(x, 0), y_t^3(x, 0) = y^{31} - \hat{y}_t^3(x, 0),$$

satisfies

$$z(x, T) = 0, z_t(x, T) = 0 \quad (3.66)$$

$$y^1(x, T) = 0, y_t^1(x, T) = 0, y^3(x, T) = 0, y_t^3(x, T) = 0. \quad (3.67)$$

Then by linearity,  $\tilde{z} = \hat{z} + z$ ,  $\tilde{y}^1 = \hat{y}^1 + y^1$ ,  $\tilde{y}^3 = \hat{y}^3 + y^3$  is the solution of (3.41)-(3.47) and it satisfies (3.48)-(3.49). For this reason it is sufficient to prove Theorem 15 in the special case

$$(z_T^1, y_T^{11}, y_T^{31}, z_T^0, y_T^{10}, y_T^{30}) = (0, 0, 0, 0, 0, 0).$$

**End of Remark 20.**

**Remark 21.** Remark 20 can also be seen as a consequence of the fact the semigroup that arises from (3.59)-(3.61) is invertible due to the conservative nature of the problem and using Theorem 2, Chapter 1.

The first idea of HUM (Hilbert's Uniqueness Method) is to seek controls of the form  $(M(t), g_1(t), g_3(t)) = (w_{xx}(1, t), v_x^1(1, t), v_x^3(1, t))$  where  $(w, v^1, v^3)$  is a solution of (3.8)-(3.14) with initial conditions  $(w^1, v^{11}, v^{31}, w^0, v^{10}, v^{30}) \in \mathcal{H}$ . Let us recall (see Theorem 10, Remarks 16-18) that given  $(w^1, v^{11}, v^{31}, w^0, v^{10}, v^{30}) \in \mathcal{H}$ , (3.8)-(3.14) has a unique solution satisfying,

$$(w_{xx}(1, t), v_x^1(1, t), v_x^3(1, t)) \in L^2[0, T],$$

(see Theorem 11) and the linear form

$$(w^1, v^{11}, v^{31}, w^0, v^{10}, v^{30}) \rightarrow (w_{xx}(1, t), v_x^1(1, t), v_x^3(1, t))$$

is continuous from  $\mathcal{H}$  to  $(L^2[0, T])^3$ .

As mentioned in Remark 20, we consider the problem of controlling every state in  $\mathcal{H}'$  to zero in time  $T$ . For clarity, we rewrite the problem again:

$$mz_{tt} + Kz_{xxxx} - N^2h_2G_2z_{xx} - NG_2(-y_x^1 + y_x^3) = 0 \quad (3.68)$$

$$h_1\rho_1y_{tt}^1 - h_1E_1y_{xx}^1 - \frac{G_2}{h_2}(-y^1 + y^3) - NG_2z_x = 0 \quad (3.69)$$

$$h_3\rho_3y_{tt}^3 - h_3E_3y_{xx}^3 + \frac{G_2}{h_2}(-y^1 + y^3) + NG_2z_x = 0, \quad (3.70)$$

$$z(0, t) = z(1, t) = z_x(0, t) = y^1(0, t) = y^3(0, t) = 0 \quad (3.71)$$

$$z_x(1, t) = w_{xx}(1, t), y^1(1, t) = v_x^1(1, t), y^3(1, t) = v_x^3(1, t) = v^3(x)(1) \quad (3.72)$$

$$z(x, T) = 0, z_t(x, T) = 0 \quad (3.73)$$

$$y^1(x, T) = 0, \quad (3.74)$$

$$y_t^1(x, T) = 0, y^3(x, T) = 0, y_t^3(x, T) = 0.$$

Using Theorem 12 we deduce that the problem (3.68)-(3.74) has a unique solution satisfying  $(z_t(0), y_t^1(0), y_t^3(0), z(0), y^1(0), y^3(0)) \in \mathcal{H}'$ . Hence we have defined a map  $\tilde{\Lambda} : \mathcal{H} \rightarrow \mathcal{H}'$  which takes the homogenous initial conditions and maps it to non-homogenous initial conditions i.e

$$\tilde{\Lambda}(w^0, v^{10}, v^{30}, w^1, v^{11}, v^{31}) = (z_t(0), y_t^1(0), y_t^3(0), z(0), y^1(0), y^3(0)).$$

If  $(w^1, v^{11}, v^{31}, w^0, v^{10}, v^{30})$  are such that

$$(z_t(0), y_t^1(0), y_t^3(0), z(0), y^1(0), y^3(0)) = (z^1, y^{11}, y^{31}, z^0, y^{10}, y^{30}),$$

then the controls given by  $M(t) = w_{xx}(1, t), g_1(t) = v_x^1(1, t), g_3(t) = v_x^3(1, t)$  drives the system (3.41) - (3.47) to rest in time  $T$ . Thus, Theorem 15 will be proved if we show the surjectivity of the map  $\tilde{\Lambda} : \mathcal{H} \rightarrow \mathcal{H}'$ . For technical reasons it is more convenient to study the surjectivity of the map

$$\Lambda(w^0, v^{10}, v^{30}, w^1, v^{11}, v^{31}) = (z_t(0), y_t^1(0), y_t^3(0), -z(0), -y^1(0), -y^3(0)).$$

Clearly, the two maps  $\tilde{\Lambda}, \Lambda$  are surjective at the same time.

In fact, we have a stronger result:

**Lemma 3.** *Assume that constants  $m, h_1\rho_1, h_3\rho_3, h_1E_1, h_3E_3, K$  are large enough so that  $F_2$  is independent of  $G_2$ . Assume that  $\frac{G_2}{F_1h_2} < \frac{\tilde{C}}{F_2}$  and  $T$  is large enough such that*

$$T > \left( \frac{\frac{2D}{F_1}}{\frac{\tilde{C}}{F_2} - \frac{G_2}{F_1h_2}} \right).$$

*Then  $\Lambda$  is an isomorphism of  $\mathcal{H}$  onto  $\mathcal{H}'$ .*

**Proof:** Clearly  $\Lambda$  is a linear map. Next we compute the following:

$$\langle \Lambda(w^0, v^{10}, v^{30}, w^1, v^{11}, v^{31}), (w^0, v^{10}, v^{30}, w^1, v^{11}, v^{31}) \rangle_{\mathcal{H}', \mathcal{H}}$$

We first multiply Equation (3.30) by  $w$  and integrate by parts.

$$\begin{aligned}
& \int_0^T \int_0^1 m z_{tt} w = \int_0^1 m(z_t w - z w_t)|_{t=0}^{t=T} + \int_0^T \int_0^1 m z w_{tt}. \\
& \int_0^T \int_0^1 k w_{xxxx} w = \int_0^T \int_0^1 K z_{xx} w_{xx} = \int_0^T K M(t) w_{xx}(1, t) + \int_0^T \int_0^1 K z w_{xxxx}. \\
& -N^2 G_2 h_2 \int_0^T \int_0^1 z_{xx} w = N^2 G_2 h_2 \int_0^T \int_0^1 z_x w_x = -N^2 G_2 h_2 \int_0^T \int_0^1 z w_{xx}. \\
& -N G_2 \int_0^T \int_0^1 (-y^1 + y^3)_x w = N G_2 \int_0^T \int_0^1 (-y^1 + y^3) w_x.
\end{aligned}$$

Next we multiply Equation (3.31) by  $v^1$  and integrate by parts.

$$\begin{aligned}
& h_1 \rho_1 \int_0^T \int_0^1 y_{tt}^1 v^1 = h_1 \rho_1 \int_0^1 (y_t^1 v^1 - y^1 v_t^1)|_{t=0}^{t=T} + h_1 \rho_1 \int_0^T \int_0^1 y^1 v_{tt}^1. \\
& -h_1 E_1 \int_0^T \int_0^1 y_{xx}^1 v^1 = h_1 E_1 \int_0^T \int_0^1 y_x^1 v_x^1 = h_1 E_1 \int_0^T g_1(t) v_x^1(1, t) - h_1 E_1 \int_0^T \int_0^1 y^1 v_{xx}^1. \\
& -N G_2 \int_0^T \int_0^1 z_x y^1 = N G_2 \int_0^T \int_0^1 z y_x^1.
\end{aligned}$$

We leave the term  $-\frac{G_2}{h_2} \int_0^T \int_0^1 (-y^1 + y^3) v^1$  unchanged.

Next we multiply Equation (3.32) by  $v^3$  and integrate by parts.

$$\begin{aligned}
& h_3 \rho_3 \int_0^T \int_0^1 y_{tt}^3 v^3 = h_3 \rho_3 \int_0^1 (y_t^3 v^3 - y^3 v_t^3)|_{t=0}^{t=T} + h_3 \rho_3 \int_0^T \int_0^1 y^3 v_{tt}^3. \\
& -h_3 E_3 \int_0^T \int_0^1 y_{xx}^3 v^3 = h_3 E_3 \int_0^T \int_0^1 y_x^3 v_x^3 = h_3 E_3 \int_0^T g_3(t) v_x^3(1, t) - h_3 E_3 \int_0^T \int_0^1 y^3 v_{xx}^3. \\
& N G_2 \int_0^T \int_0^1 z_x y^3 = N G_2 \int_0^T \int_0^1 z y_x^3.
\end{aligned}$$

We leave the term  $\frac{G_2}{h_2} \int_0^T \int_0^1 (-y^1 + y^3) v^3$  unchanged.

Adding all the terms obtained above, using the zero terminal state condition and using the controls given by  $M(t) = w_{xx}(1, t)$ ,  $g_1(t) = v_x^1(1, t)$ ,  $g_3(t) = v_x^3(1, t)$ , we get

$$\begin{aligned}
& \langle \Lambda(w^0, v^{10}, v^{30}, w^1, v^{11}, v^{31}), (w^0, v^{10}, v^{30}, w^1, v^{11}, v^{31}) \rangle_{\mathcal{H}', \mathcal{H}} \\
& = \int_0^T K w_{xx}^2(1) + h_1 E_1 (v_x^1(1, t))^2 + h_3 E_3 (v_x^3(1, t))^2 dt.
\end{aligned} \tag{3.75}$$

By Theorem 14, we have that if the parametric restrictions stated in Lemma 3 are satisfied, then

$$\int_0^T K w_{xx}^2(1) + h_1 E_1 (v_x^1(1, t))^2 + h_3 E_3 (v_x^3(1, t))^2 dt$$

defines a norm equivalent to the energy norm. The two inequalities show boundedness and coercivity of  $\Lambda$  and hence by the Lax-Milgram theorem  $\Lambda : \mathcal{H} \rightarrow \mathcal{H}'$  is an isometric isomorphism. This proves the Lemma.

**Proof of Theorem 15:** Lemma 3 proves the surjectivity of the map  $\Lambda$  and hence keeping Remark 20 in mind, we have the proof of Theorem 15.

### 3.9 Inclusion of the moment of inertia term

In this section we consider the problem (3.1)-(3.7)(see e.g [6],[30]) with an additional term namely  $-\alpha w_{xxtt}$ . We rewrite the problem with this extra term in place.

$$m w_{tt} - \alpha w_{xxtt} + K w_{xxxx} - N^2 h_2 G_2 w_{xx} - N G_2 (-v_x^1 + v_x^3) = 0 \quad (3.76)$$

$$h_1 \rho_1 v_{tt}^1 - h_1 E_1 v_{xx}^1 - \frac{G_2}{h_2} (-v^1 + v^3) - N G_2 w_x = 0 \quad (3.77)$$

$$h_3 \rho_3 v_{tt}^3 - h_3 E_3 v_{xx}^3 + \frac{G_2}{h_2} (-v^1 + v^3) + N G_2 w_x = 0. \quad (3.78)$$

The boundary conditions are as follows:

$$w(0, t) = w(1, t) = w_x(0, t) = v^1(0, t) = v^3(0, t) = 0 \quad (3.79)$$

$$w_x(1, t) = M(t), v^1(1, t) = g_1(t), v^3(1, t) = g_3(t), \quad (3.80)$$

where  $M(t)$ ,  $g_1(t)$  and  $g_3(t)$  are control inputs at the right end. The initial conditions are as follows:

$$w(x, 0) = w^0(x), w_t(x, 0) = w^1(x) \quad (3.81)$$

$$v^1(x, 0) = v^{10}(x), v_t^1(x, 0) = v^{11}(x), v^3(x, 0) = v^{30}(x), v_t^3(x, 0) = v^{31}(x). \quad (3.82)$$

### 3.10 Semigroup Formulation (with the inclusion of additional term)

We consider the homogenous problem first. We recall the homogenous problem once again.

$$mw_{tt} - \alpha w_{xxtt} + Kw_{xxxx} - N^2 h_2 G_2 w_{xx} - NG_2(-v_x^1 + v_x^3) = 0 \quad (3.83)$$

$$h_1 \rho_1 v_{tt}^1 - h_1 E_1 v_{xx}^1 - \frac{G_2}{h_2}(-v^1 + v^3) - NG_2 w_x = 0 \quad (3.84)$$

$$h_3 \rho_3 v_{tt}^3 - h_3 E_3 v_{xx}^3 + \frac{G_2}{h_2}(-v^1 + v^3) + NG_2 w_x = 0. \quad (3.85)$$

The boundary conditions are as follows:

$$w(0, t) = w_x(0, t) = w_x(1, t) = v^1(0, t) = v^3(0, t) = 0 \quad (3.86)$$

$$w(1, t) = 0, v^1(1, t) = 0, v^3(1, t) = 0, \quad (3.87)$$

The initial conditions are as follows:

$$w(x, 0) = w^0(x), w_t(x, 0) = w^1(x) \quad (3.88)$$

$$v^1(x, 0) = v^{10}(x), v_t^1(x, 0) = v^{11}(x), v^3(x, 0) = v^{30}(x), v_t^3(x, 0) = v^{31}(x). \quad (3.89)$$

Let  $Y = (w_1, w_2, w_3, w_4, w_5, w_6)^T = (w, v^1, v^3, w_t, v_t^1, v_t^3)^T = (U, V)^T$ . Then Equation (3.76 - 3.78) can be rewritten as follows.

$$\frac{dY}{dt} = \mathcal{A}Y, \quad Y(0) = (w^0, v^{10}, v^{30}, w^1, v^{11}, v^{31})^T \quad (3.90)$$

where (using  $D = \frac{\partial}{\partial x}$  and  $J = mI - \alpha D^2$ )

$$\tilde{\mathcal{A}} = \begin{pmatrix} J^{-1}(-KD^4 + N^2 h_2 G_2 D^2) & J^{-1}(-NG_2 D) & J^{-1}(NG_2 D) \\ \frac{NG_2}{h_1 \rho_1} D & \frac{E_1}{\rho_1} D^2 - \frac{G_2}{h_2 h_1 \rho_1} & \frac{G_2}{h_2 h_1 \rho_1} \\ -\frac{NG_2}{h_3 \rho_3} D & \frac{G_2}{h_2 h_3 \rho_3} & \frac{E_3}{\rho_3} D^2 - \frac{G_2}{h_2 h_3 \rho_3} \end{pmatrix}, \quad (3.91)$$

$$\mathcal{A} = \begin{pmatrix} 0_{3 \times 3} & I_{3 \times 3} \\ \tilde{\mathcal{A}} & 0_{3 \times 3} \end{pmatrix},$$

$$\mathcal{D}(\mathcal{A}) = H_w^3 \times H_{v^1}^2 \times H_{v^3}^2 \times H_w^2 \times H_{v^1}^1 \times H_{v^3}^1$$

and

$$H_w^3 = \{\phi \in H^3(0, 1) : \phi(0) = \phi(1) = \phi_x(0) = \phi_x(1) = 0\}$$

$$H_w^2 = \{\phi \in H^2(0, 1) : \phi(0) = \phi(1) = \phi_x(0) = \phi_x(1) = 0\}$$

$$H_{v^1}^2 = H_{v^3}^2 = \{\phi \in H^2(0, 1) : \phi(0) = \phi(1) = 0\}$$

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{H} = H_w^2 \times H_0^1(0, 1) \times H_0^1(0, 1) \times H_0^1(0, 1) \times L^2(0, 1) \times L^2(0, 1)$$

The energy inner product is given by the following:

$$\left\langle \begin{pmatrix} w \\ v^1 \\ v^3 \\ w_t \\ v_t^1 \\ v_t^3 \end{pmatrix}, \begin{pmatrix} \hat{w} \\ \hat{v}^1 \\ \hat{v}^3 \\ \hat{w}_t \\ \hat{v}_t^1 \\ \hat{v}_t^3 \end{pmatrix} \right\rangle_e = \int_0^1 m w_t \bar{\hat{w}}_t + \alpha w_{xt} \bar{\hat{w}}_{xt} + h_1 \rho_1 v_t^1 \bar{\hat{v}}_t^1 + h_3 \rho_3 v_t^3 \bar{\hat{v}}_t^3 + K w_{xx} \bar{\hat{w}}_{xx} + h_1 E_1 v_x^1 \bar{\hat{v}}_x^1 \\ + h_3 E_3 v_x^3 \bar{\hat{v}}_x^3 + G_2 h_2 (h_2^{-1} (-v^1 + v^3) + N w_x) (h_2^{-1} (-\hat{v}_1 + \hat{v}_3) + N \hat{w}_x) dx.$$

Consequently we have the following expression for the energy norm:

$$E(t) = \|(w, v^1, v^3, w_t, v_t^1, v_t^3)\|_{\mathcal{H}}^2 = \int_0^1 m w_t^2 + \alpha w_{xt}^2 + h_1 \rho_1 (v_t^1)^2 + h_3 \rho_3 (v_t^3)^2 \quad (3.92) \\ + \int_0^1 K w_{xx}^2 + h_1 E_1 (v_x^1)^2 + h_3 E_3 (v_x^3)^2 + G_2 h_2 (h_2^{-1} (-v^1 + v^3) + N w_x)^2.$$

As in Section 3.2, we define a new norm  $\hat{E}(t)$  which is equivalent to the energy norm and easier to work with.

$$\hat{E}(t) = \|w_{xx}\|_{L^2(0,1)}^2 + \|w_t\|_{L^2(0,1)}^2 + \|w_{xt}\|_{L^2(0,1)}^2 + \|v_t^1\|_{L^2(0,1)}^2 \quad (3.93) \\ + \|v_x^1\|_{L^2(0,1)}^2 + \|v_t^3\|_{L^2(0,1)}^2 + \|v_x^3\|_{L^2(0,1)}^2.$$

As in the previous case, we have the following theorem.

**Theorem 16.** *The norms  $\hat{E}(t)$  is equivalent to  $E(t)$ , i.e there exist constants  $\hat{F}_1 > 0$  and  $\hat{F}_2 > 0$  such that*

$$\hat{F}_1 \hat{E}(t) \leq E(t) \leq \hat{F}_2 \hat{E}(t), \quad (3.94)$$

where

$$\hat{F}_1 = \min(\alpha, h_1 \rho_1, h_3 \rho_3, h_1 E_1, h_3 E_3, m, K) \quad (3.95)$$

$$\hat{F}_2 = \max \left( m, \alpha, h_1 \rho_1, h_3 \rho_3, h_1 E_1 + 2G_2 \pi^2 \left( N + \frac{1}{h_2} \right), h_3 E_3 + 2G_2 \pi^2 \left( N + \frac{1}{h_2} \right), \right. \\ \left. K + NG_2 \pi^2 (Nh_2 + 1) \right).$$

**Proof :** The proof proceeds exactly as in Theorem 9.

We also have the following remark:

**Remark 22.** It turns out that  $\hat{E}(t)$  is an easier norm to work with. The fact that  $\hat{F}_1$  is independent of the coupling constant  $G_2$  will be needed later on while proving estimates. Furthermore, constants  $m, \alpha, h_1 \rho_1, h_3 \rho_3, h_1 E_1, h_3 E_3, K$  can be chosen large enough so that  $\hat{F}_2$  is also independent of  $G_2$ .

As in the previous case, we have the following theorem:

**Theorem 17.**  *$\mathcal{A}$  is the infinitesimal generator of a strongly continuous semigroup on  $\mathcal{H}$ .*

**Proof :** The proof follows the same lines as the proof of Theorem 10. For completeness, we show the calculations again.

We can easily check that  $\mathcal{A}^* = -\mathcal{A}$  and  $D(\mathcal{A}^*) = D(\mathcal{A})$  by direct calculation. Next we check that both  $\mathcal{A}$  and  $\mathcal{A}^*$  are dissipative on  $\mathcal{H}$ . Then the result will follow from the Lumer-Phillips theorem (see e.g. Pazy [27]). We show the calculation for  $\mathcal{A}$ . The calculation for  $\mathcal{A}^*$  is similar. Let  $Y \in \mathcal{D}(\mathcal{A})$  be given by  $Y = (x_1, x_2, x_3, y_1, y_2, y_3)$ .  $\langle \mathcal{A}Y, Y \rangle_e$  consists of various terms.

$$\text{Term 1} = \int_0^1 m J^{-1} (-KD^4 x_1 + N^2 h_2 G_2 D^2 x_1 + NG_2 (-x_2 + x_3)) \bar{y}_1 \\ \text{Term 2} = \int_0^1 \alpha D(J^{-1} (-KD^4 x_1 + N^2 g_2 h_2 D^2 x_1 + NG_2 D(-x_2 + x_3))) D \bar{y}_1$$

$$\begin{aligned}
&= - \int_0^1 \alpha D^2(J^{-1}(-KD^4x_1 + N^2g_2h_2D^2x_1 + NG_2D(-x_2 + x_3)))\bar{y}_1. \\
\text{Term 3} &= \int_0^1 h_1\rho_1\left(\frac{1}{h_1\rho_1}(h_1E_1D^2x_2 + \frac{G_2}{h_2}(-x_2 + x_3) + NG_2Dx_1)\right)\bar{y}_2 \\
&= \int_0^1 -h_1E_1Dx_2\overline{Dy_2} + \frac{G_2}{h_2}(-x_2 + x_3)\bar{y}_2 - NG_2x_1\overline{Dx_2} \\
\text{Term 4} &= \int_0^1 h_3\rho_3\left(\frac{1}{h_3\rho_3}(h_3E_3D^2x_3 - \frac{G_2}{h_2}(-x_2 + x_3) - NG_2Dx_1)\right)\bar{y}_3 \\
&= \int_0^1 (-h_3E_3Dx_3\overline{Dy_3} - \frac{G_2}{h_2}(-x_2 + x_3)\bar{y}_3 + NG_2x_1\overline{Dy_3}) \\
\text{Term 5} &= \int_0^1 KD^2y_1\overline{D^2x_1} = \int_0^1 -KDy_1\overline{D^3x_1} = \int_0^1 Ky_1\overline{D^4x_1} \\
\text{Term 6} &= \int_0^1 h_1E_1Dy_2\overline{Dx_2} \\
\text{Term 7} &= \int_0^1 h_3E_3Dy_3\overline{Dx_3} \\
\text{Term 8} &= \int_0^1 G_2h_2(h_2^{-1}(-y_2 + y_3) + NDy_1)(h_2^{-1}(-\bar{x}_2 + \bar{x}_3) + ND\bar{x}_1) \\
&= \int_0^1 \left(\frac{G_2}{h_2}(-y_2 + y_3)(-\bar{x}_2 + \bar{x}_3) - N^2G_2h_2y_1\overline{D^2x_1} - NG_2D(-y_2 + y_3)\bar{x}_1 - NG_2y_1\overline{D(-x_2 + x_3)}\right)
\end{aligned}$$

Adding Terms 1 through 8 and rearranging, we get the following. Note that the term  $J^{-1}$  cancels with  $J$  while adding all the terms.

$$\begin{aligned}
&\langle \mathcal{A}Y, Y \rangle_{\mathcal{H}} = \int_0^1 K(\overline{D^4x_1y_1} - D^4x_1\bar{y}_1) + N^2G_2h_2(D^2x_1\bar{y}_1 - \overline{D^2x_1y_1}) \\
&+ NG_2(D(-x_2 + x_3)\bar{y}_1 - \overline{D(-x_2 + x_3)y_1}) + h_1E_1(Dy_2\overline{Dx_2} - \overline{Dy_2}Dx_2) + h_3E_3(Dy_3\overline{Dx_3} - \overline{Dy_3}Dx_3) \\
&+ NG_2(x_1\overline{D(-y_2 + y_3)} - D(-y_2 + y_3)\bar{x}_1) + \frac{G_2}{h_2}((-y_2 + y_3)\overline{(-x_2 + x_3)} - (-x_2 + x_3)\overline{(-y_2 + y_3)}).
\end{aligned}$$

Hence, once again we have  $\text{Re}(\langle \mathcal{A}Y, Y \rangle_{\mathcal{H}}) = 0$  and hence  $\mathcal{A}$  is a dissipative operator and the proof is complete.

Remarks 16,17 and (18) hold true in this case also.

### 3.11 An important identity (with the additional term)

We recalculate the identity obtained in Section 3.3 with the moment of inertia term included.

**Lemma 4.** *Let  $T > 0$ . Then every smooth solution of (3.83-3.89) satisfies the following identity:*

$$\begin{aligned}
& \frac{h_1 E_1}{2} \int_0^T (v_x^1(1, t))^2 + \frac{h_3 E_3}{2} \int_0^T (v_x^3(1, t))^2 + \frac{K}{2} \int_0^T (w_{xx}(1, t))^2 = \\
& \frac{m}{2} \int_0^T \int_0^1 w_t^2 + \frac{3K}{2} \int_0^T \int_0^1 w_{xx}^2 + \frac{N^2 h_2 G_2}{2} \int_0^T \int_0^1 w_x^2 - \frac{\alpha}{2} \int_0^T \int_0^1 w_{xt}^2 \\
& + \frac{h_3 \rho_3}{2} \int_0^T \int_0^1 (v_t^3)^2 + \frac{h_3 E_3}{2} \int_0^T \int_0^1 (v_x^3)^2 \frac{h_1 \rho_1}{2} \int_0^T \int_0^1 (v_t^1)^2 + \frac{h_1 E_1}{2} \int_0^T \int_0^1 (v_x^1)^2 \\
& - \frac{G_2}{2h_2} \int_0^T \int_0^1 (-v^1 + v^3)^2 + X
\end{aligned} \tag{3.96}$$

where  $X = \int_0^1 m w_t x w_x - \alpha w_{xxt} x w_x + h_1 \rho_1 v_t^1 x v_x^1 + h_3 \rho_3 v_t^3 x v_x^3 \Big|_{t=0}^{t=T}$ .

**Proof:** The proof proceeds exactly like in Theorem 2 except for an addition calculation which results while multiplying Equation (3.83) by  $xw_x$  and integrating by parts. We show that calculation here.

$$\begin{aligned}
-\alpha \int_0^T \int_0^1 w_{xxtt} x w_x &= -\alpha \int_0^T w_{xxt} x w_x \Big|_{t=0}^{t=T} + \alpha \int_0^T \int_0^1 w_{xxt} x w_{xt} \\
\alpha \int_0^T \int_0^1 w_{xxt} x w_x &= \alpha \int_0^T \int_0^1 x \left( \frac{1}{2} w_{xt}^2 \right)_x = -\frac{\alpha}{2} \int_0^T \int_0^1 w_{xt}^2
\end{aligned}$$

Hence we have

$$-\alpha \int_0^T \int_0^1 w_{xxtt} x w_x = -\alpha \int_0^T w_{xxt} x w_x \Big|_{t=0}^{t=T} - \frac{\alpha}{2} \int_0^T \int_0^1 w_{xt}^2.$$

The rest of the calculations remain the same and the Theorem is proved.

### 3.12 Estimating $X$ (with the additional term)

We estimate the term  $X$  in a manner similar to Section 3.4.

$$X = \int_0^1 m w_t x w_x + h_1 \rho_1 v_t^1 x v_x^1 + h_3 \rho_3 v_t^3 x v_x^3 - \alpha \int_0^1 w_{xxt} x w_x \Big|_{t=0}^{t=T}$$

We first look at

$$X(t) = \int_0^1 m w_t x w_x + h_1 \rho_1 v_t^1 x v_x^1 + h_3 \rho_3 v_t^3 x v_x^3 - \alpha \int_0^1 w_{xxt} x w_x.$$

The only additional estimate as compared to Section 3.4 is due to the moment of inertia term. We only show that part here.

$$\alpha \int_0^1 w_{xxt} x w_x = \alpha \int_0^1 x \left( \frac{1}{2} w_{xt}^2 \right)_x = -\frac{\alpha}{2} \int_0^1 w_{xt}^2$$

Hence following the same lines as in Section 3.4, we have  $|X| \leq \hat{D} \hat{E}(t) \leq \frac{2\hat{D}}{\hat{F}_1} E(0)$  where

$$\hat{D} = \max (m\pi^2, h_1 \rho_1, h_3 \rho_3, h_1 E_1, h_3 E_3, \alpha).$$

### 3.13 Hidden regularity estimate (with the additional term)

Following the same lines as in Section 3.5, we derive an analogous hidden regularity estimate for the homogenous problem (3.83)-(3.89). This estimate will allow us to define a weak solution to the controlled equations (3.76)-(3.82).

**Theorem 18.** *Let  $T > 0$ . Then every smooth solution of (3.83)-(3.89) satisfies the following "hidden" regularity estimate:*

$$\frac{h_1 E_1}{2} \int_0^T (v_x^1(1, t))^2 + \frac{h_3 E_3}{2} \int_0^T (v_x^3(1, t))^2 + \frac{K}{2} \int_0^T w_{xx}^2(1) \leq \left( \frac{TC + 2\hat{D}}{\hat{F}_1} \right) E(0), \quad (3.97)$$

where

$$\hat{D} = \max (m\pi^2, h_1 \rho_1, h_3 \rho_3, h_1 E_1, h_3 E_3, \alpha),$$

$$C = \max (m, 3K, N^2 h_2 G_2 \pi^2, h_1 E_1, h_1 \rho_1, h_3 E_3, h_3, \rho_3),$$

$$\hat{F}_1 = \min (\alpha, h_1 \rho_1, h_3 \rho_3, h_1 E_1, h_3 E_3, m, K).$$

As a consequence, the linear form  $(w^1, v^{11}, v^{31}, w^0, v^{10}, v^{30}) \rightarrow (w_{xx}(1, t), v_x^1(1, t), v_x^3(1, t))$  is continuous from  $\mathcal{H}$  to  $(L^2[0, T])^3$ .

**Proof :** The proof follows the same steps as in Theorem 11.

### 3.14 Solution of the non-homogenous problem (with the additional term)

In this section, we use the regularity estimate proved in Theorem 18 to show the existence of weak solutions for the non-homogenous problem. To maintain consistency of notation, we rename the variables in Equations (3.76)-(3.82) as follows:

$$mz_{tt} - \alpha z_{xxtt} + Kz_{xxxx} - N^2 h_2 G_2 z_{xx} - NG_2(-y_x^1 + y_x^3) = 0 \quad (3.98)$$

$$h_1 \rho_1 y_{tt}^1 - h_1 E_1 y_{xx}^1 - \frac{G_2}{h_2}(-y^1 + y^3) - NG_2 z_x = 0 \quad (3.99)$$

$$h_3 \rho_3 y_{tt}^3 - h_3 E_3 y_{xx}^3 + \frac{G_2}{h_2}(-y^1 + y^3) + NG_2 z_x = 0. \quad (3.100)$$

The boundary conditions are as follows:

$$z(0, t) = z(1, t) = z_x(0, t) = y^1(0, t) = y^3(0, t) = 0 \quad (3.101)$$

$$z_x(1, t) = M(t), y^1(1, t) = g_1(t), y^3(1, t) = g_3(t), \quad (3.102)$$

where  $M(t)$ ,  $g_1(t)$  and  $g_3(t)$  are control inputs at the right end. The initial conditions are as follows:

$$z(x, 0) = z^0(x), z_t(x, 0) = z^1(x) \quad (3.103)$$

$$y^1(x, 0) = y^{10}(x), y_t^1(x, 0) = y^{11}(x), y^3(x, 0) = y^{30}(x), y_t^3(x, 0) = y^{31}(x). \quad (3.104)$$

In order to define a solution for Equations (3.98)-(3.104), we first multiply Equation (3.76) by  $z$  and formally integrate by parts. We have the same terms as in Section 3.6 except for the following:

$$\begin{aligned} -\alpha \int_0^T \int_0^1 w_{xxtt} z &= -\alpha \int_0^1 (w_{xxt} z - w_{xx} z_t) \Big|_{t=0}^{t=T} - \alpha \int_0^T \int_0^1 w_{xx} z_{tt} \\ \alpha \int_0^T \int_0^1 w_{xx} z_{tt} &= \alpha \int_0^T \int_0^1 w_x z_{xtt} = -\alpha \int_0^T \int_0^1 w z_{xxtt}. \end{aligned}$$

So in all, the extra term amounts to the following:

$$-\alpha \int_0^T \int_0^1 w_{xxtt} z = -\alpha \int_0^1 (w_{xxt} z - w_{xx} z_t) \Big|_{t=0}^{t=T} - \alpha \int_0^T \int_0^1 w z_{xxtt}.$$

Now we multiply Equation (3.77) with  $y^1$ , Equation (3.78) with  $y^2$  and formally integrate by parts. Since we get exactly the same terms as in Section 3.6 we choose not to repeat the terms.

Adding everything from above and rearranging, we get the following:

$$\begin{aligned}
& \int_0^1 ((mI - D^2)w_t(T)z(T) - (mI - D^2)w(T)z_t(T)) \\
& + h_1\rho_1(v_t^1(T)y^1(T) - v^1y_t^1(T)) + h_3\rho_3(v_t^3(T)y^3(T) - v^3(T)y_t^3(T)) \\
= & \int_0^1 ((mI - D^2)w^1z^0 - (mI - D^2)w^0z^1) + h_1\rho_1(v^{11}y^{10} - v^{10}y^{11}) + h_3\rho_3(v^{31}y^{30} - v^{30}y^{31}) \\
& + \int_0^T Kw_{xx}(1,t)M(t) + h_1E_1v_x^1(1,t)g_1(t) + h_3E_3v_x^3(1,t)g_3(t).
\end{aligned}$$

We define the following linear form.

$$\begin{aligned}
L_T((w^0, v^{10}, v^{30}, w^1, v^{11}, v^{31})) = & \tag{3.105} \\
& \int_0^T Kw_{xx}(1,t)M(t) + h_1E_1v_x^1(1,t)g_1(t) + h_3E_3v_x^3(1,t)g_3(t) \\
& + \langle -(mI - D^2)z^1, -y^{11}, -y^{31}, (mI - D^2)z^0, y^{10}, y^{30}, (w^0, v^{10}, v^{30}, w^1, v^{11}, v^{31}) \rangle_{\mathcal{H}', \mathcal{H}},
\end{aligned}$$

where

$$\begin{aligned}
& \langle -(mI - D^2)z^1, -y^{11}, -y^{31}, (mI - D^2)z^0, y^{10}, y^{30}, (w^0, v^{10}, v^{30}, w^1, v^{11}, v^{31}) \rangle_{\mathcal{H}', \mathcal{H}} \\
= & \int_0^1 ((mI - D^2)w^1z^0 - (mI - D^2)w^0z^1) + h_1\rho_1(v^{11}y^{10} - v^{10}y^{11}) + h_3\rho_3(v^{31}y^{30} - v^{30}y^{31}).
\end{aligned}$$

Using the above defined linear form, we can rewrite everything from above as follows:

$$\begin{aligned}
L_T((w^0, v^{10}, v^{30}, w^1, v^{11}, v^{31})^T) = & \tag{3.106} \\
& \langle -(mI - D^2)z_t, -y_t^1, -y_t^3, (mI - D^2)z, y^1, y^3 \rangle_{t=T}, (w, v^1, v^3, w_t, v^1, v^3)_{t=T} \rangle_{\mathcal{H}', \mathcal{H}}
\end{aligned}$$

**Definition 12.** We say that  $(z, y^1, y^3, z_t, y_t^1, y_t^3)$  is a solution of (3.30)-(3.36) if

$$(z, y^1, y^3, z_t, y_t^1, y_t^3) \in H_0^1(0, 1) \times L^2(0, 1) \times L^2(0, 1) \times L^2(0, 1) \times H^{-1}(0, 1) \times H^{-1}(0, 1)$$

and (3.106) is satisfied  $\forall t \in \mathbb{R}$  and

$$\forall (w^0, v^{10}, v^{30}, w^1, v^{11}, v^{31}) \in \mathcal{H}.$$

**Theorem 19.** *Given*

$$(z^0, y^{10}, y^{30}, z^1, y^{11}, y^{31}) \in H_0^1(0, 1) \times L^2(0, 1) \times L^2(0, 1) \times L^2(0, 1) \times H^{-1}(0, 1) \times H^{-1}(0, 1),$$

and  $M(t), g_1(t), g_3(t) \in L^2[0, T]$ , the problem (3.98)-(3.104) has a unique solution.

**Proof :** It is clear from Theorem 18 that the linear form defined in Equation (3.105) is bounded on  $\mathcal{H}$  for every  $t \in \mathbb{R}$ . Furthermore, due to conservation of energy, the linear map  $(w, v^1, v^3, w_t, v^1, v^3)|_{t=T} \rightarrow (w^0, v^{10}, v^{30}, w^1, v^{11}, v^{31})$  is an isomorphism from  $\mathcal{H}$  onto itself. Hence the linear form  $(w, v^1, v^3, w_t, v^1, v^3)|_{t=T} \rightarrow L_T((w^0, v^{10}, v^{30}, w^1, v^{11}, v^{31}))$  is also bounded on  $\mathcal{H}$ . Therefore, by the Riesz representation theorem, there exists a unique element  $(\hat{z}_t, y_t^1, y_t^3, \hat{z}, y^1, y^3)|_{t=T} \in \mathcal{H}'$  satisfying Equation (3.106). Next, we note that the operators  $(mI - \alpha D^2) : H_0^1(0, 1) \rightarrow H^{-1}(0, 1)$  and  $(mI - \alpha D^2) : L^2(0, 1) \rightarrow H^{-2}(0, 1)$  are isometric isomorphisms. Hence given  $(\hat{z}, \hat{z}_t)|_{t=T} \in H^{-1}(0, 1) \times H^{-2}(0, 1)$ , we can solve for  $(z, z_t)|_{t=T} \in H_0^1(0, 1) \times L^2(0, 1)$  from the following equations:

$$(mI - D^2)z(T) = \hat{z}(T), -(mI - D^2)z_t(T) = \hat{z}_t(T), \forall t \in \mathbb{R}$$

This proves the theorem.

### 3.15 Observability Estimate (with the additional term)

In this section, we prove an observability estimate for solutions of the homogenous problem (3.83)-(3.89) for initial conditions in a smooth enough space as discussed in Remark 16. This estimate will aid us to solve the problem of controlling any initial state in  $\mathcal{H}'$  to zero terminal state.

**Theorem 20.** *Let  $T > 0$ . Then every smooth solution of (3.83)-(3.89) satisfies the following observability estimate:*

$$\int_0^T \frac{K}{2} w_{xx}^2(1) + \frac{h_1 E_1}{2} (v_x^1(1, t))^2 + \frac{h_3 E_3}{2} (v_x^3(1, t))^2 \geq (T(\frac{\tilde{C}}{\hat{F}_2} - (\frac{G_2}{\hat{F}_1 h_2} + \frac{\alpha}{2\hat{F}_1})) - \frac{2\hat{D}}{\hat{F}_1})E(0),$$

where

$$\begin{aligned}\tilde{C} &= \min(m, 3K, h_1 E_1, h_1 \rho_1, h_3 E_3, h_3 \rho_3) \\ \hat{D} &= \max(m\pi^2, h_1 \rho_1, h_3 \rho_3, h_1 E_1, h_3 E_3, \alpha) \\ \hat{F}_1 &= \min(\alpha, h_1 \rho_1, h_3 \rho_3, h_1 E_1, h_3 E_3, m, K) \\ \hat{F}_2 &= \max\left(m, \alpha, h_1 \rho_1, h_3 \rho_3, h_1 E_1 + 2G_2\pi^2\left(N + \frac{1}{h_2}\right), h_3 E_3 + 2G_2\pi^2\left(N + \frac{1}{h_2}\right), \right. \\ &\quad \left. K + NG_2\pi^2(Nh_2 + 1)\right).\end{aligned}$$

**Proof:** We first recall the identity proved in Lemma 2:

$$\begin{aligned}& \frac{h_1 E_1}{2} \int_0^T (v_x^1(1, t))^2 + \frac{h_3 E_3}{2} \int_0^T (v_x^3(1, t))^2 + \frac{K}{2} \int_0^T (w_{xx}(1, t))^2 = \\ & \frac{m}{2} \int_0^T \int_0^1 w_t^2 + \frac{3K}{2} \int_0^T \int_0^1 w_{xx}^2 + \frac{N^2 h_2 G_2}{2} \int_0^T \int_0^1 w_x^2 + \frac{h_3 \rho_3}{2} \int_0^T \int_0^1 (v_t^3)^2 \\ & + \frac{h_3 E_3}{2} \int_0^T \int_0^1 (v_x^3)^2 \frac{h_1 \rho_1}{2} \int_0^T \int_0^1 (v_t^1)^2 + \frac{h_1 E_1}{2} \int_0^T \int_0^1 (v_x^1)^2 \\ & - \frac{G_2}{2h_2} \int_0^T \int_0^1 (-v^1 + v^3)^2 + X\end{aligned}$$

From above, as in Section 3.7, we omit the term  $\frac{N^2 h_2 G_2}{2} \int_0^T \int_0^1 w_x^2$  as we are only looking for a lower bound for the right hand side. We also recall the estimate on the term  $X$  as derived in Section 3.12 i.e

$$|X| \leq \frac{2\hat{D}}{\hat{F}_1} E(0),$$

and hence

$$X \geq -\frac{2\hat{D}}{\hat{F}_1} E(0).$$

We use the above estimate along with similar estimating techniques as in Theorem 18. We first estimate as follows:

$$\left| \frac{G_2}{2h_2} \int_0^T \int_0^1 (-v^1 + v^3)^2 \right| \leq T \frac{G_2}{h_2} \hat{E}(t) \leq T \frac{G_2}{\hat{F}_1 h_2} E(0)$$

Hence

$$\frac{G_2}{2h_2} \int_0^T \int_0^1 (-v^1 + v^3)^2 \geq -T \frac{G_2}{\hat{F}_1 h_2} E(0).$$

We have an additional estimate due to the moment of inertia term

$$\left| \frac{\alpha}{2} \int_0^T \int_0^1 w_{xt}^2 \right| \leq \frac{\alpha}{2} \hat{E}(t) \leq \frac{\alpha}{2\hat{F}_1} E(0)$$

and hence

$$-\frac{\alpha}{2} \int_0^T \int_0^1 w_{xt}^2 \geq -\frac{\alpha}{2\hat{F}_1} E(0).$$

We also have

$$\begin{aligned} & \frac{m}{2} \int_0^T \int_0^1 w_t^2 + \frac{3K}{2} \int_0^T \int_0^1 w_{xx}^2 + \frac{N^2 h_2 G_2}{2} \int_0^T \int_0^1 w_x^2 + \frac{h_3 \rho_3}{2} \int_0^T \int_0^1 (v_t^3)^2 \\ & + \frac{h_3 E_3}{2} \int_0^T \int_0^1 (v_x^3)^2 + \frac{h_1 \rho_1}{2} \int_0^T \int_0^1 (v_t^1)^2 + \frac{h_1 E_1}{2} \int_0^T \int_0^1 (v_x^1)^2 \\ & \geq T \tilde{C} \hat{E}(t) \geq \frac{\tilde{C}}{\hat{F}_2} E(0), \end{aligned}$$

where

$$\tilde{C} = \min(m, 3K, h_1 E_1, h_1 \rho_1, h_3 E_3, h_3 \rho_3).$$

Putting all of the above together, we have the proof of the theorem.

Combining Theorems 18 and 20 and using a density argument as in Remark 17, we have the following theorem:

**Theorem 21.** *Assume that constants  $m, \alpha, h_1 \rho_1, h_3 \rho_3, h_1 E_1, h_3 E_3, K$  are large enough so that  $\hat{F}_2$  is independent of  $G_2$ . Assume that*

$$\frac{\tilde{C}}{\hat{F}_2} > \left( \frac{G_2}{\hat{F}_1 h_2} + \frac{\alpha}{2\hat{F}_1} \right)$$

and  $T$  is large enough such that

$$T > \left( \frac{\frac{2\hat{D}}{\hat{F}_1}}{\frac{\tilde{C}}{\hat{F}_2} - \left( \frac{G_2}{\hat{F}_1 h_2} + \frac{\alpha}{2\hat{F}_1} \right)} \right).$$

Then

$$\int_0^T \frac{K}{2} w_{xx}^2(1) + \frac{h_1 E_1}{2} (v_x^1(1, t))^2 + \frac{h_3 E_3}{2} (v_x^3(1, t))^2 \quad (3.107)$$

defines a norm equivalent to the energy norm  $\|\cdot\|_{\mathcal{H}}$ .

### 3.16 The HUM principle (with the additional term)

As in Section 3.8, we will apply Hilbert's Uniqueness Method (see se.g [19]) to derive an exact controllability result. First, we look at (3.98)-(3.104) with terminal state conditions and with controls given by  $M(t), g_1(t), g_3(t)$  i.e

$$mz_{tt} - \alpha z_{xxtt} + Kz_{xxxx} - N^2 h_2 G_2 z_{xx} - N G_2 (-y_x^1 + y_x^3) = 0 \quad (3.108)$$

$$h_1 \rho_1 y_{tt}^1 - h_1 E_1 y_{xx}^1 - \frac{G_2}{h_2} (-y^1 + y^3) - N G_2 z_x = 0 \quad (3.109)$$

$$h_3 \rho_3 y_{tt}^3 - h_3 E_3 y_{xx}^3 + \frac{G_2}{h_2} (-y^1 + y^3) + N G_2 z_x = 0. \quad (3.110)$$

The boundary conditions are as follows:

$$z(0, t) = z(1, t) = z_x(0, t) = y^1(0, t) = y^3(0, t) = 0 \quad (3.111)$$

$$z_x(1, t) = M(t), y^1(1, t) = g_1(t), y^3(1, t) = g_3(t), \quad (3.112)$$

$$z(x, 0) = z^0(x), z_t(x, 0) = z^1(x) \quad (3.113)$$

$$y^1(x, 0) = y^{10}(x), y_t^1(x, 0) = y^{11}(x), y^3(x, 0) = y^{30}(x), y_t^3(x, 0) = y^{31}(x). \quad (3.114)$$

It follows from Theorem 19 that for initial conditions

$$((mI - \alpha D^2)z^1, y^{11}, y^{31}, (mI - \alpha D^2)z^0, y^{10}, y^{30}) \in \mathcal{H}',$$

the problem (3.108)-(3.114) has a unique solution  $(z_t, y_t^1, y_t^3, z, y^1, y^3) \in \mathcal{H}'$ .

**Definition 13.** The problem (3.108)-(3.114) is exactly controllable if for any given

$$((z^0, y^{10}, y^{30}, z^1, y^{11}, y^{31}), (z_T^0, y_T^{10}, y_T^{30}, z_T^1, y_T^{11}, y_T^{31})) \in$$

$$H_0^1(0, 1) \times L^2(0, 1) \times L^2(0, 1) \times L^2(0, 1) \times H^{-1}(0, 1) \times H^{-1}(0, 1)$$

there exist controls  $(M(t), g_1(t), g_3(t))$  in  $L^2[0, T]$  such that the solution of (3.108)-(3.114) satisfies

$$z(x, T) = z_T^0, z_t(x, T) = z_T^0 \quad (3.115)$$

$$y^1(x, T) = y_T^{10}, y_t^1(x, T) = y_T^{11}, y^3(x, T) = y_T^{30}, y_t^3(x, T) = y_T^{31}. \quad (3.116)$$

**Theorem 22.** Assume that constants  $m, \alpha, h_1\rho_1, h_3\rho_3, h_1E_1, h_3E_3, K$  are large enough so that  $\hat{F}_2$  is independent of  $G_2$ . Assume that

$$\frac{\tilde{C}}{\hat{F}_2} > \left( \frac{G_2}{\hat{F}_1 h_2} + \frac{\alpha}{2\hat{F}_1} \right)$$

and  $T$  is large enough such that

$$T > \left( \frac{\frac{2\hat{D}}{\hat{F}_1}}{\frac{\tilde{C}}{\hat{F}_2} - \left( \frac{G_2}{\hat{F}_1 h_2} + \frac{\alpha}{2\hat{F}_1} \right)} \right)$$

where

$$\tilde{C} = \min(m, 3K, h_1E_1, h_1\rho_1, h_3E_3, h_3\rho_3)$$

$$\hat{D} = \max(m\pi^2, h_1\rho_1, h_3\rho_3, h_1E_1, h_3E_3, \alpha)$$

$$\hat{F}_1 = \min(\alpha, h_1\rho_1, h_3\rho_3, h_1E_1, h_3E_3, m, K)$$

$$\hat{F}_2 = \max\left(m, \alpha, h_1\rho_1, h_3\rho_3, h_1E_1 + 2G_2\pi^2\left(N + \frac{1}{h_2}\right), h_3E_3 + 2G_2\pi^2\left(N + \frac{1}{h_2}\right), K + NG_2\pi^2(Nh_2 + 1)\right).$$

Then for any pair

$$\left( (z^0, y^{10}, y^{30}, z^1, y^{11}, y^{31}), (z_T^0, y_T^{10}, y_T^{30}, z_T^1, y_T^{11}, y_T^{31}) \right) \in$$

$$H_0^1(0, 1) \times L^2(0, 1) \times L^2(0, 1) \times L^2(0, 1) \times H^{-1}(0, 1) \times H^{-1}(0, 1)$$

there exist controls  $M(t), g_1(t), g_3(t)$  in  $L^2[0, T]$  such that the solution of (3.108)-(3.114) satisfies

$$z(x, T) = z_T^0, z_t(x, T) = z_T^0 \tag{3.117}$$

$$y^1(x, T) = y_T^{10}, y_t^1(x, T) = y_T^{11}, y^3(x, T) = y_T^{30}, y_t^3(x, T) = y_T^{31}. \tag{3.118}$$

**Remark 23.** For a similar reason as in Remark 20, it is sufficient to prove Theorem 22 for the special case

$$(z_T^1, y_T^{11}, y_T^{31}, z_T^0, y_T^{10}, y_T^{30}) = (0, 0, 0, 0, 0, 0).$$

As in Section 3.8, the first idea of HUM (Hilbert's Uniqueness Method) is to seek controls of the form  $(M(t), g_1(t), g_3(t)) = (w_{xx}(1, t), v_x^1(1, t), v_x^3(1, t))$  where  $(w, v^1, v^3)$  is a solution of (3.83)-(3.89) with initial conditions  $(w^1, v^{11}, v^{31}, w^0, v^{10}, v^{30}) \in \mathcal{H}$ . Let us recall (see Theorem 17), Remarks 16-18) that given  $(w^1, v^{11}, v^{31}, w^0, v^{10}, v^{30}) \in \mathcal{H}$ , (3.83)-(3.89) has a unique solution,  $w_{xx}(1, t), v_x^1(1, t), v_x^3(1, t)$  belonging to  $L^2[0, T]$  (see Theorem 18) and the linear form

$$(w^1, v^{11}, v^{31}, w^0, v^{10}, v^{30}) \rightarrow (w_{xx}(1, t), v_x^1(1, t), v_x^3(1, t))$$

is continuous from  $\mathcal{H}$  to  $(L^2[0, T])^3$ .

As mentioned in Remark 23, we consider the problem of controlling every state in  $H_0^1(0, 1) \times L^2(0, 1) \times L^2(0, 1) \times L^2(0, 1) \times H^{-1}(0, 1) \times H^{-1}(0, 1)$  to zero in time  $T$ . For clarity, we rewrite the problem again:

$$mz_{tt} - \alpha z_{xxtt} + Kz_{xxxx} - N^2 h_2 G_2 z_{xx} - N G_2 (-y_x^1 + y_x^3) = 0 \quad (3.119)$$

$$h_1 \rho_1 y_{tt}^1 - h_1 E_1 y_{xx}^1 - \frac{G_2}{h_2} (-y^1 + y^3) - N G_2 z_x = 0 \quad (3.120)$$

$$h_3 \rho_3 y_{tt}^3 - h_3 E_3 y_{xx}^3 + \frac{G_2}{h_2} (-y^1 + y^3) + N G_2 z_x = 0. \quad (3.121)$$

$$z(0, t) = z(1, t) = z_x(0, t) = y^1(0, t) = y^3(0, t) = 0 \quad (3.122)$$

$$z_x(1, t) = M(t), y^1(1, t) = g_1(t), y^3(1, t) = g_3(t), \quad (3.123)$$

$$z(x, T) = 0, z_t(x, T) = 0 \quad (3.124)$$

$$y^1(x, T) = 0, y_t^1(x, T) = 0, \quad (3.125)$$

$$y^3(x, T) = 0, y_t^3(x, T) = 0.$$

Using Theorem 19 we deduce that the problem (3.119)-(3.125) has a unique solution satisfying  $((mI - \alpha D^2)z_t(0), y_t^1(0), y_t^3(0), -(mI - D^2)z(0), y^1(0), y^3(0)) \in \mathcal{H}'$ . Hence we have defined the following map  $\tilde{\Lambda} : \mathcal{H} \rightarrow \mathcal{H}'$

$$\tilde{\Lambda}(w^0, v^{10}, v^{30}, w^1, v^{11}, v^{31}) = ((mI - \alpha D^2)z_t(0), y_t^1(0), y_t^3(0), (mI - \alpha D^2)z(0), y^1(0), y^3(0)).$$

If  $(w^1, v^{11}, v^{31}, w^0, v^{10}, v^{30})$  are such that

$$\begin{aligned} & ((mI - \alpha D^2)z_t(0), y_t^1(0), y_t^3(0), (mI - D^2)z(0), y^1(0), y^3(0)) = \\ & ((mI - \alpha D^2)z^1, y^{11}, y^{31}, (mI - \alpha D^2)z^0, y^{10}, y^{30}), \end{aligned}$$

then the controls given by  $M(t) = w_{xx}(1, t), g_1(t) = v_x^1(1, t), g_3(t) = v_x^3(1, t)$  drive the system (3.108) - (3.114) to rest in time  $T$ . Thus, Theorem 15 will be proved if we show the surjectivity of the map  $\tilde{\Lambda} : \mathcal{H} \rightarrow \mathcal{H}'$ . For technical reasons it is more convenient to study the surjectivity of the map

$$\begin{aligned} & \Lambda(w^0, v^{10}, v^{30}, w^1, v^{11}, v^{31}) \\ & = ((mI - \alpha D^2)z_t(0), y_t^1(0), y_t^3(0), -(mI - \alpha D^2)z(0), -y^1(0), -y^3(0)). \end{aligned}$$

Clearly, the two maps  $\tilde{\Lambda}, \Lambda$  are surjective at the same time.

Once again, we have a stronger result:

**Lemma 5.** *Assume that constants  $m, \alpha, h_1\rho_1, h_3\rho_3, h_1E_1, h_3E_3, K$  are large enough so that  $\hat{F}_2$  is independent of  $G_2$ . Assume that*

$$\frac{\tilde{C}}{\hat{F}_2} > \left( \frac{G_2}{\hat{F}_1 h_2} + \frac{\alpha}{2\hat{F}_1} \right)$$

and  $T$  is large enough such that

$$T > \left( \frac{\frac{2\hat{D}}{\hat{F}_1}}{\frac{\tilde{C}}{\hat{F}_2} - \left( \frac{G_2}{\hat{F}_1 h_2} + \frac{\alpha}{2\hat{F}_1} \right)} \right).$$

Then  $\Lambda$  is an isomorphism of  $\mathcal{H}$  onto  $\mathcal{H}'$ .

**Proof:** Clearly  $\Lambda$  is a linear map. We calculate the following:

$$\langle \Lambda(w^0, v^{10}, v^{30}, w^1, v^{11}, v^{31}), (w^0, v^{10}, v^{30}, w^1, v^{11}, v^{31}) \rangle_{\mathcal{H}', \mathcal{H}}.$$

As in the proof of Lemma 3, we multiply Equation (3.98) by  $w$ , Equation (3.99) by  $v^1$ , Equation (3.100) by  $v^3$  and integrate by parts. We get exactly the same terms except for the following additional term due to moment of inertia:

$$-\alpha \int_0^T \int_0^1 z_{xxtt} w = -\alpha \int_0^1 (z_{xxt} w - z_{xx} w_t) \Big|_{t=0}^{t=T} - \alpha \int_0^T \int_0^1 z w_{xxtt}.$$

Adding all the terms from Lemma 3 with the term above, using the zero terminal state condition and using the controls given by  $M(t) = w_{xx}(1, t)$ ,  $g_1(t) = v_x^1(1, t)$ ,  $g_3(t) = v_x^3(1, t)$ , we get the same right hand side as in Equation (3.75).

$$\begin{aligned} & \langle \Lambda(w^0, v^{10}, v^{30}, w^1, v^{11}, v^{31}), (w^0, v^{10}, v^{30}, w^1, v^{11}, v^{31}) \rangle_{\mathcal{H}', \mathcal{H}} \\ &= \int_0^T K w_{xx}^2(1) + h_1 E_1(v_x^1(1, t))^2 + h_3 E_3(v_x^3(1, t))^2 dt. \end{aligned} \quad (3.126)$$

By Theorem 21, we have that if the parametric restrictions stated in Lemma 5 are satisfied, then

$$\int_0^T K w_{xx}^2(1) + h_1 E_1(v_x^1(1, t))^2 + h_3 E_3(v_x^3(1, t))^2 dt$$

defines a norm equivalent to the energy norm. The two inequalities show boundedness and coercitivity of  $\Lambda$  and hence  $\Lambda : \mathcal{H} \rightarrow \mathcal{H}'$  is an isometric isomorphism. This proves the Lemma.

**Proof of Theorem 22:** Lemma 5 proves the surjectivity of the map  $\Lambda$  and hence keeping Remark 23 in mind, we have the proof of Theorem 22.

**Remark 24.** Put in words, Theorem 15 and 22 state that for certain parametric restrictions and a sufficiently large control time  $T$ , the two problems presented in this chapter are exactly controllable in appropriate spaces.

**Remark 25.** We have illustrated in this chapter that observability of the homogenous problems (3.8)-(3.14),(3.83)-(3.89) implies controllability of the non-homogenous problems (3.30)-(3.36),(3.98)-(3.104). We have also seen that the controls that drive the non-homogenous problems (3.30)-(3.36),(3.98)-(3.104) from one state to another in appropriately defined spaces were explicitly constructed using the solution of the homogenous problems (3.8)-(3.14),(3.83)-(3.89). This is the essence of the HUM principle.

**CHAPTER 4. Exact controllability results for a Rao-Nakra sandwich  
beam using the moment method**

**4.1 Introduction**

In this chapter, some results from [14],[15],[16] are summarized. The following formulation of the Rao-Nakra model (adapted here for a beam model) is described in Hansen[6].

$$\begin{aligned} m\ddot{w} - \alpha D_x^2 \ddot{w} + K D_x^4 w - D_x N h_2 (G_2 \varphi + \tilde{G}_2 \dot{\varphi}) &= 0 \quad \text{on } (0, L) \times (0, \infty) \\ \mathbf{h}_O \mathbf{p}_O \ddot{\mathbf{v}}_O - \mathbf{h}_O \mathbf{E}_O D_x^2 \mathbf{v}_O + \mathbf{B}^T (G_2 \varphi + \tilde{G}_2 \dot{\varphi}) &= 0 \quad \text{on } (0, L) \times (0, \infty) \end{aligned} \quad (4.1)$$

where  $\varphi = h_2^{-1} \mathbf{B} v_O + N w_x$ . In addition we consider the following controlled boundary conditions:

$$\begin{aligned} w(0, t) = D_x^2 w(0, t) = D_x \mathbf{v}_O(0, t) = w(L, t) &= 0 \quad t > 0, \\ K D_x^2 w(L, t) = M(t), \quad \mathbf{h}_O \mathbf{E}_O D_x \mathbf{v}_O(L, t) &= \mathbf{g}_O(t) \quad t > 0 \end{aligned} \quad (4.2)$$

In the above,  $w$  denotes the transverse displacement of the beam,  $\varphi$  denotes the shear angle of the core layer,  $\mathbf{v}_O = (v_1, v_3)^T$  is the vector of longitudinal displacement along the neutral axis of the outer layers. ( $i = 1, 3$  is for the outer layers,  $i = 2$  is for the core layer.) The density of the  $i$ th layer is denoted  $\rho_i$ , the thickness  $h_i$ , the Young's modulus  $E_i$ , the shear modulus of the core layer is  $G_2$ . We let  $m = \sum h_i \rho_i$  denote the mass density per length,  $\alpha = \rho_1 h_1^3/12 + \rho_3 h_3^3/12$  denote a moment of inertia parameter,  $K = E_1 h_1^3/12 + E_3 h_3^3/12$  denote the bending stiffness. In addition,

$$\begin{aligned} \mathbf{p}_O &= \text{diag}(\rho_1, \rho_3), \quad \mathbf{h}_O = \text{diag}(h_1, h_3), \quad \mathbf{E}_O = \text{diag}(E_1, E_3) \\ \mathbf{B} &= (-1, 1), \quad N = \frac{h_1 + h_2 + h_3}{h_2}. \end{aligned}$$

The boundary control functions acting at the right end of the beam are  $M(t)$ , the applied moment, and  $\mathbf{g}_{\mathcal{O}}(t) = (g_1(t), g_3(t))^T$ , the longitudinal force. (See Hansen[6] for a precise definition of the applied forces.)

To keep things brief, we will assume that the wave speeds  $\sqrt{K/\alpha}$ ,  $\sqrt{E_1/\rho_1}$  and  $\sqrt{E_3/\rho_3}$  are distinct. Similar results for the case of identical wave speeds are discussed in [14],[15] and [16].

We summarize the results below.

## 4.2 Semigroup Formulation

Let  $(u, v)_{\Omega} = \int_0^L u \cdot \bar{v} dx$ , where  $u$  may be either scalar or vector valued. Define quadratic forms  $a$  and  $c$  by

$$\begin{aligned} c(w, \mathbf{v}_{\mathcal{O}}) &= (mw, w)_{\Omega} + \alpha(w_x, w_x)_{\Omega} + (\mathbf{h}_{\mathcal{O}} \mathbf{p}_{\mathcal{O}} \mathbf{v}_{\mathcal{O}}, \mathbf{v}_{\mathcal{O}})_{\Omega} \\ a(w, \mathbf{v}_{\mathcal{O}}) &= K(w_{xx}, w_{xx})_{\Omega} + (\mathbf{h}_{\mathcal{O}} \mathbf{E}_{\mathcal{O}} \mathbf{v}_{\mathcal{O}x}, \mathbf{v}_{\mathcal{O}x})_{\Omega} + (G_2 h_2 \varphi, \varphi)_{\Omega}. \end{aligned} \quad (4.3)$$

The energy of the beam is given by

$$\mathcal{E}(t) = \frac{R}{2} (c(\dot{w}, \dot{\mathbf{v}}_{\mathcal{O}}) + a(w, \mathbf{v}_{\mathcal{O}}))$$

where  $R$  is the width of the beam. The homogenous problem can be written as follows:

$$\frac{dY}{dt} = \mathcal{A}Y := \begin{pmatrix} 0 & I \\ A_1 & A_2 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}, \quad (4.4)$$

$$\text{where } A_1 U = \begin{pmatrix} J^{-1}(-KD_x^4 u + D_x N h_2 G_2 [h_2^{-1}(\mathbf{B}u + h_2 N D_x u)]) \\ \mathbf{h}_{\mathcal{O}}^{-1} \mathbf{p}_{\mathcal{O}}^{-1} [\mathbf{h}_{\mathcal{O}} \mathbf{E}_{\mathcal{O}} D_x^2 \mathbf{u} - \mathbf{B}^T G_2 [h_2^{-1}(\mathbf{B}u + h_2 N D_x u)]] \end{pmatrix} \quad (4.5)$$

$$A_2 V = \begin{pmatrix} J^{-1}(D_x N h_2 \tilde{G}_2 [h_2^{-1}(\mathbf{B}v + h_2 N D_x v)]) \\ \mathbf{h}_{\mathcal{O}}^{-1} \mathbf{p}_{\mathcal{O}}^{-1} [-\mathbf{B}^T \tilde{G}_2 [h_2^{-1}(\mathbf{B}v + h_2 N D_x v)]] \end{pmatrix}. \quad (4.6)$$

The energy inner product is defined by

$$\langle Y, \hat{Y} \rangle_e = a(U; \hat{U}) + c(V; \hat{V}), \quad (\hat{Y} = (\hat{U}, \hat{V})),$$

where

$$a(\cdot; \cdot), c(\cdot; \cdot)$$

are the bilinear forms that coincide with the previously defined quadratic forms  $a(\cdot)$ ,  $c(\cdot)$  on the diagonal. Let

$$\begin{aligned} X_1 &= \{u, \mathbf{u}\} \in H^2(0, L) \cap H_0^1(0, L) \times (H^1(0, L))^2 \\ X_0 &= \{u, \mathbf{u}\} \in H_0^1(0, L) \times (L^2(0, L))^2. \end{aligned}$$

It can be shown [6] that the equations of motion are well-posed on the energy space  $(U, \dot{U}) \in C([0, T]; X_1 \times X_0)$ . It is not hard to prove the same for semigroup solutions. The domain of this semigroup is  $\mathcal{D}(\mathcal{A}) = X_2 \times X_1$ , where

$$X_2 = \{(u, \mathbf{u}) \in X_1 : u \in H^3(0, L), \mathbf{u} \in (H^2(0, L))^2 + BC's\}$$

where “+BC’s” means  $D_x^2 u$  and  $D_x \mathbf{u}$  vanish at each end. The non homogenous problem can be written as follows:

$$\frac{d}{dt} \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 0 & I \\ A_1 & A_2 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} + \begin{pmatrix} 0 \\ \mathcal{B}\{M, \mathbf{g}_\mathcal{O}\} \end{pmatrix}; \quad (4.7)$$

where

$$\mathcal{B}\{M, \mathbf{g}_\mathcal{O}\} = \begin{pmatrix} J^{-1}M(t)\delta'_L(x) \\ \mathbf{h}_\mathcal{O}^{-1}\mathbf{p}_\mathcal{O}^{-1}\mathbf{g}_\mathcal{O}\delta_L(x) \end{pmatrix}. \quad (4.8)$$

$$Y = (U, V)^T \in C([0, T]; X_1 \times X_0). \quad (4.9)$$

In order to define solutions of (4.7) one first extends the semigroup  $e^{\mathcal{A}t}$  to a weaker space defined by duality. However for the inputs defined in (4.7)-(4.8), using the Carleson’s measure criterion it can be shown (see [15]) that (4.7) is well posed on  $X_1 \times X_0$ .

### 4.3 Spectral Analysis

**Proposition 1.** *Assume the wave speeds are distinct. The spectrum of  $\mathcal{A}$  consists of the eigenvalues*

$$\sigma(\mathcal{A}) = \bigcup_{k=0}^{\infty} S_k,$$

where  $S_0$  consists of the double eigenvalue 0 and the roots of (4.10).

$$\lambda^2 + \lambda R\tilde{G}_2/h_2 + RG_2/h_2 = 0, \quad R = \mathbf{B}\mathbf{h}_\mathcal{O}^{-1}\mathbf{p}_\mathcal{O}^{-1}\mathbf{B}^T > 0, \quad (4.10)$$

Eigenvectors and generalized eigenvectors associated with  $S_0$  are given in (4.12)-(4.13) and (4.11).

$$U = (0, h_3 \rho_3, -h_1 \rho_1)^T, V = \lambda U. \quad (4.11)$$

$$\mathbf{u} = \vec{\mathbf{1}}_{\mathcal{O}}, \quad u = 0, \quad V = 0. \quad (4.12)$$

$$U = 0, \quad v = 0, \quad \mathbf{v} = \vec{\mathbf{1}}_{\mathcal{O}}. \quad (4.13)$$

For  $k = 1, 2, 3, \dots$  and  $j = 0, 1, 3$  we have

$$S_k = \{\lambda_{k,0}^+ := s_k^+, \lambda_{k,0}^- := s_k^-, \lambda_{k,1}^+, \lambda_{k,1}^-, \lambda_{k,3}^+, \lambda_{k,3}^-\},$$

where for  $k$  sufficiently large,  $\lambda_{k,j}^+, \lambda_{k,j}^-$  are complex conjugate roots which are given below:

$$s_k^\pm = -\frac{\tilde{b}}{2\alpha} \pm i\sigma_k \sqrt{\frac{K}{\alpha}} + \mathcal{O}(\sigma_k^{-1}). \quad (4.14)$$

$$\lambda_{k,j}^\pm = i\sigma_k \sqrt{\frac{E_j}{\rho_j}} - \frac{\tilde{\gamma}}{2h_j \rho_j} + \mathcal{O}(k^{-1}), j = 1, 3. \quad (4.15)$$

For  $\lambda = \lambda_{k,j}^\pm \neq 0$ , the corresponding eigenvectors (and possibly generalized eigenvectors)  $Y_\lambda$  exist and are of the form

$$Y_\lambda = \begin{pmatrix} \frac{U_\lambda}{\lambda} \\ U_\lambda \end{pmatrix}; U_\lambda = \begin{pmatrix} u_{k,j} \\ \mathbf{u}_{k,j} \end{pmatrix}; \begin{pmatrix} u_{k,j} \\ \mathbf{u}_{k,j} \end{pmatrix} = \begin{pmatrix} A_{k,j} \sin(\sigma_k x) \\ \vec{B}_{k,j} \cos(\sigma_k x) \end{pmatrix},$$

where  $k$  sufficiently large the  $Y_{\lambda_{k,j}^\pm}$  are eigenfunctions given by

$$\begin{pmatrix} u_{k,0} \\ \mathbf{u}_{k,0} \end{pmatrix} = \frac{1}{\sigma_k} \begin{pmatrix} \sin(\sigma_k x) \\ -[(\mathbf{h}_0 \mathbf{p}_0 + \mathbf{h}_0 \mathbf{E}_0 y_k^2)^{-1} \mathbf{B}^T \tilde{G}_2 y_k + \mathcal{O}(k^{-1})] \cos(\sigma_k x) \end{pmatrix},$$

where  $y_k = \frac{\sigma_k}{s_k^\pm}$ ,

$$\begin{pmatrix} u_{k,1} \\ \mathbf{u}_{k,1} \end{pmatrix} = \begin{pmatrix} \mathcal{O}(k^{-2}) \sin(\sigma_k x) \\ \cos(\sigma_k x) \\ \mathcal{O}(k^{-1}) \cos(\sigma_k x) \end{pmatrix}, \quad \begin{pmatrix} u_{k,3} \\ \mathbf{u}_{k,3} \end{pmatrix} = \begin{pmatrix} \mathcal{O}(k^{-2}) \sin(\sigma_k x) \\ \mathcal{O}(k^{-1}) \cos(\sigma_k x) \\ \cos(\sigma_k x) \end{pmatrix}.$$

**Theorem 23.** *The eigenvectors of  $\mathcal{A}$  form a Riesz basis for the finite energy space  $X_1 \times X_0$ .*

The Riesz basis property Theorem 23 is proved using Bari's theorem [32], applied to the eigenvector estimates of Proposition 1. Next, we consider the problem of controlling an initial finite energy state to another in time  $T$  with controls  $\mathbf{g}_{\mathcal{O}}$ ,  $M(t)$  belonging to  $L^2(0, T)$ . We prove that if  $T > \tau$  where

$$\tau = 2L \left[ \min \left( \sqrt{\frac{K}{\alpha}}, \sqrt{\frac{E_1}{\rho_1}}, \sqrt{\frac{E_3}{\rho_3}} \right) \right]^{-1}, \quad (4.16)$$

then the system (4.1)–(4.2) is exactly controllable modulo a finite dimensional quotient. If the coupling terms  $G_2$  and  $\tilde{G}_2$  are sufficiently small, this quotient consists of the the space determined by “zero energy” uncontrollable state  $w = 0$ ,  $(v_1, v_3) = (1, 1)$ .

#### 4.4 The moment problem and its solution

After suitably defining new controls  $\tilde{f}_1(t)$ ,  $\tilde{f}_2(t)$  and  $\tilde{f}_3(t)$ , we have the following moment problem: For  $k \in \mathbb{N}$

$$c_{0,k}^{\pm} = \int_0^T e^{\lambda_{0,k}t} (\tilde{f}_0(t) + \mathcal{O}(k^{-1})\tilde{f}_1(t) + \mathcal{O}(k^{-1})\tilde{f}_3(t)) dt \quad (4.17)$$

$$c_{1,k}^{\pm} = \int_0^T e^{\lambda_{1,k}t} (\mathcal{O}(k^{-1})\tilde{f}_0(t) + \tilde{f}_1(t) + \mathcal{O}(k^{-1})\tilde{f}_3(t)) dt \quad (4.18)$$

$$c_{3,k}^{\pm} = \int_0^T e^{\lambda_{3,k}t} (\mathcal{O}(k^{-1})\tilde{f}_0(t) + \mathcal{O}(k^{-1})\tilde{f}_1(t) + \tilde{f}_3(t)) dt. \quad (4.19)$$

The finite system corresponding to the nullvector (4.12), the generalized nullvector (4.13), and the eigenvectors described in (4.11) are

$$c_{0,0} = \int_0^T 0 dt \quad (4.20)$$

$$c_{0,1} = \int_0^T \tilde{g}_1 + \tilde{g}_3 dt \quad c_{0,3}^{\pm} = \int_0^T e^{\lambda_{0,3}^{\pm}t} (h_3\rho_3\tilde{g}_1 - h_1\rho_3\tilde{g}_3) dt \quad (4.21)$$

**Remark 26.** As is easy to see from (4.20), it is not possible to steer a solution of (4.1) from the origin to the state corresponding to the null vector solution  $w = 0$ ,  $\mathbf{v}_{\mathcal{O}} = (1, 1)^T$ .

We state the result which guarantees a solution of the moment problem in terms of the original controls  $M(t)$ ,  $g_1(t)$  and  $g_3(t)$ .

**Theorem 24.** *Assume the wave speeds are distinct. Given any  $\{c_\lambda\} \in \ell^2$  there exists functions  $M(t)$  and  $\mathbf{g}_O(t)$  in  $L^2(0, T)$  which solve the three moment problems (4.17)–(4.19), for all  $k > K$ , where  $K$  is sufficiently large in any time  $T > \tau$ , where the control time  $\tau$  is given in (4.16).*

**Remark 27.** Keeping Remark 26 in mind, one can ask whether it is possible to solve (4.17)–(4.19) for all  $k$  together with (4.21). If possible, then exact controllability of (4.1)–(4.2) holds modulo the one dimensional uncontrollable quotient described in Remark 26. A sufficient condition for this result is that the eigenvalues grow (are not repeated) along each branch. It is not hard to show that this will hold if the coupling between the equations is sufficiently small and  $\left\{ \frac{K\sigma_k^4}{m + \alpha\sigma_k^2} \right\}_{k=1}^\infty, \sigma = \frac{k\pi}{L}$  is a sequence of distinct numbers.

#### 4.5 Controllability results

Let  $\mathcal{P}_\infty$  denote the spectral projection operator defined on  $X_1 \times X_0$  by

$$\mathcal{P}_\infty \left( \sum_{k=1}^\infty \sum_{\lambda \in S_k} c_\lambda Y_\lambda \right) = \sum_{k \geq K} \sum_{\lambda \in S_k} c_\lambda Y_\lambda,$$

where  $K$  is the integer defined in Theorem 24.

**Theorem 25.** *Given any initial data  $Y_0 \in X_1 \times X_0$  and  $T > \tau$  ( $\tau$  as defined in (4.16)), there exists  $\{M, \tilde{\mathbf{g}}_O\} \in (L^2(0, T))^3$  such that the solution  $Y(t)$  of (4.7) satisfies (4.9) and  $\mathcal{P}_\infty Y(t) = 0, \forall t \geq T$ .*

In view of Remark 27, we also have the following corollary.

**Corollary 1.** *If  $G_2$  and  $\tilde{G}_2$  are sufficiently small and  $\left\{ \frac{K\sigma_k^4}{m + \alpha\sigma_k^2} \right\}_{k=1}^\infty, \sigma_k = \frac{k\pi}{L}$  is a sequence of distinct numbers, then for  $T > \tau$  Equation (4.1) is exactly controllable in the quotient space  $(X_1 \times X_0)/(0, 1, 1)^T$ .*

**Remark 28.** Theorems 24 and 25 remain valid for the case of identical wave speeds [15], however the analysis is somewhat different.

#### 4.6 Simultaneous boundary control

We give sufficient conditions for a similar result as in Theorem 25 to hold with a reduced number of boundary controls. In particular, we consider two physically motivated choices of boundary controls. In the case that the top and bottom of the beam at the endpoint  $x = L$  are subject to surface tractions  $\hat{g}_1(t)$ , and  $\hat{g}_3(t)$ , the controls in (4.2) take the form (see [6]),

$$\mathbf{g}_{\mathcal{O}} = \left( \frac{\hat{g}_1(t)}{E_1 h_1}, \frac{\hat{g}_3(t)}{E_3 h_3} \right)^T, M(t) = \left( \frac{h_1}{2K} \hat{g}_1(t) - \frac{h_3}{2K} \hat{g}_3(t) \right). \quad (4.22)$$

In the case that the surface tractions are applied in equal and opposite amounts, the controls take the form

$$\mathbf{g}_{\mathcal{O}} = \left( \frac{u(t)}{h_1 E_1}, -\frac{u(t)}{h_3 E_3} \right)^T, M(t) = \frac{h_1 + h_3}{2K} u(t). \quad (4.23)$$

#### 4.7 The moment problem and controllability results

Using controls of the form (4.23), the moment problem (4.17)-(4.19) can be rewritten using  $\tilde{u} = u(T - t)$  as follows:

$$c_{k,0}^{\pm} = \int_0^T e^{\lambda_{0,k}^{\pm} t} A_k \tilde{u}(t) ds \quad (4.24)$$

$$c_{k,1}^{\pm} = \int_0^T e^{\lambda_{1,k}^{\pm} t} B_k \tilde{u}(t) ds \quad (4.25)$$

$$c_{k,3}^{\pm} = \int_0^T e^{\lambda_{3,k}^{\pm} t} C_k \tilde{u}(t) ds, \quad (4.26)$$

where

$$A_k = \frac{h_1 + h_3}{2K} + \frac{\varepsilon_{12}^k}{h_1 E_1} - \frac{\varepsilon_{13}^k}{h_3 E_3}, \quad (4.27)$$

$$B_k = \frac{(h_1 + h_3) \varepsilon_{21}^k}{2K} + \frac{1}{h_1 E_1} - \frac{\varepsilon_{23}^k}{h_3 E_3}, \quad (4.28)$$

$$C_k = \frac{(h_1 + h_3) \varepsilon_{31}^k}{2K} + \frac{\varepsilon_{32}^k}{h_1 E_1} - \frac{1}{h_3 E_3}. \quad (4.29)$$

The constants  $A_k$ ,  $B_k$  and  $C_k$  defined above are bounded and bounded away from zero if  $k \geq K$  where  $K$  is sufficiently large. Hence dividing, (4.24)-(4.26) by  $A_k$ ,  $B_k$ ,  $C_k$  respectively, we get

$$d_{k,j}^{\pm} = \int_0^T e^{\lambda_{k,j}^{\pm} t} \tilde{u}(t) dt, \quad (4.30)$$

where  $\{d_{k,j}^\pm\} \in l^2$ . Under the assumptions of the theorem, for  $k \geq K$  sufficiently large, there exists a  $\delta > 0$  such that

$$|\lambda_{k_0,j_0}^{m_0} - \lambda_{k_1,j_1}^{m_1}| \geq \delta \text{ for } (m_0, k_0, j_0) \neq (m_1, k_1, j_1). \quad (4.31)$$

**Theorem 26.** *Assume the wave speeds  $\sqrt{\frac{E_j}{\rho_j}}, j = 1, 3$  are distinct and the numbers*

$$\left\{ \rho_1 h_1, \rho_3 h_3, \frac{\alpha}{(h_1 + h_2 + h_3)} \right\}$$

*are distinct. Then the eigenvalues  $\lambda_{k,j}^\pm$  have the following asymptotic form:*

$$\lambda_{k,j}^+ = -a_j + i\mu_j \sigma_k + \mathcal{O}(k^{-1}), \quad (4.32)$$

*as  $k \rightarrow \infty$ , where  $\mu_0 = \sqrt{\frac{K}{\alpha}}, \mu_j = \sqrt{\frac{E_j}{\rho_j}}, j = 1, 3$ , and  $a_0, a_1, a_3$  are distinct non-negative numbers. Furthermore, there exists a control  $u(t)$  of the form (4.23) that solves all but finitely many of the equations in (4.17)-(4.19) with  $T > \tau$  where*

$$\tau = (2L(\frac{1}{\mu_0} + \frac{1}{\mu_1} + \frac{1}{\mu_3})).$$

The proof of Theorem 26 relies on the following proposition.

**Proposition 2.** *Assume that  $\mu_j > 0, j = 1, \dots, n$ ,  $0, a_1 < a_2 < \dots < a_n$ ,  $\lambda_{k,j}^\pm = -a_j \pm i\mu_j k + z_{k,j}^\pm, j = 1, \dots, n, k \in \mathbb{N}$ ,  $z_{k,j}^\pm \in l^2$ , and  $\{\lambda_{k,j}^\pm\}$  are pairwise distinct. Let  $T > \sum_{j=1}^n \frac{2\pi}{\mu_j}$ . Then  $\{e^{\lambda_{k,j}^\pm t}\}$  forms a Riesz basis for its closed span in  $L^2(0, T)$ .*

The proof of Proposition 2 relies on some ideas from [7] along with some standard perturbation techniques from the theory of non-harmonic Fourier series [32] and will appear in a future publication.

We have the following controllability result:

**Theorem 27.** *For the case of distinct wave speeds, assume the hypothesis of Theorem 26 holds. Then given any initial data  $Y_0 \in X_1 \times X_0$  and  $T > \tau$  ( $\tau$  as defined in Theorem 26), there exists  $u \in L^2(0, T)$  such that the solution of (4.1), (4.1), (4.2) (as defined by  $Y(t)$  of (4.7)) satisfies  $Y(t) \in C([0, T]; X_1 \times X_0)$  and  $\mathcal{P}_\infty Y(t) = 0, \forall t \geq T$ .*

**Remark 29.** In the case that the wave speeds  $\sqrt{E_1/\rho_1}$ ,  $\sqrt{E_1/\rho_1}$ ,  $\sqrt{K/\alpha}$  are equal, the minimum gap condition fails since two branches of eigenvalues are asymptotically the same. Hence a single control cannot be used to solve the moment problem. However, by choosing two controls as in (4.22), all but finitely many moments can be solved. Hence a result similar to Theorem 27 is true for the case of equal wave speeds. For a detailed explanation see [16].

## CHAPTER 5. Conclusion

We have seen the application of two methods in this thesis, namely the moment method in Chapter 2 and the multiplier method in Chapter 3 to control of beams. In this chapter, we attempt to draw a parallel between the two methods and also outline future research topics.

In both cases, we were attempting to solve a control problem involving a differential equation as follows:

$$\dot{x} = \mathcal{A}x + bu(t) \tag{5.1}$$

$$x(0) = x_0. \tag{5.2}$$

### 5.1 Solution of the differential equation

#### 5.1.1 The homogenous problem (the common step)

The first step in both methods is same, namely finding an appropriate Hilbert space  $\mathcal{H}$  in which the homogenous problem (Equation (5.1) with  $u(t) = 0$ ) is well-posed. The techniques involve the usage of semigroup theory ([2],[27]), the usage of Lumer-Phillips theorem in particular to show that the operator  $\mathcal{A}$  generates a semigroup on  $\mathcal{H}$ .

#### 5.1.2 The non-homogenous problem

After showing the semigroup property of the operator  $\mathcal{A}$ , we try to find an appropriate space in which the non-homogenous problem is well-posed. It turns out that due to the boundary action, the regularity of the space in which the non-homogenous problem is well-posed, is weaker than the original space  $\mathcal{H}$ . We also have that (Remark 16), Chapter 3) there is an infinite array  $\{\mathcal{X}_k\}_{k \in \mathbb{Z}}$  of spaces in which the extensions or restrictions of  $\mathcal{A}$  generate

a semigroup. In the moment method, we project these spaces onto the space of sequences using Hilbert space isomorphisms and endeavor to choose the optimal sequence space from the infinite array  $\{\mathcal{X}_k\}_{k \in \mathbb{Z}}$  of spaces. The projections onto the sequence space is justified due to the Riesz basis property of the eigenfunctions  $\{\phi_k\}$  of the operator  $\mathcal{A}$ . The main result used here concerning regularity is the Carleson's measure criterion [31]. In the multiplier method, to prove regularity we choose an appropriate multiplier  $m(x)$  and multiply the underlying PDE and integrate by parts to obtain a certain "hidden regularity" result. This result along with the application of the Riesz representation theorem gives us the space in which the non-homogenous problem is well-posed. In both cases, the solution of the non-homogenous problem belongs to a weaker (or rougher) space than the space  $\mathcal{H}$ .

## 5.2 Solution of the control problem

### 5.2.1 The moment method

In the moment method, we restate the problem of control as a moment problem i.e the problem of determining a control  $u(t)$  given the non-harmonic Fourier coefficients  $c_k$ . Explicitly

$$c_k = \int_0^T u(\tau) e^{\lambda_k \tau} d\tau, \quad (5.3)$$

where  $\{c_k\}$  is a known sequence related to the initial and terminal data,  $\{\lambda_k\}$  is a sequence of eigenvalues of the operator  $\mathcal{A}$  and  $T, u(t)$  are the unknown control time and control input to be determined. The solution of the moment problem heavily relies on spectral analysis of the operator  $\mathcal{A}$  and estimates on the growth and separation of the eigenvalues and the norm of the eigenfunctions. The key result to prove is that the sequence  $\{e^{\lambda_k t}\}$  forms a Riesz basis of  $L^2(0, T)$  for sufficiently large  $T$ . This guarantees the existence of functions  $\{q_k(t)\}$  that are biorthogonal to  $\{e^{\lambda_k t}\}$  and the solution to the moment problem (5.3) is given by  $u(t) = \sum_k c_k q_k(t)$ . The techniques involved in the proofs are adapted from the theory of non-harmonic Fourier series (see [32]).

### 5.2.2 The multiplier method (The HUM principle)

The key ingredient in solving the control problem in the multiplier method is to prove an observability estimate. This estimate allows us to show that a certain map  $\Lambda$  from the space of initial conditions of the homogenous problem  $\mathcal{X}$  to the space of initial conditions of the non-homogenous problem  $\mathcal{X}'$  is an isometric isomorphism, thereby implying the exact controllability of the non-homogenous problem in the space  $\mathcal{X}'$ , the dual of  $\mathcal{X}$ . The regularity and the observability estimates imply the boundedness and coercitivity of the bilinear form generated by  $\Lambda$  which allows us to use the Lax-Milgram lemma to our advantage. This technique has been referred to in the literature as the HUM (Hilbert's Uniqueness Method) principle. In [19] it has been shown that the observability estimate is equivalent to showing that the moment operator  $\mathcal{M} : l^2 \rightarrow L^2(0, T)$  given by  $\{c_k\} \rightarrow \sum_k c_k e^{\lambda_k t}$  is onto. This guarantees the existence of a control  $u(t) \in L^2(0, T)$  which solves the moment problem. Hence there is no harm in saying that the multiplier method is an operator theoretic approach towards solving the moment problem ( 5.3).

**Remark 30.** In conclusion, we remark that the moment method is applicable to situations where the spectral analysis of the operator  $\mathcal{A}$  can be done easily, whereas the multiplier method is applicable to situations where the spectral analysis is difficult.

### 5.3 Future research

1. In this thesis we have restricted our interests to one-dimensional models of three layer sandwich beam. A next step would be to consider the case of plates i.e two dimensional theory.
2. The multiplier method has been used to prove an exact controllability result for the conservative Rao-Nakra beam. It remains to see how this technique can be adapted to the damped situation.
3. The multiplier and moment methods are equivalent to each other. Each multiplier problem yields a moment problem and vice versa. The usage of multiplier method always

has in its background a moment problem which arises out of projecting the observability estimate onto an appropriate Hilbert space of sequences. It remains to show that such moment problems can actually be solved directly using techniques from non-harmonic Fourier series. An example of such a problem is boundary control of the wave equation on a unit square (see Appendix A). Likewise, there are boundary conditions that work well for the moment method that do not seem to work so well using the multiplier method. (see Appendix C for a detailed discussion of one such case.)

## APPENDIX A. Boundary control of the wave equation on a unit square

### Introduction

In this section we formally derive the moment problem that arises out of controlling the wave equation on the boundary of a unit square. Let us assume that the control acts on the bottom and left corners of the unit square. Let us denote  $\Omega = [0, 1] \times [0, 1]$ . The equations of motion are as follows:

$$w_{tt} - \Delta w = 0, \quad (x, y, t) \in \Omega \times [0, T] \quad (\text{A.1})$$

The homogenous boundary conditions are as follows:

$$w(1, y, t) = 0, w(x, 1, t) = 0. \quad (\text{A.2})$$

The controlled boundary conditions are as follows:

$$w(x, 0, t) = u(x, t), w(0, y, t) = v(y, t). \quad (\text{A.3})$$

The initial conditions are as follows:

$$w(x, y, 0) = w^0(x, y); w_t(x, y, 0) = w^1(x, y) \text{ on } \Omega. \quad (\text{A.4})$$

We consider the homogenous problem at first. We make the following variable substitution and rewrite Equation (A.1) as follows:

$$m = \begin{pmatrix} w \\ w_t \end{pmatrix} \quad (\text{A.5})$$

$$\begin{aligned} \dot{m} &= \mathcal{A}m & (\text{A.6}) \\ m(0) &= \begin{pmatrix} w^0 \\ w^1 \end{pmatrix} \text{ on } \Omega, \end{aligned}$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}, \quad (\text{A.7})$$

$$\mathcal{D}(\mathcal{A}) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega). \quad (\text{A.8})$$

Also  $\mathcal{A} : \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{H}$ , where  $\mathcal{H}$  is the natural energy space with the following energy inner product:

$$\left\langle \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} h \\ l \end{pmatrix} \right\rangle_e = \int_{\Omega} f' \bar{h}' + g \bar{l}. \quad (\text{A.9})$$

### Eigenvalues and Eigenvectors of $\mathcal{A}$

Let  $e_k(x) = \sin(k\pi x)$  and  $e_j(y) = \sin(j\pi y)$ . The eigenvalues and eigenvectors of  $\mathcal{A}$  are easily seen to be sinusoidal and are given by the following:

$$\lambda_{k,j}^{\pm} = \pm i\pi \sqrt{k^2 + j^2}, \quad \phi_{k,j}^{\pm}(x, y) = \frac{1}{2} \begin{pmatrix} \frac{e_k(x)e_j(y)}{\lambda_{k,j}^{\pm}} \\ e_k(x)e_j(y) \end{pmatrix}, \quad \forall k, j \in \mathbb{N}. \quad (\text{A.10})$$

It can also be easily seen that the eigenvectors  $\{\phi_{k,j}^{\pm}\}_{k,j \in \mathbb{N}}$  in Equation (A.10) form an orthonormal basis for the energy space  $\mathcal{H}$ . By using standard PDE techniques, it can be shown that the homogenous problem (i.e with  $u(x, t) = v(y, t) = 0$ ) is well posed in  $H_0^1(\Omega) \times L^2(\Omega)$ . We multiply Equation (A.1) by a test function satisfying homogenous boundary conditions and use integration by parts to transfer the controls  $u(x, t)$  and  $v(y, t)$  to the right hand side. We obtain the following:

$$\dot{z} = \mathcal{A}z + \begin{pmatrix} 0 \\ -(\delta'(y)u(x, t) + \delta'(x)v(y, t)) \end{pmatrix} \quad (\text{A.11})$$

$$z(0) = \begin{pmatrix} w^0 \\ w^1 \end{pmatrix} \text{ on } \Omega, \quad (\text{A.12})$$

By using Carleson's measure criterion as in Chapter 2 or the method of transposition as in Chapter 3, we can show that the controlled problem (A.11)-(A.12) is well posed in  $L^2(\Omega) \times H^{-1}(\Omega)$ . We seek solutions of the following form:

$$v(x, y, t) = \sum_{k,j \in \mathbb{N}} z_{k,j}^{\pm}(t) \phi_k^{\pm}(x, y). \quad (\text{A.13})$$

We also seek to write :

$$\begin{pmatrix} w^0 \\ w^1 \end{pmatrix} = \sum_{k,j \in \mathbb{N}} z_{k,j}^{\pm}(0) \phi_{k,j}^{\pm}(x, y) \quad (\text{A.14})$$

$$\begin{pmatrix} 0 \\ -(\delta'(y)u(x, t) + \delta'(x)v(y, t)) \end{pmatrix} = \sum_{k,j \in \mathbb{N}} f_{k,j}^{\pm} \phi_{k,j}^{\pm}(x, y), \quad (\text{A.15})$$

where

$$\begin{aligned} f_{k,j}^{\pm} &= \frac{1}{2} (k\pi \int_0^1 u(x, t) \sin(k\pi x) dx + j\pi \int_0^1 v(y, t) \sin(j\pi y) dy) \\ &= \frac{1}{2} (k\pi u_k(t) + j\pi v_j(t)). \end{aligned} \quad (\text{A.16})$$

### The control problem

We consider the problem of reaching the all states in  $L^2(\Omega) \times H^{-1}(\Omega)$  from the origin in time  $T$  using controls  $u(x, t)$  and  $v(y, t)$ . By Theorem 2, Chapter 1, this implies exact controllability in  $L^2(\Omega) \times H^{-1}(\Omega)$  as we have a conservative system. Substituting (A.13) in (A.11) and using (A.14),(A.15), we follow the same steps as in Chapter 2 to obtain the following moment problem:

$$d_{k,j}^{\pm} = \frac{1}{2} \int_0^T e^{\lambda_{k,j}^{\pm} t} (k\pi u_k(t) + j\pi v_j(t)) dt, \quad (\text{A.17})$$

where

$$d_{k,j} = w_{k,j}(T) \in l^2(\mathbb{N}), \lambda_{k,j}^{\pm} = \pm i\pi \sqrt{k^2 + j^2}. \quad (\text{A.18})$$

$$u_k(t) = \int_0^1 u(x, t) \sin(k\pi x) dx, \quad v_j(t) = \int_0^1 v(y, t) \sin(j\pi y) dy$$

In the above problem the unknowns are  $T$  and  $u_k(t), v_j(t)$ .

On the other hand using the multiplier method as in Chapter 3, we can prove the following observability and regularity estimate:

$$C_1 E(0) \leq \int_0^T \int_0^1 |w_x(0, y, t)|^2 dy dt + |w_y(x, 0, t)|^2 dx dt \leq C_2 E(0), \quad \forall T > 2\sqrt{2}, \quad (\text{A.19})$$

where  $E(0) = \int_{\Omega} |w^0(x, y)|^2 + |\nabla w^1(x, y)|^2 dx dy$  is the initial energy which is conserved for all time and  $C_1, C_2 > 0$  are constants that depend only on the initial data and  $T$ . Hence we can use the HUM principle to conclude that the wave equation on the unit square is exactly controllable in the space  $L^2(\Omega) \times H^{-1}(\Omega)$  provided the control time satisfies  $T > 2\sqrt{2}$ .

### Open problem

Using the series representation as in Equations (A.13) and (A.14), Equation A.19) can be rewritten as follows:

$$\begin{aligned} C_1 \sum_{k,j \in \mathbb{N}} |z_{k,j}^{\pm}(0)|^2 &\leq \int_0^T \int_0^1 \left| \sum_{k,j \in \mathbb{N}} \frac{k}{\pm\sqrt{k^2+j^2}} \sin(jy) e^{(\pm i\pi\sqrt{k^2+j^2}t)} \right|^2 dy dt \\ &+ \int_0^T \int_0^1 \left| \sum_{k,j \in \mathbb{N}} \frac{j}{\pm\sqrt{k^2+j^2}} \sin(kx) e^{(\pm i\pi\sqrt{k^2+j^2}t)} \right|^2 dx dt \leq C_2 \sum_{k,j \in \mathbb{N}} |z_{k,j}^{\pm}(0)|^2. \end{aligned} \quad (\text{A.20})$$

An inequality of the form (A.20) in one dimension is called Ingham's inequality, the proof of which uses the theory of non-harmonic Fourier series and is very clever (see [12]). This is an indication that an independent proof of (A.20) using non-harmonic Fourier series might require a clever trick in two dimensions. Furthermore Equation (A.20) implies that the moment operator  $\mathcal{M}$  as described in Chapter 5, Section 5.2.2, is onto and hence the moment problem (A.17) is solvable. However, it remains to be seen whether (A.17) can be solved for  $u_k(t)$  and  $v_j(t)$  provided  $T > 2\sqrt{2}$  directly using methods of non-harmonic Fourier series. The solution also makes physical sense since the supremum of the lengths of geometric optics that do not touch a controlled surface is  $2\sqrt{2}$ .

## APPENDIX B. An unusual observability estimate

### Introduction

In this appendix, we consider the same problem as in Chapter 2 and try to use the multiplier method instead of the moment method as in [14],[15] and [16].

$$mw_{tt} - \alpha w_{xxtt} + Kw_{xxxx} - N^2 h_2 G_2 w_{xx} - NG_2(-v_x^1 + v_x^3) = 0 \quad (\text{B.1})$$

$$h_1 \rho_1 v_{tt}^1 - h_1 E_1 v_{xx}^1 - \frac{G_2}{h_2}(-v^1 + v^3) - NG_2 w_x = 0 \quad (\text{B.2})$$

$$h_3 \rho_3 v_{tt}^3 - h_3 E_3 v_{xx}^3 + \frac{G_2}{h_2}(-v^1 + v^3) + NG_2 w_x = 0. \quad (\text{B.3})$$

The boundary conditions are as follows:

$$w(0, t) = w(1, t) = w_{xx}(0, t) = v_x^1(0, t) = v_x^3(0, t) = 0 \quad (\text{B.4})$$

$$w_{xx}(1, t) = M(t), v_x^1(1, t) = g_1(t), v_x^3(1, t) = g_3(t), \quad (\text{B.5})$$

where  $M(t)$ ,  $g_1(t)$  and  $g_3(t)$  are control inputs at the right end. The initial conditions are as follows:

$$w(x, 0) = w^0(x), w_t(x, 0) = w^1(x) \quad (\text{B.6})$$

$$v^1(x, 0) = v^{10}(x), v_t^1(x, 0) = v^{11}(x), v^3(x, 0) = v^{30}(x), v_t^3(x, 0) = v^{31}(x). \quad (\text{B.7})$$

Using semigroup theory, we can show that  $\mathcal{A}$  generates a semigroup in the natural energy space  $\mathcal{H} = H_w^2 \times H_0^1(0, 1) \times H_0^1(0, 1) \times L^2(0, 1) \times L^2(0, 1) \times L^2(0, 1)$ , where  $H_w^2 = \{\phi \in H^2(0, 1) : \phi(0) = \phi(1) = 0\}$ . We omit the well-posedness issue here and focus on obtaining the regularity and observability estimates. As usual, we consider the homogenous problem first i.e (B.1)-(B.7) with  $M(t) = g_1(t) = g_3(t) = 0$ . We omit writing  $dx$  and  $dt$  as it is understood from the

limits of integration. We first multiply Equation (B.1) by  $xw_x$  and integrate by parts. We get the following:

$$\begin{aligned}
& \int_0^T \int_0^1 mw_{tt}xw_x = m \int_0^1 w_t x w_x |_{t=0}^{t=T} + \frac{1}{2} \int_0^T \int_0^1 w_t^2. \\
-N^2 h_2 G_2 \int_0^T \int_0^1 w_{xx} x w_x &= \frac{-N^2 h_2 G_2}{2} \int_0^T (w_x(1, t))^2 + \frac{N^2 h_2 G_2}{2} \int_0^T \int_0^1 w_x^2. \\
-\alpha \int_0^T \int_0^1 w_{xxtt} x w_x &= -\alpha \int_0^1 w_{xxt} x w_x |_{t=0}^{t=T} + \alpha \int_0^T \int_0^1 w_{xxt} x w_{xt} \\
&= -\alpha \int_0^1 w_{xxt} x w_x |_{t=0}^{t=T} + \alpha \int_0^T w_{xt}^2 - \alpha \int_0^T \int_0^1 w_{xt} (x w_{xt})_x \\
-\alpha \int_0^T \int_0^1 w_{xxtt} x w_x &= -\alpha \int_0^1 w_{xxt} x w_x |_{t=0}^{t=T} + \frac{\alpha}{2} \int_0^T w_{xt}^2 - \frac{\alpha}{2} \int_0^T \int_0^1 w_{xt}^2. \\
K \int_0^T \int_0^1 w_{xxxx} x w_x &= K \int_0^T w_{xxx}(1, t) w_x(1, t) - K \int_0^T \int_0^1 w_{xxx} (x w_x)_x \\
&= K \int_0^T \int_0^1 w_{xxx} w_x = -K \int_0^T \int_0^1 w_{xx}^2 \\
K \int_0^T \int_0^1 w_{xxxx} x w_x &= -\frac{K}{2} \int_0^T \int_0^1 w_{xx}^2.
\end{aligned}$$

Hence we have

$$K \int_0^T \int_0^1 w_{xxxx} x w_x = K \int_0^T w_{xxx}(1, t) w_x(1, t) + \frac{3K}{2} \int_0^T \int_0^1 w_{xx}^2.$$

So from Equation (B.1) without changing the coupled terms, in all we have:

$$\begin{aligned}
& \int_0^T (-K w_{xxx}(1, t) w_x(1, t) + \frac{N^2 h_2 G_2}{2} (w_x(1, t))^2) = \frac{N^2 G_2 h_2}{2} \int_0^T \int_0^1 w_x^2 \quad (\text{B.8}) \\
& + \alpha \int_0^T \int_0^1 w_{xxt} x w_{xt} + \int_0^T \int_0^1 N G_2 (-v_x^1 + v_x^3) x w_x \\
& + \frac{m}{2} \int_0^T \int_0^1 w_t^2 + \int_0^1 (-\alpha w_{xxt} x w_x + m w_t x w_x) |_{t=0}^{t=T}.
\end{aligned}$$

Next we multiply (B.2) by  $xv_x^1$  and integrate by parts.

$$\begin{aligned}
h_1 \rho_1 \int_0^T \int_0^1 v_{tt}^1 x v_x^1 &= h_1 \rho_1 \int_0^1 v_t^1 x v_x^1 |_{t=0}^{t=T} - \frac{h_1 \rho_1}{2} \int_0^T \int_0^1 (v_t^1)^2 \\
-h_1 E_1 \int_0^T \int_0^1 v_{xx}^1 x v_x^1 &= \frac{-h_1 E_1}{2} \int_0^T (v_x^1(1, t))^2 + \frac{h_1 E_1}{2} \int_0^T \int_0^1 (v_x^1)^2.
\end{aligned}$$

Finally from Equation (B.2) without changing the coupled terms, we have:

$$\begin{aligned} \frac{h_1 E_1}{2} \int_0^T (v_x^1(1, t))^2 &= \frac{h_1 \rho_1}{2} (v_t^1)^2 + \frac{h_1 E_1}{2} \int_0^T \int_0^1 (v_x^1)^2 \\ &- \int_0^T \int_0^1 \left( \frac{G_2}{h_2} (-v^1 + v^3) + N G_2 w_x \right) x v_x^1 + h_1 \rho_1 \int_0^1 v_t^1 x v_x^1 \Big|_{t=0}^{t=T}. \end{aligned} \quad (\text{B.9})$$

Similarly from Equation (B.3), we have

$$\begin{aligned} \frac{h_3 E_3}{2} \int_0^T (v_x^3(1, t))^2 &= \frac{h_3 \rho_3}{2} \int_0^T \int_0^1 (v_t^3)^2 + \frac{h_3 E_3}{2} \int_0^T \int_0^1 (v_x^3)^2 \\ &+ \int_0^T \int_0^1 \left( \frac{G_2}{h_2} (-v^1 + v^3) + N G_2 w_x \right) x v_x^3 + h_3 \rho_3 \int_0^1 v_t^3 x v_x^3 \Big|_{t=0}^{t=T}. \end{aligned} \quad (\text{B.10})$$

Adding Equations (B.8), (B.9) and (B.10) and noting that all the coupling terms are gone, we have the following:

$$\begin{aligned} &\int_0^T \left( -K w_{xxx}(1, t) w_x(1, t) + \frac{N^2 h_2 G_2}{2} (w_x(1, t))^2 \right) + \frac{h_1 E_1}{2} \int_0^T (v_x^1(1, t))^2 \\ &+ \frac{h_3 E_3}{2} \int_0^T (v_x^3(1, t))^2 = \frac{N^2 G_2 h_2}{2} \int_0^T \int_0^1 w_x^2 + \alpha \int_0^T \int_0^1 w_{xxt} x w_{xt} \\ &+ \frac{m}{2} \int_0^T \int_0^1 w_t^2 \frac{h_1 \rho_1}{2} \int_0^T \int_0^1 (v_t^1)^2 + \frac{h_1 E_1}{2} \int_0^T \int_0^1 (v_x^1)^2 \\ &+ \frac{h_3 \rho_3}{2} \int_0^T \int_0^1 (v_t^3)^2 + \frac{h_3 E_3}{2} \int_0^T \int_0^1 (v_x^3)^2 + X \end{aligned} \quad (\text{B.11})$$

where

$$X = h_1 \rho_1 \int_0^1 v_t^1 x v_x^1 \Big|_{t=0}^{t=T} + h_3 \rho_3 \int_0^1 v_t^3 x v_x^3 \Big|_{t=0}^{t=T} + \int_0^1 \left( -\alpha w_{xxt} x w_x + m w_t x w_x \right) \Big|_{t=0}^{t=T} \quad (\text{B.12})$$

We can estimate  $X$  as before to get  $|X| \leq C_3 E(0)$ . We can also estimate the right hand side of (B.11) to get regularity and observability estimates.

$$\begin{aligned} &\int_0^T \left( -K w_{xxx}(1, t) w_x(1, t) + \frac{N^2 h_2 G_2}{2} (w_x(1, t))^2 \right) + \frac{h_1 E_1}{2} \int_0^T (v_x^1(1, t))^2 \\ &+ \frac{h_3 E_3}{2} \int_0^T (v_x^3(1, t))^2 \leq C_1 E(0) \end{aligned} \quad (\text{B.13})$$

$$\begin{aligned} &\int_0^T \left( -K w_{xxx}(1, t) w_x(1, t) + \frac{N^2 h_2 G_2}{2} (w_x(1, t))^2 \right) + \frac{h_1 E_1}{2} \int_0^T (v_x^1(1, t))^2 \\ &+ \frac{h_3 E_3}{2} \int_0^T (v_x^3(1, t))^2 \geq (C_2 T - C_3) E(0), \end{aligned} \quad (\text{B.14})$$

where  $C_1, C_2$  and  $C_3$  are constants that depend on the parameters. From (B.13) and (B.14) we have the  $\int_0^T (-K w_{xxx}(1, t) w_x(1, t) + \frac{N^2 h_2 G_2}{2} (w_x(1, t))^2 + \frac{h_1 E_1}{2} \int_0^T (v_x^1(1, t))^2 + \frac{h_3 E_3}{2} \int_0^T (v_x^3(1, t))^2$  is a norm equivalent to the energy norm as in Chapter 2.

### Open problem

It is not clear whether the term

$$\int_0^T (-K w_{xxx}(1, t) w_x(1, t) + \frac{N^2 h_2 G_2}{2} (w_x(1, t))^2 + \frac{h_1 E_1}{2} \int_0^T (v_x^1(1, t))^2 + \frac{h_3 E_3}{2} \int_0^T (v_x^3(1, t))^2$$

defines an equivalent norm for the energy space  $\mathcal{H} = H_w^2 \times H_0^1(0, 1) \times H_0^1(0, 1) \times L^2(0, 1) \times L^2(0, 1) \times L^2(0, 1)$ , where  $H_w^2 = \{\phi \in H^2(0, 1) : \phi(0) = \phi(1) = 0\}$ . The two estimates above seem to yield a trace theorem which needs further investigation. Even if the term above can be shown to be an equivalent norm on  $\mathcal{H}$ , it is not clear how to use the regularity and observability estimates above to construct the HUM controls explicitly as we did in Chapter 3 to show exact controllability in  $\mathcal{H}'$ . This is an example where the moment method seems to work well (see [14],[15] and [16]) whereas the applicability of the multiplier method to this situation remains an open question.

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