

**Greedy quasigroups and greedy algebras with applications to combinatorial
games**

by

Theodore Allen Rice

A dissertation submitted to the graduate faculty
in partial fulfillment of the requirements for the degree of
DOCTOR OF PHILOSOPHY

Major: Mathematics

Program of Study Committee:
Jonathan D.H. Smith, Major Professor
Elgin Johnston
Scott Hansen
Giora Slutski
Sun Yell Song

Iowa State University

Ames, Iowa

2007

Copyright © Theodore Allen Rice, 2007. All rights reserved.

DEDICATION

This thesis is dedicated to my mother who passed away in 1996. She always suspected that I would go into “pure math.” Her love and encouragement are still with me today. If she were alive today, she would be very proud of me. I also dedicate this thesis to my father who has always supported me whatever I have done. I know he is very proud of me. I dedicate this to him because of the support he gives me.

TABLE OF CONTENTS

LIST OF TABLES	vii
ABSTRACT	viii
CHAPTER 1. Motivation: Combinatorial Games	1
1.1 Introduction	1
1.2 Definitions	1
1.3 Some basic facts about games	4
1.3.1 Sums of games	5
1.4 Values and outcomes	6
1.5 Examples of games	7
1.5.1 Nim	7
1.5.2 Wythoff's Nim	10
1.5.3 Fibonacci representations	11
1.6 Winning strategy	12
1.6.1 Digital Deletions	14
1.6.2 Nim in disguise	16
1.7 Playing misère games	17
1.8 Sequential compounds	18
1.8.1 Determining outcomes	19
1.9 Conclusion	19
CHAPTER 2. Further Results on Wythoff's Game	20
2.1 Introduction	20
2.2 Finding the zero values	20

2.3	Using Fibonacci numbers to find a winning strategy	21
2.4	Fibonacci-like sequences in Wythoff's game	21
2.5	The WSG algorithm for the \mathcal{G} function	22
2.5.1	Time and Space complexity of WSG	24
2.6	Implications of WSG	24
2.7	Additive periodicity	25
2.8	Conclusion	28
CHAPTER 3. Quasigroup Theory		29
3.1	Introduction	29
3.2	Definitions	29
3.3	Quasigroup homomorphisms	30
3.4	Quasigroup congruences	31
3.5	Conjugates	33
3.6	Isotopy	33
CHAPTER 4. Latin Squares		35
4.1	Introduction	35
4.2	Orthogonality	37
4.3	Pandiagonal latin squares	39
CHAPTER 5. Greedy Quasigroups		42
5.1	Introduction	42
5.2	Generation of greedy quasigroups	42
5.3	Column structure of greedy quasigroups.	43
5.4	Multiplication groups	52
5.4.1	Permutation Groups	53
5.4.2	Basic results	54
5.4.3	2-transitivity	57
5.4.4	High transitivity	58
5.5	Subquasigroups	58

5.6	Homomorphisms	59
5.7	Generalized greedy quasigroups	62
5.8	Transfinite extensions of greedy quasigroups	64
5.8.1	Infinite seeds	65
5.9	The greedy idempotent quasigroup	66
5.10	Conclusion	67
CHAPTER 6. Wythoff Quasigroups		68
6.1	Introduction	68
6.2	Definition and basic properties	68
6.3	Some calculations on columns	70
6.4	Subquasigroups	93
6.5	Non-isomorphism	93
6.6	Conclusion	94
CHAPTER 7. Game Theory Applications		95
7.1	Introduction	95
7.2	Playing greedy quasigroups as games	95
7.3	Analysis of Digital Deletions.	96
7.4	Conclusion	98
CHAPTER 8. Pandiagonal Latin Squares as Algebras		99
8.1	Introduction	99
8.2	Latin squares with transversals	99
8.3	Identities in tri-quasigroups	101
8.4	Restriction to isotopy classes	104
8.5	Conclusion	105
CHAPTER 9. Greedy Rings		106
9.1	Greedy ring table	106
CHAPTER 10. Summary		110

APPENDIX A. Prover9 Generated Proofs	113
BIBLIOGRAPHY	130
ACKNOWLEDGEMENTS	134

LIST OF TABLES

Table 1.1	Outcome classes	5
Table 1.2	Nim addition	8
Table 1.3	Misère Nim	10
Table 1.4	Wythoff's Nim	11
Table 1.5	Digital Deletions	14
Table 2.1	The first 20 0-values	21
Table 2.2	Some \mathcal{G} -values for Wythoff's game	23
Table 5.1	Part of the table for Q_2	43
Table 5.2	Transfinite extension of Q_0	65
Table 5.3	Q_ω	66
Table 5.4	Q_I : the greedy idempotent quasigroup	66
Table 6.1	Part of the multiplication table for W_5	69
Table 8.1	A latin square with 4 transversals	100
Table 9.1	Step 1 for the greedy ring	107
Table 9.2	Step 2 for the greedy ring	107
Table 9.3	Step 3 for the greedy ring	108
Table 9.4	Step 4 for the greedy ring	108

ABSTRACT

Greedy quasigroups and Wythoff Quasigroups arose out of a desire to better understand certain combinatorial games. Greedy and Wythoff quasigroups have remarkable algebraic properties. In particular, I will investigate the existence of subquasigroups and isomorphism classes. Natural generalizations of greedy quasigroups are also investigated and it is shown that the “greedy” property extends nicely to conjugates. Since Wythoff quasigroups have more structure than ordinary quasigroups, it is natural to ask whether they are an example of a variety of quasigroups. This question is investigated by introducing the idea of tri-quasigroups. Tri-quasigroups are investigated and some remarkable identities are proven. Finally, in the spirit of Conway, a greedy ring is investigated. The construction and characterization are given.

CHAPTER 1. Motivation: Combinatorial Games

1.1 Introduction

This chapter introduces combinatorial games which provide the motivation and inspiration for the research in this thesis. The basic definition of combinatorial games is given along with a description of construction of combinatorial games in general. Nim is introduced as the primary motivating example and it is shown that a large class of games are equivalent to nim. Wythoff's game which is a modification of nim is explained to motivate Wythoff quasigroups. Digital Deletions, which appears to be quite different from nim, is explained and is shown to be equivalent to nim.

Since combinatorial games are of interest to amateur mathematicians, more detail is given in this chapter so that anyone interested in such games. Certain results are given that are rather trivial, but are necessary to understand the main results of the thesis.

1.2 Definitions

In John H. Conway's book "On Numbers and Games" (ONAG) he introduces the theory of combinatorial games. Several games are introduced with their theory explained. First it should be specified what is meant by a combinatorial game.

Definition 1.2.1. A *combinatorial game* is a game which satisfies the following conditions:

1. The game is between exactly two players, often they are called Left and Right.
2. There are several positions and a given starting position. Usually there are only finitely many positions.

3. There is a set of rules which determine the allowable legal moves. It is possible, and frequently the case, that Left can have a different set of options than Right at a given position.
4. Left and Right move alternately.
5. The first player to be unable to move loses. This is referred to as *normal play*.
(One can also specify that the last player to move loses; that is the first player without a move wins. This is called *misère play*.)
6. The game is such that it must end with one player the winner: there are no draws.
7. There is complete information about the game; there is no bluffing.
8. Nothing is left to chance.

(28, p.2).

Most games people are familiar with are not combinatorial games, since they violate one of the conditions. Most games played with cards has chance built into the game. Games like Tic-Tac-Toe and chess can end in a draw. Games like Go are not properly combinatorial games since the winner of Go is not the last player to move, but the player with the most space. (However, Go can be analyzed using the techniques of combinatorial games).

Games are an extension of Conway's expression of numbers, which is a generalization of Dedekind's cuts. All numbers defined in this way are also games, but there are games which are not numbers. Historically, games were developed first, and numbers came out of the definition of games.

To express a game, one can write a given position in a combinatorial game as

$$\left\{ L_1, L_2, \dots, L_k \mid R_1, R_2, \dots, R_n \right\}, k, n \geq 0,$$

where each L_i is a position Left could put the game into if it were his turn, and each R_j is a position that Right could put the game in if it were her turn. Write x^L for a typical *left option* of the game and x^R for a typical *right option*. One can write $G = \left\{ G^L \mid G^R \right\}$.

The simplest game is $\{\emptyset|\emptyset\} \equiv \{\mid\} \equiv 0$. As Conway says, “I courteously offer you the first move, and call upon you to make it.” (14, p.72). The next simplest game is $\{\{\mid\}\mid\} = \{0\mid\} \equiv 1$. In this game, Left can move to the 0 game on his turn, but Right can’t do anything if it is his turn. In this case Left wins no matter who starts, so the game is, by convention, a positive game. Similarly, $\{\mid\{\mid\}\} = \{\mid 0\} \equiv -1$. A *positive* game is a game that Left can win no matter who starts, and a *negative* game is a game that Right can win no matter who starts. Write $G > 0$ for a positive game, $G < 0$ for a negative game, and $G = 0$ for the zero game.

One can also create the game $\{0\mid 0\}$. In this game, the first player must move the game to 0, causing the other player to lose. This game is not positive, since Right can win if he starts. Similarly, it is not negative. It is not zero, since the first player wins rather than loses. Thus need a fourth category of games is needed. Such games are called *fuzzy*. Define $*$ $\equiv \{0\mid 0\}$.

For a fuzzy game, one writes $G \parallel 0$; write $G \parallel > 0$ for a game that is positive or fuzzy, and $G \geq 0$ has the usual meaning. So $G \geq 0$ means that Left will always win provided Right starts, and $G \parallel > 0$ means that Left wins provided Left starts. Fuzzy games are examples of games that are not numbers. While all numbers are games, not all games are numbers. The game $*$ is such an example.

Games can be inductively built up from games already in existence. So far I have discussed two iterations of this creation process. There are 22 games created in the next creation process. I will not discuss them all, but will give some insight into where the theory is going.

Consider the new games created next: $\{0\mid 1\}$, $\{1\mid 0\}$, $\{-1\mid 1\}$ and $\{1\mid -1\}$. In game $\{0\mid 1\}$ Left wins no matter what. This game is given the value $\frac{1}{2}$, similarly $\{-1\mid 0\} \equiv -\frac{1}{2}$. There is a sense in which there is a half move advantage in these games. See Example 1.3.7 for an explanation.

In game $\{-1\mid 1\}$, when either player moves, the game is put into a position that is a win for the other player. Thus this game has the same outcome as the 0 game, so $\{-1\mid 1\} = 0$. In this game each player wants to give the other one the move. However, in the game $\{1\mid -1\}$ it is to a player’s benefit to move, as the moving player’s position will improve. Consider $\{100\mid -100\}$. This game has a lot at stake, since both players stand to gain 100.

1.3 Some basic facts about games

All games are constructed from simpler games. Game 0 came into existence on the 0th iteration. Games 1, -1, *, came into existence on the first iteration and so on. Each game that comes in to existence on the n th iteration can only have games that have come into existence on previous iterations as its options. I will define some relations on games inductively based on the options of the games.

Definition 1.3.1. The *negative* of a game G is $-G \equiv \{-G^R \mid -G^L\}$.

I will now prove some basic facts about combinatorial games. The proofs rely on the fact that the games are finite and are thus inductively built up from the 0 game. When induction is applied, the base case is always the empty set and the claims are vacuously true for the empty set. Suppose the claim is true for all options of G and reason from there.

Theorem 1.3.2. *In any game G either the game is a win for Left, a win for Right, a win for the first player, or a loss for the first player.*

Proof. This is equivalent to the statement: For every game G , either $G \geq 0$ or $G <|| 0$ and either $G \leq 0$ or $G >|| 0$.

Suppose the claim is true for all G^L, G^R . I want to show that either there is a winning first move for Left ($G >|| 0$) or that there is not ($G \leq 0$). Then if any $G^L \geq 0$, Left can win by moving to this G^L , and following the winning strategy that exists since the game is either positive or a 0 game with Right starting. If not, then every $G^L <|| 0$, and Right has a winning strategy since the position is either fuzzy or in Right's favor. Right simply waits for Left to move and applies the winning strategy. The other pair is proven in the same way. The four classes are all available and there are no other possibilities. \square

The following table explains the outcome classes.

		Right Starts	
		Left has a winning strategy	Right has a winning strategy
Left starts	Right has a winning strategy	$G = 0$	$G < 0$
	Left has a winning strategy	$G > 0$	$G \parallel 0$

Table 1.1 Outcome classes

1.3.1 Sums of games

One can imagine playing two or more games at once. This leads us to the idea of a sum of games. Imagine two games, G, H are placed on a table. When it is Left's turn to move, he selects one game and makes a legal move in it. Then Right does the same. She selects one of the games and makes a legal move in it. This sum is called the *disjunctive sum* or just *sum*. Write this as $G + H \equiv \{G^L + H, G + H^L \mid G^R + H, G + H^R\}$. Note that this is an inductive definition. Left will leave one game alone and move to a left option in the other.

Some basic facts about sums of games are proven. The proofs usually are based on discussing the strategies of playing the games.

Theorem 1.3.3. *For all games G , $G - G = 0$.*

Proof. The moves for one player in G become legal for the opponent in $-G$, and vice versa. The second player can always win in $G - G$, by playing the corresponding move in the other game. If Left moves to some G^L , Right then is able to move to $-G^L$ in the other part. \square

Theorem 1.3.4. *If $G \geq 0$ and $H \geq 0$ then $G + H \geq 0$.*

Proof. The assumptions say that if Right starts in either component, Left can win in that component. (See Table 1.1). The claim is that if Right starts in $G + H$, Left can still win. Left's strategy is to play in the component Right moves in, following the winning strategy. In this way, Left will always have a response to Right. Thus Left wins. \square

Theorem 1.3.5. *If H is a zero game, then $G + H$ has the same outcome as G .*

Proof. The player that can win in G always responds appropriately if his opponent plays in G , only playing in H if his opponent plays there first and then following the winning strategy. This strategy guarantees that the winner of G will win $G + H$. \square

Definition 1.3.6. Two games G, H are said to be *equivalent* if $G - H$ is a zero game. These games are said to have the same *value*.

Example 1.3.7. Consider the situation $\{0|1\} + \{0|1\} + \{0\}$. By the definitions this should be a 0 game, i.e. a second player win. I will now verify this by looking possible moves and responses. If Left starts, the game becomes $\{0|1\} + \{0\}$, Right moves to $1 + \{0\} = 1 + -1 = 0$ with Left to move, so Left loses. If Right starts, moving to $\{0|1\} + \{0|1\}$ lets Left move to $\{0|1\}$ where Right moves to 1 and loses. So Right moves to $\{0|1\} + 1 + \{1\}$, where Left can move to $0 + 1 + \{0\} = 1 + -1 = 0$, so Right loses. This method of determining the winner by looking at the game-tree is cumbersome. Knowing the value of games allows one to determine winning strategies more easily.

1.4 Values and outcomes

One must distinguish between the value of a game and its *form*. The games $\{-1|1\}, \{\}\{\}$ both have value 0, but are in different forms. There are situations where the form of the game makes a difference.

From the above theorems, one can see that games form a commutative group under addition if one considers the values of games. Games also form a partially ordered set. Note that since it is possible to have $G \parallel H$, there are games that are not comparable, so one does not get a total order.

The game $*$ = $\{0|0\}$ is important and interesting. As remarked above, $*$ \parallel 0. Note that $* + * = \{*\mid*\} = 0$ since either player moves to $*$, giving the win to the other player.

Combinatorial games are divided into two types, *partisan games* where Left and Right have different options, and *impartial games* where Left and Right have the same options.

An impartial game is thus in the form $G = \{A, B, C, \dots \mid A, B, C, \dots\}$ Simplify the notation

to $G = \{A, B, C, \dots\}$. The game $*$ = $\{0|0\}$ is a typical impartial game. The games I will discuss are impartial games.

The value of an impartial game is neither positive or negative since one cannot distinguish players by looking at their options. Designate the *outcome* of a game, $o(G)$, by \mathcal{P} , if the previous player wins, or \mathcal{N} , if the next player wins. Of course, it is assumed both players know how to play and don't make mistakes. A player wants to be able to move into a \mathcal{P} position on his move.

From these definitions, it must be the case that in a \mathcal{P} -position, every move is into a \mathcal{N} -position and in every \mathcal{N} -position, there is a move into a \mathcal{P} -position.

1.5 Examples of games

1.5.1 Nim

One of the most important and best known games is Nim. Nim is played with piles of sticks or counters. A player may take as many counters as desired from any one pile or "nim-heap". Then the next player moves the same way. Since Nim is a combinatorial game, in ordinary play, the last player to make a move wins.

The strategy for Nim with two piles is well-known: make the piles equal, once this is done, a player can always equalize the piles, thus assuring that if his opponent can take a counter, there is a corresponding one for him to take in the other heap, thus assuring he will always be able to move. However, if someone is stuck with the move and equal heaps, then there is no hope against a knowledgeable opponent. If this is the case, then one has a lost position.

When there are more than two piles, one has to use more sophisticated strategy. To see what this is, first analyze the 2-heap version of Nim further. Characterize positions by the sense of advantage the player to move has. The empty position is given a value 0 since the player to move has no advantage. With just one stone on the table, the player with the move has the advantage, so this is a *fuzzy* position. One can give it the value $1*$. Similarly for a pile of k stones, the player to move takes all k counters and wins. Note that this is the only winning move, taking fewer allows the other player to take the rest and win. Give this position

the value k^* . Now what if there are two piles? Compute the nim-sum of the piles. This is given in Table 1.2. I have left the *'s off the values to make the table more readable.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	0	3	2	5
2	2	3	0	1	6
3	3	2	1	0	7
4	4	5	6	7	0

Table 1.2 Nim addition

Note that the nim-sum is commutative. This makes sense since the order of the piles on the table doesn't matter. This sum tells whether the first player can win and helps decide the winning move.

It has already remarked that two equal piles is a loss, so each of these has value 0 and this is reflected in the table. The nim sum tells us that the two piles confer the same advantage as a single pile with that many counters. How is the table constructed? The construction is very simple. Place a 0 in the upper left corner and apply the “mex-rule”

The *mex* of a set of natural numbers is the minimal excluded natural number of the set, that is the least number not in the set. By the well-ordering principle, the mex must exist and is well-defined. To fill in the table, put the mex of all numbers to the left and above the entry in question. Formally this may be written $q_{ij} = \text{mex}(\{q_{kj}\}_{k=0}^{i-1} \cup \{q_{ik}\}_{k=0}^{j-1})$.

Sometimes this operation is referred to as the *Sprague-Grundy function*. The Sprague-Grundy function, \mathcal{G} , of a position g , is the mex of the Sprague Grundy function of all the followers of g ($F(g)$), that is of all the possible moves from g . Thus

$$\mathcal{G}(g) = \text{mex}\mathcal{G}(F(g)).$$

Nim-addition also corresponds to binary addition without carrying. Since binary addition is associative, nim-addition is associative as well. Note that with this fact nim-addition is an abelian group with 0 as the identity and each element is its own inverse. It makes sense that

nim-sums should be associative since there is not a grouping imposed on the heaps at the start of the game.

Finding the nim-sum tells us how many to put in a single heap next to the other piles to make a 0 game. If the nim-sum of a game is $n*$, adding a pile of n counters makes the new game have value $n* + n* = 0$ because nim addition is associative.

This tells us how to play multi-heap Nim: find the nim-sum of any two heaps, combining them into a single heap with the same value, and repeat the process until all heaps are accounted for. If the value is 0, one is lost against a clever opponent. Any move one makes will cause the position to have a non-zero value, which is a win for the opponent. The opponent can then reduce the position back to a position with value 0. If the value is positive, there is a way to reduce the value to 0, since the game is a first player win. Thus one must be able to put the game into a position that is a second player win with one's opponent to move, the definition of a zero game. Thus the \mathcal{P} -positions in nim are the positions with value 0, and the \mathcal{N} -positions are all of the others.

One can also play Nim under the misère condition. One way to look at this is to imagine that one of the counters is poisoned and that taking it poisons the player. One might think that this version would be totally different or that the strategy is somehow “opposite” that of regular nim. In fact it isn't. Keeping the piles even is the correct strategy, until there are two piles of size two. In this case, respond the opposite way one would in normal play. In normal play, if one's opponent took both counters, he would take both and win, in misère play he takes one, leaving the poisoned one behind, and if the opponent took one, he would take one in normal play, but in misère play, he would take two leaving the poisoned one behind.

To construct the table this time note that having an empty game is good, since then the opponent lost. Call this 1. If there is an empty heap and a heap of one pile, one must take the last counter and lose, so this position is also lost. Note that two piles of one heap is a win, and the remaining states have the same value as in nim. Construct the table using the mex-rule.

Now, each of these tables for Nim form a quasigroup. (Background information on quasi-

+	0	1	2	3	4
0	1	0	2	3	4
1	0	1	3	2	5
2	2	3	0	1	6
3	3	2	1	0	7
4	4	5	6	7	0

Table 1.3 Misère Nim

groups is given in a later chapter). The table for nim-addition is actually an abelian group as remarked above. A natural question is whether or not the misère table is a group, and if it is, whether it is isomorphic to ordinary nim-addition. This is a reasonable conjecture, since the strategy for both games is the same, except at the key moment.

An equivalent characterization of Nim is the following game. Place a rook on any square of a quarter-infinite chess board with the corner of the in the northwest. On his move, a player may move the rook north or west as far as desired. The first player who can't move, because the Rook is in the corner loses. It is easy to see that the row coordinate corresponds to one nim-heap and the column coordinate another. Of course one can add several rooks to the board to correspond to several piles.

1.5.2 Wythoff's Nim

Wythoff's Nim is played with two piles of counters as in Nim. A player may take any number of counters from one pile, as in Nim, but may also take the same number of counters from both piles. This game is similar to nim, but with a different strategy. If the piles are ever of equal size, one player can simply remove both of them and win the game. This game has the equivalent characterization of placing a queen on a quarter-infinite chessboard. The added ability to remove the same number of counters from each heap corresponds to the diagonal move of the queen. Of course, one can play this game with several queens at once. This version is called Wyt Queens in *Winning Ways*. Let's look at the table of nim values for this game.

The zero entries are the most important, since a 0 game is a second-player win. The zero entries appear at $(0, 0)$, $(1, 2)$, $(3, 5)$, $(4, 7)$, $(6, 10)$, $(8, 13)$, ... and their compliments.

	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	2	0	4	5	3	7	8	6
2	2	0	1	5	3	4	8	6	7
3	3	4	5	6	2	0	1	9	10
4	4	5	3	2	7	6	9	0	1
5	5	3	4	0	6	8	10	1	2
6	6	7	8	1	9	10	3	4	5
7	7	8	6	9	0	1	4	5	3
8	8	6	7	10	1	2	5	3	4

Table 1.4 Wythoff's Nim

Determining the correct course of play is not as easy as it is in nim.

However, it can be done after making a couple observations.

1.5.3 Fibonacci representations

Every integer can be expressed as the sum of Fibonacci numbers.

Definition 1.5.1. A *Fibonacci representation* is a finite sequence of 0's and 1's. A 1 in the i th position indicates the presence of the i th Fibonacci number, where $F_1 = F_2 = 1$. A number is determined by summing the Fibonacci numbers present.

As an example, consider $100000 = 8$. However 10101 and 11000 also denote 8. So, unlike binary representations, Fibonacci representations are not unique.

Notice that this representation can be accomplished so that no two consecutive Fibonacci numbers are used. If two consecutive would need to be used, say F_i, F_{i-1} , they could be replaced by F_{i+1} . Also since the first two Fibonacci numbers are 1, only the second one, F_2 , is needed.

Definition 1.5.2. A Fibonacci representation is said to be *canonical* if the representation contains no adjacent 1's and F_1 is not present.

A Fibonacci representation is said to be *second canonical* if there are no adjacent 1's and the right most 1 is in an odd numbered position.

The canonical and second canonical representations exist and both are unique. For 8, the canonical representation is 100000 and the second canonical representation is 10101. Both are also lexicographic, that is if $m < n$ then the representation of m appears before that of n in the lexicographic order.

Definition 1.5.3. Let n be in represented canonical form. n is said to be an *A-number* if the rightmost 1 is in an even numbered position. Otherwise, n is a *B-number*.

A positive integer must be either an A-number or a B-number. These are used to devise a winning strategy. Given (a, b) with $a < b$, (a, b) is said to be a *safe pair* if a is an A-number and the canonical representation of b is that of a with a zero adjoined on the end. For safe pairs other than $(0,0)$ b is a B-number. It will be shown that the safe pairs are exactly the 0-values of Wythoff's nim.

1.6 Winning strategy

Proposition 1.6.1. *Characterization of safe pairs*

- If (a, b) is a safe pair, every pair (c, d) which is reachable from (a, b) is not a safe pair.
- If (c, d) is a safe pair, there is a safe pair, (a, b) which is reachable from (c, d) by a legal move.

The following lemmas are useful. Details can be found in (36).

Lemma 1.6.1. *If (a, b) is a safe pair, deleting the last zero of a yields the second canonical representation of $b - a$.*

Lemma 1.6.2. *For each $n > 0$, there is exactly one pair safe pair (a_n, b_n) such that $b_n - a_n = n$. One can find a_n by adjoining a 0 to the second-canonical representation of n , and b is found by adjoining a 0 to the canonical representation of a .*

Corollary 1.6.3. *If $m < n$, $a_m < a_n$ and $b_m < b_n$.*

Corollary 1.6.4. *Each n belongs to exactly one safe pair.*

Now the proposition can be proven.

Proof. If (a_n, b_n) is a safe pair, reducing a_n give another pair containing a_n , which cannot be safe by Corollary 1.6.4 and similarly if b_n is reduced. If both a_n, b_n are reduced, there is a new pair (c, d) such that $d - c = n$, but this cannot be safe by Lemma 1.6.2. Thus no move leads to a safe pair

Now suppose one has an unsafe pair (a, b) . If $a = b$ reducing both a, b to 0, gives the safe pair $(0, 0)$. Otherwise represent a, b canonically. If a is a B-number, reduce b to the corresponding A-number which is a with the last digit deleted. If a is an A-number and b is greater than the corresponding B-number (a with a 0 appended) reduce b to that B-number. If a is an A-number and b is less than the corresponding B-number, (say) a' , let $m = b - a$ and $n = a' - a$. Thus $m < n$ and $a_m < a_n = a$. Reduce a to a_m , an equal reduction in b produces b_m , since (a_m, b_m) is the unique safe pair with difference m . \square

The winning strategy as described by Silber is thus as follows

1. Given (a, b) , ($a < b$) represent each in canonical form. If the position is a safe position, one is lost against a knowledgeable opponent. Silber suggests conceding, I suggest making a small move, removing one counter from a pile, and trying to prolong the game and hoping for a mistake.
2. If the smaller number, a , is in B , reduce the larger to the corresponding number in A in a safe position, i.e. that of the a without its last 0, do so.
3. If the smaller number, a , is in A , and the larger can be reduced to the corresponding number in B , i.e that of a with a zero on the end do so.
4. If none of the above hold, determine the second canonical representation of the difference $b - a$. Appending a zero to this number gives an element $a' \in A$ and appending a second zero gives $b' \in B$. The pair (a', b') is a safe pair obtained by subtracting the same value from each of a, b .

This strategy is only a little more complicated than that of nim.

Of course, this game can be played in misère fashion as well. The P positions are almost the same. Remove $0, 0$; change $(1, 2)$ to $(0, 1), (1, 0)$ and add $(2, 2)$. The rest remain the same. So the misère play of Wythoff's Nim is the same as the normal play except at the end just as was found with nim.

1.6.1 Digital Deletions

The game of *Digital Deletions* is played on a string of digits. In Conway the theory is described for decimal digits, but there is no reason to make the restriction to decimal digits in general. For now, however, the decimal digit game will be discussed. (The theory for the general game is essentially the same). The game is played on a string of digits, say 314159. The player to move may strictly decrease any one digit or may delete a zero and all digits to the right. This game is clearly an impartial game. Its values are values of Nim-heaps.

If one precedes some string with value $*x$, (the value of a Nim-heap with x counters), with some digit n , the resulting string is a function of x . Call this function f_n . $xf_n = \{x'f_n, xf_{n-1}, \dots, xf_1, xf_0\}$, where x' is any option of a position of value x . One can build a table for these values. The inductive definition for f_n is such that each entry is the mex of the entries above it and to the left with the exception that 0 can never appear in the first row.

x	f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9
0	1	0	2	3	4	5	6	7	8	9
1	2	1	0	4	3	6	5	8	7	10
2	3	2	1	0	5	4	7	6	9	8
3	4	3	5	1	0	2	8	9	6	7
4	5	4	3	2	1	0	9	10	11	6
5	6	5	4	7	2	1	0	3	10	11
6	7	6	8	5	9	3	1	0	2	4
7	8	7	6	9	10	11	2	1	0	3
8	9	8	7	6	11	10	3	2	1	0
9	10	9	11	8	6	7	4	5	3	1

Table 1.5 Digital Deletions

Of course if one is playing in base 16, for instance, he would need 6 more rows. The most obvious feature of this table is that it is not symmetric. Conway says this about the game: “We can deduce that the entries in each line are ultimately arithmetico-periodic, so that the game has in principle a complete theory. Perhaps the reader will find out exactly where the periodicity occurs. But apart from the formulae $xf_0 = x+1$, $xf_2 = x+3_2$, $(x+9)f_3 = (x+9)f_3$ for $x \geq 3$ there seem to be no easy answers” (14, p.192). (Here $+_3$ is addition base 3 without carrying.)

One can use these to find the value of any position and thus the right move to make.

Example 1.6.5. Consider 314159. To compute the value, realize that the empty position has value 0. Append a 9 to this position, getting a value of 9. Then append a 5 to this position getting a value of 7, and so forth. The value is $0f_9f_5f_1f_4f_1f_3$. To evaluate:

$$0 \xrightarrow{f_9} 9 \xrightarrow{f_5} 7 \xrightarrow{f_1} 7 \xrightarrow{f_4} 10 \xrightarrow{f_1} 10 \xrightarrow{f_3} 12 \quad (1.1)$$

The position has value 12. To get the right move, imagine that the position really has a value 0 and work backwards getting a new chain.

$$0 \xleftarrow{f_3} 2 \xleftarrow{f_1} 2 \xleftarrow{f_4} 5 \xleftarrow{f_1} 5 \xleftarrow{f_5} 0 \xleftarrow{f_9} 8 \quad (1.2)$$

Now one needs to find a way onto the second chain by finding a legal move.

$$\begin{array}{cccccccc} 12 & \xleftarrow{f_3} & 10 & \xleftarrow{f_1} & 10 & \xleftarrow{f_4} & 7 & \xleftarrow{f_1} & 7 & \xleftarrow{f_5} & 9 & \xleftarrow{f_9} & 0 \\ & & & \swarrow f_9 & & \swarrow f_6 & & & & \swarrow f_7 & & \swarrow f_1 & \\ 0 & \xleftarrow{f_3} & 2 & \xleftarrow{f_1} & 2 & \xleftarrow{f_4} & 5 & \xleftarrow{f_1} & 5 & \xleftarrow{f_5} & 0 & \xleftarrow{f_9} & 8 \end{array} \quad (1.3)$$

While that most transitions from the top row to the bottom row force increases in numbers, but one can reduce the 9 to a 1. This move puts the player in the bottom row which has value 0 assuring victory.

It is interesting that there is only one good move and 22 bad moves. In longer strings the difference between the number of good moves and bad gets larger. In 8315553613086720000 the only two good moves are to decrease the 7 to a 6 and to delete the last two zeros while there are 65 losing moves! It would certainly be nice to have a simple rule for determining what move to make.

1.6.2 Nim in disguise

I will eventually show that all short non-partisan games are equivalent to Nim. (A *short* game is a game with finitely many positions.) First some motivation for this theorem will be given. Several games look like a different game, but are really a poorly disguised version of Nim. The first is *Poker Nim*. The game is played exactly like Nim, except each player has a *finite* reserve of counters which may be added to any heap. This has no effect since if a player is winning, he doesn't need to ever add counters. If his opponent adds counters, he can simply remove the counters added, restoring the position, and reducing the number of reserve counters in his opponent's cache.

Northcutt's Game is played with checkers on the rows of a chessboard between White and Black, each only moving their own color. Players may only move back and forth along the rows without jumping. A checker can not move along the columns. The game ends when one player can't move, since all his checkers are pinned against the edge of the board. This game is like Poker Nim, where the spaces between the checkers are the sizes of the nim-heaps, and the spaces behind each checker are the reserve counters. (This is a slightly restricted version of Poker-Nim).

Northcutt's Game is potentially infinite in length, since players could alternately advance and retreat a particular checker. In Poker Nim, if one allows a player to replace counters he has removed, the same situation may arise. However, this is not a significant problem in analysis. The key factor is that one player may not indefinitely prolong the game. In Poker Nim, eventually the chip reserve will be exhausted, and the player must take from the board again. The player with the advantage can always force a win in finite time.

Now these games give us intuition that maybe any such game is Nim in disguise.

Theorem 1.6.6. *Let G be any game played with a finite collection of non-negative integers so that each move affects exactly one of the numbers and changes the number to a different one. Any decrease of the number is allowed. Additionally, one may be able to increase the value of a number. However, the game is such that it always ends. (That is one can not increase and decrease the same number infinitely often.) The outcome of any position in G is the same as*

that of the position in Nim with the same number value.

Proof. This game is a generalization of Poker Nim. As in Poker Nim, the player with the winning strategy does not need to increase any number. He can simply follow the winning strategy for Nim. If his opponent adds to a number, he can simply reduce it to restore the position. Since the rules guarantee an eventual end to the game, this insures he can win. \square

Remark 1.6.7. In Poker Nim, the ending condition can be guaranteed by making the counter reserves finite and specifying that removed counters are out of play.

The increases in the above game are called *reversible* moves. The general theorem is now proven.

Theorem 1.6.8. *Each (short) impartial game G is equivalent in play to some Nim-heap.*

Proof. Let $G = \{A, B, C, \dots\}$. Suppose the claim is true for all the options, A, B, C, \dots of G . Thus these positions are equivalent to Nim-heaps of sizes a, b, c, \dots respectively. Let $n = \text{mex}\{a, b, c, \dots\}$. It is now shown that G is equivalent to a Nim-heap of size n .

Certainly all the numbers $0, 1, \dots, n-1$ appear among the numbers a, b, c, \dots , so any decrease is possible. It is not possible to move to n ; but perhaps some of a, b, c, \dots are greater than n . In any case, there is now have the situation from Theorem 1.6.6. By that theorem, one has a situation like that of a nim-heap of size n . \square

Note that these theorems are slightly more restricted than Northcutt's Game and Poker Nim. Each of these *could* be infinite if played poorly. The analysis still holds, however.

The above theorem says that Digital Deletions is really a cleverly disguised version of Nim. Knowing this, one could move to a 0 position if possible and win the game. Any game one devises can be played well if it is possible to convert positions into nim-values.

1.7 Playing misère games

The tables for nim and misère nim, are almost identical. The only differences are in the $(0, 0); (0, 1); (1, 0); (1, 1)$ places. This suggests that the strategy in misère nim is similar to that

of ordinary nim. In fact, it is. Keep the piles even until the end, and then instead of moving to $(1, 1)$ move to $(1, 0)$. Thus the strategy is almost identical. In Wythoff's Nim, the strategy is also similar. (See the section on Wythoff's Nim for more details.)

The natural question is whether this is always the case. Since all impartial games reduce to nim, it seems like the misère version of an impartial game should be similar to the normal play version. Unfortunately, this is not the case. Misère versions of games are typically more difficult to analyze than normal play versions of games. One reason in particular for this is that the game $G + G$ may or may not be a \mathcal{P} position. In normal play $G + G$ is always a \mathcal{P} position since the second player can always mimic the first in the other copy of G . However, consider the games $*2 + *2$ and $*1 + *1$. The former is a \mathcal{P} position in misère nim, while $*1 + *1$ is an \mathcal{N} position in misère nim. This suggests that a \mathcal{P} position in normal play is not always one in misère play.

Characterize games into the following classes:

- PP Previous player wins, normal or misère
- PN Previous player wins normal, next wins misère
- NP Next player wins normal, previous wins normal
- NN Next player wins, normal or misère

Another characterization of NP is that the game takes an even number of moves, and a game in the form PN takes an odd number of moves. Games of the form PP and NN are called *firm* (sometimes called *frigid*). Games of the form NP and PN are called *fickle* (sometimes called *frisky*).

1.8 Sequential compounds

Definition 1.8.1. Given two games, G, H , Stromquist and Ullman in (39) define the *sequential compound* of G and H , denoted $G \rightarrow H$ as the game whose options are in to form $G' \rightarrow H$ if $G \neq 0$ and all the options of H is $G = 0$. That is, the play is in G until there are no more moves in G , and then the play moves to H .

Example 1.8.2. The game $G \rightarrow *$ is simply G played under the misère rule. Once play in G is

finished, the next player plays in the game $*$, making the only legal move leaving his opponent with out a move, and thus wins the game. Thus, the last player to move in G , loses $G \rightarrow *$, which is identical to the misère rule.

1.8.1 Determining outcomes

In combinatorial games, one wants to know the outcome, that is the winner, of the game. The outcome of a sequential compound, $o(G \rightarrow H)$ cannot be determined from the outcomes of G and H . That is, even knowing $o(G), o(H)$, nothing can be said about $o(G \rightarrow H)$. Since knowing the winner of the game under normal play, does not help figure out the winner under misère play. However, there is the following.

Lemma 1.8.3. *If $o(H_1)$ and $o(H_2)$, then $o(G \rightarrow H_1) = o(G \rightarrow H_2)$ for every game G .*

Proof. Use induction on G . If $G = 0$, the result is trivial. Otherwise, suppose the claim holds for all options, G' of G .

$$o(G \rightarrow H_1) \Leftrightarrow o((G \rightarrow H_1)') = N \tag{1.4}$$

$$\Leftrightarrow o(G' \rightarrow H_1) = N \tag{1.5}$$

$$\Leftrightarrow o(G' \rightarrow H_2) = N \tag{1.6}$$

$$\Leftrightarrow o((G \rightarrow H_2)') = N \tag{1.7}$$

$$\Leftrightarrow o(G \rightarrow H_2) = P \tag{1.8}$$

Thus $o(G \rightarrow H_1) = o(G \rightarrow H_2)$ for all G . □

Thus if $o(H) = P$, $o(G \rightarrow H) = o(G \rightarrow 0) = o(G)$. If $o(H) = N$, $o(G \rightarrow H) = o(G \rightarrow *)$.

1.9 Conclusion

Combinatorial games come in a myriad of forms and new combinatorial games are constantly being developed. Many such games are modifications of nim, others are entirely new ideas. The ideas in this thesis could be applied to many of the former, and perhaps even the latter. Combinatorial games remain an active field of new and interesting research.

CHAPTER 2. Further Results on Wythoff's Game

2.1 Introduction

Wythoff's nim has been studied extensively. This section represents parts of the literature that are applicable to later sections. Areas of interest include computing the nim-values, in particular finding the locations of values that have value 0 and developing algorithms to make such computations. The reader should recall that computing values in nim is very straightforward. This is not the case for Wythoff's game. The results in this section apply to Wythoff quasigroups and later sections will refer to these results. As with the previous chapter, results are given here which will be of interest to amateur mathematicians.

2.2 Finding the zero values

There are several characterizations of the 0 values in Wythoff's Nim. One can generate them using the following method: At each step, the first number is the smallest natural number not already used and the second number is such that the difference between it and the first of the n th pair is n . The disadvantage to this formula is that to discover whether a pair is a 0 pair, one must compute all the smaller pairs. Wythoff gave a closed formula for the safe pairs "out of a hat." The following proof was devised by J. Hyslop and A. Ostrowski and is given by Fraenkel in (22). First, the following fact: If x, y are positive irrational numbers with $x^{-1} + y^{-1} = 1$, the sequences $[x], [2x], [3x], \dots$ and $[y], [2y], [3y], \dots$ together include each positive integer once.

Let x, y be positive irrationals with $x^{-1} + y^{-1} = 1$. Then for integer N , $\frac{N}{x}, \frac{N}{y}$ are irrationals

whose sum is N . So

$$\left\lfloor \frac{N}{x} \right\rfloor + \left\lfloor \frac{N}{y} \right\rfloor = N - 1$$

This is the number of members of the union of the two sequences that are less than N . Taking $N = 1, 2, 3, \dots$ deduce that one of the multiples of either x or y appears in each interval $[n, n + 1]$. Thus the integral parts, $[nx], [ny]$ are exactly the natural numbers. Thus one of the requirements for the zero positions in Wythoff's Nim is satisfied. The other, that the difference shall be n is satisfied by taking $y = x + 1$. Then, since $x^{-1} + y^{-1} = 1$, one has $x^2 - x - 1 = 0$. Thus $x = \frac{1 + \sqrt{5}}{2} = \phi$, the golden ratio. Then $y = x + 1 = x^2 = \phi^2$. Thus the zero positions are $[n\phi], [n\phi^2]$.

n	a_n	b_n	n	a_n	b_n
1	1	2	11	17	28
2	3	5	12	19	31
3	4	7	13	21	34
4	6	10	14	22	36
5	8	13	15	24	39
6	9	15	16	25	41
7	11	18	17	27	44
8	12	20	18	29	47
9	14	23	19	30	49
10	16	26	20	32	52

Table 2.1 The first 20 0-values

2.3 Using Fibonacci numbers to find a winning strategy

Given the occurrence of ϕ is it not surprising that the Fibonacci numbers appear in the winning strategy discussed earlier. To find the winning move from (a, b) , use the *Fibonacci representation* of a, b to determine the winning move. (Compare with Nim, where binary representation is used.)

2.4 Fibonacci-like sequences in Wythoff's game

The 0 positions are sometimes referred to as *Wythoff pairs*. Looking at the table, one sees that the Wythoff pairs $(a_1, b_1), (a_2, b_2), (a_5, b_5), (a_{13}, b_{13})$ form the sequence $(1, 2), (3, 5), (8, 13), (21, 34)$,

which are the Fibonacci numbers. There are also Wythoff pairs whose members are not Fibonacci numbers. The first is $(a_3, b_3) = (4, 7)$. Suppose one generates a fibonacci sequence using this pair. One gets $(4, 7), (11, 18), (29, 47)$. This is exactly the sequence $(a_3, b_3), (a_7, b_7), (a_{18}, b_{18})$. This suggests the following theorem:

Theorem 2.4.1. *Let G_1, G_2, G_3 be the Fibonacci sequence generated by a Wythoff pair (a_n, b_n) . Then every pair $(G_1, G_2), (G_3, G_4), \dots$ is also a Wythoff pair.*

The proof can be found in (35).

2.5 The WSG algorithm for the \mathcal{G} function

Let j be a non-negative integer. The WSG algorithm is a recursive algorithm for constructing the sequence T_j of all pairs (a, b) so that $\mathcal{G}(a, b) = j$ in Wythoff's game, where \mathcal{G} is the Sprague-Grundy function. See (12) for details. Assume that $a \leq b$ since Wythoff's game is symmetric. Write $T_j = \{(a_0, b_0), (a_1, b_1), \dots\}$. Let $D_j = \{b_0 - a_0, b_1 - a_1, \dots\}$. Thus D_j is the set of differences of the pairs in T_j .

To construct the next pair in T_j , assume the initial segment has already been computed, up to (a_{k-1}, b_{k-1}) . Let $m = \text{mex}\{a_i, b_i | 0 \leq i < k\}$. Also assume that all the sequences T_i with $i < j$ have been constructed up to the point where T_i contains m in one of its pairs. Construct the next pair in T_j by the *WSG Algorithm* (Wythoff-Sprague-Grundy):

1. Set $\text{mex}\{a_i, b_i | 0 \leq l < k\} = m$ and $\text{mex}\{b_i - a_i | 0 \leq i < k\} = d$.
2. If $(m, m + d)$ does not appear in any $T_i, i < j$ and $m + d$ does not appear as a second term in any pair already in T_j , then set $(m, m + d) = (a_k, b_k)$ and terminate.
3. Otherwise, let $d = d + r$ where r is the smallest positive integer so that $d + r$ is not already in D_j . Go back to step 2.

It is now proven that the WSG-algorithm is accurate. The algorithm clearly terminates since for sufficiently large d , as produced in step 3 $(m, m + d)$ is not in any $T_i, i < j$ and $m + d$ is not a b_k in T_j .

No move from any $(a, b) \in T_j$ produces a pair in T_j . Since the difference $d = b - a$ can only appear once in D_j by construction, a move from $(a, b) \in T_j$ to $(a - k, b - k)$ forces $(a - k, b - k) \notin T_j$. Now for a move $(a, b) \in T_j$ to (a, b_k) , one cannot have $(a, b_k) \in T_j$ since the a_j 's in T_j are distinct. Similarly for a move from (a, b) to $(a - k, b)$.

Now it is shown that T_j contains only the pairs that have value j and contains every such pair. Assume by induction that T_i contains all and only all pairs (a, b) such that $\mathcal{G}(a, b) = i$ for all $i < j$. To show the same for T_j it suffices to show that if $(c, d) \notin T_i$ for all $i \leq j$, then (c, d) has a follower $(x, y) \in T_j$. That is there is a move from (c, d) to (x, y) . This is sufficient since if there is a pair, $(a, b) \notin T_j$ with $\mathcal{G}(a, b) = j$, then $(a, b) \notin T_i$ for all $i \leq j$, then (a, b) has a follower, $(a', b') \in T_j$. Now $\mathcal{G}(a', b') \geq j$ (by the induction hypothesis) and $\mathcal{G}(a', b') \neq j$ since it is a follower of (a, b) with value j . So $\mathcal{G}(a', b') > j$ and it must have a follower, (a'', b'') , with $\mathcal{G}(a'', b'') = j$. Now one may pick (a, b) from all such pairs so that $\mathcal{G}(a, b) = j$, $(a, b) \notin T_j$ so that $a + b$ is the smallest. However, $a'' + b'' < a + b$ so $(a'', b'') \in T_j$. But it is a follower of $(a', b') \in T_j$, a contradiction. Thus all (a, b) with $\mathcal{G}(a, b) = j$ are in T_j . This also shows that only (a, b) with $\mathcal{G}(a, b) = j$ can be in T_j . \square

Some new sequences are now defined. Let A_j be the sequence of a values such that $(a, b) \in T_j$ and B_j be the corresponding b values.

It is clear that A_j is strictly increasing for all j since it is constructed using the mex function.

T_0			T_1			T_2			T_3		
D_0	A_0	B_0	D_1	A_1	B_1	D_2	A_2	B_2	D_3	A_3	B_3
0	0	0	1	0	1	2	0	2	3	0	3
1	1	2	0	2	2	0	1	1	4	1	5
2	3	5	3	3	6	1	3	4	2	2	4
3	4	7	4	4	8	3	5	8	0	6	6
4	6	10	2	5	7	5	6	11	1	7	8
5	8	13	5	9	14	6	7	13	8	9	17
6	9	15	6	10	16	7	9	16	5	10	15
7	11	18	8	11	19	4	10	14	9	11	20
8	12	20	9	12	21	10	12	22	6	12	18

Table 2.2 Some \mathcal{G} -values for Wythoff's game

However, B_j is not strictly increasing (except for B_0). One also has $A_j \cup B_j = \mathbb{Z}$ and $|A_j \cap B_j| = 1$. The use of the mex function assures there is no repetition in A_j and the step 2 requirement that $p + q$ is not already in B_j assures there is no repetition in B_j . Now since A_j is increasing, p is the largest term in A_j , $p + q$ is not yet in A_j , unless $q = 0$ which does happen, specifically for the smallest mex(p) such that (p, p) did not occur in any $T_i, i < j$.

2.5.1 Time and Space complexity of WSG

The size of any position (s, t) with $s \leq t$ in Wythoff's game is $O(\log st)$. The construction of T_j by WSG up to a point where one can decide whether a pair (s, t) is in T_j or not takes time and space linear in j . For $(a_n, b_n) \in T_j, a_n \geq n$. The mex can be computed by maintaining a linear array of bits, $W(n)$ with $W(n) = 0$ if $n \notin \{A_i \cup B_i\}$ and $W(n) = 1$ otherwise. The mex is then the smallest place where the value is a zero.

2.6 Implications of WSG

Theorem 2.6.1. *Every diagonal parallel to the main diagonal $x = y$ of the table for Wythoff's nim contains every non-negative integer*

Proof. If a diagonal at horizontal distance d from the main diagonal does not contain j then T_j does not contain any pair (a, b) with $b - a = j$. There is some integer $N \geq 0$ so that for every $a \in A_j, a \geq N$, the pair $(a, a + d) \notin T_i, i < j$ since one can set $N > k$ for all the pairs $(a_k, b_k) \in T_i$ with $b_k - a_k = d$. Now it was already stated that $(a, a + d) \notin T_j$, so $a + d$ must have already been placed in B_j by the algorithm. Thus for $x \geq N$, either $x \in B_j$ or $x + d \in B_j$.

Now for $T_j = \{(a_0, b_0), (a_1, b_1), \dots\}$ and $A_j = \{a_0, a_1, \dots\}, B_j = \{b_0, b_1, \dots\}$. Sort the elements of B_j to form an increasing sequence $B'_j = \{b'_0, b'_1, \dots\}$. Now both A_j, B'_j are increasing, one has $a_n \geq a_N \geq N, b'_n \geq b'_N \geq N$ for $n \geq N$. For any positive integer k , define

$$U_k = \{a_n + l \mid 0 \leq L \leq 2kd - 1\} \quad V_k = \{b'_k + l \mid 0 \leq k \leq 2kd - 1\}.$$

Now one can rewrite these sets by sorting into sets of size $2d - 2$ and pairing the elements off:

$$U_k = \bigcup_{t=0}^{k-1} \{a_n + i + 2td, a_n + i + 2td + d \mid 0 \leq i \leq d - 1\}$$

$$V_k = \bigcup_{t=0}^{k-1} \{b'_n + i + 2td, b'_n + i + 2td + d \mid 0 \leq i \leq d-1\}$$

Each of these sets has kd pairs of integers.

Now in U_k , one of each of $a_n + i + 2td, a_n + i + 2td + d$ is in B_k for each i, t , so U_k contains at most kd elements that are in A_j . Similarly, one of each of $a_n + i + 2td, a_n + i + 2td + d$ is in B_j so at least kd elements of V_k are in B_j . In particular, $a_{n+kd} \notin U_k$, since if it were, then all $kd+1$ elements $a_n, a_{n+1}, \dots, a_{n+kd}$ would be in A_j . Thus $a_{n+kd} \geq a_n + 2kd$. Similarly, $b'_{n+kd-1} \in V_k$, so $b'_{n+kd-1} \leq b'_n + 2kd - 1$. Also, since one of $b_{n+kd-1} + 1, b_{n+kd-1} + 1$, is in B'_j , one has $b'_{n+kd} \leq b_{n+kd-1} + d + 1$.

Putting the inequalities together:

$$b_{n+kd} - a_{n+kd} \leq b'_{n+kd-1} + d + 1 - a_n - 2kd \leq b'_n - a_n + d \quad (2.1)$$

Let $n = N + l$ for $l = 1, 2, \dots, d$ one has

$$b_{N+l+kd} - a_{N+l+kd} \leq d + \max_{1 \leq i \leq d} (b'_{N+i} - a_{N+i}) \equiv c \quad (2.2)$$

where c is a constant that depends on d . Thus $b'_i - a_i \leq c$ for $i > N + d$.

Now the differences $b_i - a_i$ tend to infinity, there is some $M > 0$ so that $b_i - a_i > c$ for all $i \geq M$, and one may take $M > N + d$. Then $b'_i - a_i \leq c$ for all $i \geq M$. Thus $b_i > b'_i$ for all $i \geq M$.

Now there is a contradiction. The number of b'_i that are less than b'_M is $M + 1$. However, there are at most M b_i less than b'_M since if $i \geq M$, $b_i > b'_i \geq b'_M$. But as sets $B_j = B'_j$. \square

Corollary 2.6.1. *D_j contains every non-negative integer for every $J \geq 0$. That is for all $j \geq$ and $d \geq 0$, there is a pair $(a, b) \in T_j$ so that $a - b = d$.*

However, no direct formula for computing the \mathcal{G} values is known. (32).

2.7 Additive periodicity

It is obviously impossible to compute arbitrary rows of \mathcal{G} on a finite state machine. The machine would have to remember values that grow without bound. The number of bits to

store would eventually be more than the memory of the machine. It also appears that the FSM would have to remember an every growing number of values in order to take their mex. This is not the case. The following lemmas will be useful overcoming these obstacles and will allows us to use an FSM to analyze Wythoff's nim.

Lemma 2.7.1. $\mathcal{G}(m, n) \leq m + n$.

Proof. Use induction on $m + n$. First, $\mathcal{G}(0, 0) = 0 \leq 0 + 0$. Now assume that for all i, j such that $i + j < m + n$, $\mathcal{G}(i, j) \leq i + j$. When calculating $\mathcal{G}(m, n)$, the set of values excluded, E , contains only values that are less than $m + n$. Thus $\mathcal{G}(m, n) = \text{mex}(A) \leq \max(A) + 1 \leq (m + n - 1) + 1 = m + n$ \square

Lemma 2.7.2. $\mathcal{G}(m, n) \geq m - 2n$.

Proof. Suppose $g = \mathcal{G}(m, n) < m - 2n$. Thus g did not appear as any $\mathcal{G}(k, n)$ for $k < m$ (a total of m times). For a given k , there are three reasons that g would not appear. Either $\mathcal{G}(k, n) < g$, which can happen at most g tomes, or g cannot appear because some $\mathcal{G}(k, j) = g$, for $j < n$, or g cannot appear because some $\mathcal{G}(k - i, n - i) = g$ for $0 \leq i \leq \min(k, n)$. But g can appear once in any row, the second and third reasons can happen at most n times each. The total number of times g does not occur must be no more than $g + 2n < m - 2n + 2n = m$, a contradiction. \square

Proposition 2.7.1. *Every g must appear in every column. That is for every pair g, m there is an n such that $\mathcal{G}(m, n) = g$.*

Proof. Suppose for some g , there is a column, j such that g does not appear in column j . In order not to place g at $i \cdot j$, either g is already in row column j , i , or the diagonal or ij or there is an $m < g$ not already in column j . Since it is necessary to avoid placing g in column, j , look at the later three cases. The most times one can avoid g using the fact it has already appeared is g times, since there j columns before column j , but perhaps, they are aligned so that the next j entries of column j are excluded by the fact that their diagonal contains a g . Next, one can exhaust each element less than g . Thus, after $2j + g$ entries in column j , one must have g as an entry. Thus g appears in each column, for all g . \square

Remark 2.7.3. Not only does each g appear in each column, but the above lemma bound the row, m it appears in column n : $g - n \leq m \leq g + 2n$.

Theorem 2.7.2. *The n th row of the table for Wythoff's nim name be computed by an FSM with $O(n^2)$ bits of state.*

Proof. Define $\mathcal{H}(m, n) = (G)(m, n) - m + 2n$. The above lemmas tell us that $0 \leq \mathcal{H}(m, n) \leq 3n$. Clearly, knowing \mathcal{H} allows us to compute \mathcal{G} and \mathcal{G} is additively periodic if and only if \mathcal{H} is. The values of \mathcal{H} are bounded. Using \mathcal{H} allows us to solve the problem of the unboundedness of \mathcal{G} . It also overcomes the problem of having to store more and more numbers in order to take their mex.

Define the Left, Slanting and Down sets as follows. Let $L(m, n)$ be the set of integers which appear to the left of $\mathcal{G}(m, n)$. That is $L(m, n) = \{\mathcal{G}(m - k, n) | 1 \leq k \leq m\}$. Similarly, define $S(m, n) = \{\mathcal{G}(m - k, n - k) | 1 \leq k \leq \min(m, n)\}$ and $D(m, n) = \{\mathcal{G}(m, n - k) | 0 \leq k \leq n\}$. So L contains the \mathcal{G} values that correspond to removing counters from pile m , D to values that correspond to removing counters from pile n , and S to values corresponding to removing from both piles.

\mathcal{G} and \mathcal{H} can be calculated from L, S, D since $E(m, n) = L(m, n) \cup S(m, n) \cup D(m, n)$ is the set of elements excluded when calculating \mathcal{G} . That is $\mathcal{G} = \text{mex}(E)$. Both S, D have no more than n elements each. However, $L(m, n)$ grows arbitrarily large as n increases, therefore it cannot be directly held in an FSM, as S, D can.

By the lemmas the candidates for \mathcal{G} are in $\{m - 2n, \dots, m + n\}$ and are not in $L(m, n)$, thus define $L'(m, n) = \{m - 2n, \dots, m + n\} - L(m, n)$. This is the set of all numbers less than $m + n$ that are not in $L(m, n)$. Now $L'(m, n)$ can be represented as a bit array of $3n + 1$ bits for all m . This bit array indicates with a 1 the elements of $\{m - 2n, \dots, m + n\}$ that are in $L'(m, n)$. Similarly, construct $S'(m, n), D'(m, n)$. By definition, $G = \text{mex}(L(m, n) \cup D(m, n) \cup S(m, n)) = \min(L'(m, n) \cap D'(m, n) \cap S'(m, n))$.

To compute $\mathcal{H}(m, n)$ it is necessary to keep track of $L'(m, 0), L'(m, 1), \dots, L'(m, n), S'(m, 0), S'(m, 1), \dots, S'(m, n), D'(m, 0), \dots, D'(m, n)$. However $D'(m, 0) \subset D'(m, 1) \subset \dots \subset D'(m, n)$, so one only needs to look at $D'(m, n)$ among the D' 's.

There are $O(n)$ sets consisting of $O(n)$ bits each, for a total number of $O(n^2)$ bits.

Only $L'(m, n), S'(m, n), D'(m, n)$ are needed to compute $\mathcal{H}(m, n)$ and $L'(m + 1, n)$ but the others are needed to compute $S'(m + 1, n)$ and $D'(m + 1, n)$ which are in turn necessary to compute $\mathcal{H}(m + 1, n)$. To compute $\mathcal{H}(m, n)$ find the first entry in each of L', D', S' that is a one in each. To compute $L'(m + 1, k)$ take $L'(m, k)$ as stored in the bit array, unset the bit corresponding to $\mathcal{H}(m, k)$, shift over 1 and set the end bit, which corresponds to $m + k + 1$ to 1. By the Lemma 2.7.2, this never shifts a 1 off the end of the other end, so the number of set bits in $L'(m, k) = k + 1$ is constant as m varies. Similarly, $S'(m + 1, k)$ is calculated from $S'(m, k - 1)$ and $\mathcal{H}(m, k)$. Finally $D'(m + 1, k)$ can be computed starting with $k = 0$ and working up.

In addition to the above requirements, one may require some counting states which count $k = 0$ to n , but each of these only requires $O(\log(n))$ bits. Therefore the n th row can be computed by a FSM using $O(n^2)$ bits of state. \square

Corollary 2.7.4. $\mathcal{H}(m, n)$ must be eventually periodic for fixed n .

Proof. Let b be the total number of states in the machine. After 2^b steps, the machine must reenter a configuration previously visited and loop afterwards. \square

Corollary 2.7.5. $\mathcal{G}(m, n)$ is additively periodic for fixed n and m varying.

Proof. Since $\mathcal{H}(m, n)$ is periodic, $\mathcal{G}(m, n) = \mathcal{H}(m, n) + m - 2n$ must be additively periodic. \square

2.8 Conclusion

Although Wythoff's game is a rather simple modification of nim, it appears to be more difficult to fully analyze. The computation of nim values poses a particular problem. Research is ongoing. The results in this chapter demonstrate a minimal difficulty for Wythoff quasigroups.

CHAPTER 3. Quasigroup Theory

3.1 Introduction

This chapter provides the necessary background on quasigroups so that the ordinary reader can understand the results of later chapters. Quasigroups are defined and the multiplication group associated with quasigroups are explained. The concept of isotopy is introduced and quasigroup conjugates are explained.

3.2 Definitions

In algebra there is the concept of a group, a set with an associative binary operation with inverses. This concept can be generalized by only requiring the operation to be bijective both from the right and left. The operation does not need to be associative. Such a set with the operation, typically called *multiplication*, denoted \cdot or by juxtaposition, is called a *quasigroup*. All groups are quasigroups. However, the set of integers with the subtraction operation is not a group, since it is not associative, but it is a quasigroup. A quasigroup can be expressed as the ordered pair consisting of the set and the operation, (Q, \cdot) . A quasigroup with a two-sided identity is called a *loop*.

One can define two maps on a quasigroup, left and right multiplication by an element.

Definition 3.2.1. Let (Q, \cdot) be a quasigroup. The map $R : Q \rightarrow Q!; x \mapsto R(x)$ defines the *right multiplication*. Similarly one can define *left multiplication*.

$R(x), L(x)$ are permutations of the quasigroup for all $x \in Q$.

Proposition 3.2.2. *The maps R and L is are injections.*

Proof. First note that from the definition of a quasigroup left multiplication is a bijection.

Now:

$$qR(x) = qR(y) \Rightarrow qx = qy \Rightarrow xL(q) = yL(q) \Rightarrow x = y$$

Similarly since right multiplication is a bijection, L is an injection. \square

The disjoint union of the images of R and L denoted $R(Q) \uplus L(Q)$ is the generating set for a free group. We denote this free group \tilde{G} or $\text{UMlt}(Q, \cdot)$. This group is known as the *universal multiplication group* of Q .

We can extend the embeddings of $R(Q) \hookrightarrow Q!$ and $L(Q) \hookrightarrow Q!$ by extending their disjoint union $(L(Q) \hookrightarrow Q!) \uplus (R(Q) \hookrightarrow Q!)$ to a group homomorphism $\tilde{G} \rightarrow Q!$ by using the freeness of \tilde{G} . The image of this homomorphism may not be all of $Q!$. The image of the homomorphism is called the multiplication group of Q , denoted by $\text{Mlt}(Q, \cdot) = G$. G is the subgroup of $Q!$ generated by $L(Q) \cup R(Q)$. (Note that this union is not necessarily disjoint.)

3.3 Quasigroup homomorphisms

Although homomorphic images of groups are groups, this is not true in general for quasigroups. In order to study quasigroups together with homomorphism, another, equivalent definition of a quasigroup is necessary.

Definition 3.3.1. Consider a quasigroup (Q, \cdot) , one can introduce two new operations on the quasigroup, *left division* and *right division*.

Right division is the operation $/ : Q^2 \rightarrow Q; (x, y) \mapsto x/y = xR(y)^{-1}$

Left division is the operation $\backslash : Q^2 \rightarrow Q; (y, x) \mapsto y \backslash x = xL(y)^{-1}$

Right division undoes multiplication on the right, while left division undoes multiplication on the left. If Q is commutative, $x/y = y \backslash x$. But it is not true in general that $x/y = x \backslash y$.

Consider a set Q equipped with the operations $\cdot, /, \backslash$.

Proposition 3.3.2. *The set $(Q, \cdot, /, \backslash)$ satisfies the following:*

$$IL : y \backslash (y \cdot x) = x \quad IR : x = (x \cdot y) / y$$

$$SL : y \cdot (y \setminus x) = x \quad SR : x = (x/y) \cdot y$$

The quasigroup (Q, \cdot) is called an *combinatorial quasigroup*; and the quasigroup $(Q, \cdot, /, \setminus)$ is called a *equational quasigroup*. The concepts are equivalent.

Proposition 3.3.3. *A set with multiplication is a quasigroup if and only if it carries left and right divisions satisfying Proposition 3.3.2.*

It is useful to consider the left and right divisions as well as the multiplication operation when considering a quasigroup.

Definition 3.3.4. A *quasigroup homomorphism* $\phi : Q \rightarrow P$ is a set map between Q and P so that $(xy)\phi = x\phi \cdot y\phi$; $(x/y)\phi = x\phi/y\phi$ and $(x \setminus y)\phi = x\phi \setminus y\phi$

One can look at a subset of Q and see if it is still a quasigroup, but closure is needed in all three operations.

Definition 3.3.5. A subset S of Q is a *subquasigroup* if and only if S is closed under all three operations, $\cdot, /, \setminus$, of Q . We write $S \leq Q$.

Quasigroups are often referred to as “non-associative groups.” This is actually a fairly accurate description.

Proposition 3.3.6. *An associative quasigroup is a group.*

Proof. Let Q be an associative quasigroup and suppose $a \in Q$. Let $a \setminus a = e$. Now consider ex . Since Q is a quasigroup there is a b so that $x = ab$. So $ex = (a \setminus a)x = (a \setminus) \cdot ab = ((a \setminus a)a)b = ab = x$. So e is a left identity for Q . Similarly, there is a right identity f . Now $e = ef = f$ so the identity is a two-sided identity. Now since Q is a quasigroup, given x there is a y so that $xy = e$. Also $(yx)y = y(xy) = ye = y$ so $yx = e$ and y is a two-sided inverse to x . Thus Q is a group since it has a two-sided identity with inverses. \square

3.4 Quasigroup congruences

Definition 3.4.1. A *congruence* on a quasigroup Q is an equivalence relation, α on Q so that $\alpha \leq Q^2$. The *quotient* Q^α of quasigroup Q by congruence α forms the quasigroup $(Q^\alpha, \cdot, /, \setminus)$

on the equivalence classes of Q with well defined operations $x^\alpha \cdot y^\alpha = (x \cdot y)^\alpha$; $x^\alpha / y^\alpha = (x / y)^\alpha$ and $x^\alpha \backslash y^\alpha = (x \backslash y)^\alpha$. A quasigroup is *simple* if Q^2, \hat{Q} are the only congruences of Q .

It is desirable to show that quasigroups behave nicely. Look at congruence relations on quasigroups. To discuss the congruence relations on a quasigroup, introduce a special class of elements of the multiplication group of Q . Let

$$\rho(y, z) = R(y \backslash y)^{-1} R(y \backslash z)$$

Now since $yR(y \backslash y) = y \cdot (y \backslash y) = y$ one has that $y = yR(y \backslash y)R(y \backslash y)^{-1} = yR(y \backslash y)^{-1}$. Thus $y\rho(y, z) = yR(y \backslash y)^{-1}R(y \backslash z) = yR(y \backslash z) = z$. Also $\rho(y, y) = R(y \backslash y)^{-1}R(y \backslash y) = 1$.

We are ready for a new operation based on ρ :

$$(x, y, z)P = x\rho(y, z)$$

From this definition it can be seen that $(y, y, z)P = (z, y, y) = z$ for all $y, z \in Q$.

Lemma 3.4.2. *The operation P preserves quasigroup congruences. That is if $x_i \alpha y_i$ for $1 \leq i \leq 3$, then $(x_1, x_2, x_3)P \alpha (y_1, y_2, y_3)P$.*

Proof. $(x_1, x_2, x_3)P = x_1\rho(x_2, x_3) = x_1R(x_2 \backslash x_2)^{-1}R(x_2 \backslash x_3) = (x_1 / (x_2 \backslash x_2)) \cdot (x_2 \backslash x_3)$. Now since congruences are preserved by each $\cdot, /$ and \backslash , the lemma is proven. \square

We can find the *relation product* \circ of two relations, α, β , as follows:

$$x\alpha \circ \beta y \Leftrightarrow \exists z. x\alpha z \beta y$$

Definition 3.4.3. Congruence relations are said to be *permutable* if $\alpha \circ \beta = \beta \circ \alpha$.

Proposition 3.4.4. *The congruence relations on a quasigroup are permutable.*

Proof. Let Q be a quasigroup and let α and β be congruence relations on Q with $x\alpha y$ and $y\beta z$.

Now since $x\alpha x\beta x$ and $z\alpha z\beta z$, one has that $z = (x, x, z)P\alpha(x, y, z)p\beta(x, z, z)P = x$, so $z\alpha \circ \beta x$. Thus $x\alpha \circ \beta z$ implies $x\beta \circ \alpha z$. Similarly, $x\beta \circ \alpha z$ implies $x\alpha \circ \beta z$, so $\alpha \circ \beta = \beta \circ \alpha$. \square

An interesting effect of the P operation is the following characterization of quasigroup congruences.

Proposition 3.4.5. *Let Q be a quasigroup. Then a subquasigroup of Q^2 is a congruence of Q if and only if it contains the diagonal subquasigroup \hat{Q} .*

Proof. A congruence is a reflexive relation and therefore contains the diagonal. Conversely, suppose that $\hat{Q} \leq \alpha \leq Q^2$. It must be shown that α is symmetric and transitive. If $x\alpha y$, one has $y = (x, x, y)P\alpha(x, y, y)P = x$, so $y\alpha x$. Lastly, if $x\alpha y$ and $y\alpha z$, one has $x = (x, y, y)P\alpha(y, y, z)P = z$, so $x\alpha z$. Thus α is a congruence. \square

3.5 Conjugates

The combinatorial quasigroup (Q, \cdot) gives an equational quasigroup $(Q, \cdot, /, \backslash)$ which in turn gives two combinatorial quasigroups $(Q, /)$ and (Q, \backslash) . One can also consider multiplication in the opposite order: $x \circ y = y \cdot x$ which is also a quasigroup: $(Q, circ)$ and is denoted $(Q, \cdot)^{op}$. This gives two more combinatorial quasigroups corresponding to the left and right divisions: $(Q, //), (Q, \backslash\backslash)$. Given $x \cdot y = z$ in (Q, \cdot) and permutation in S_3 corresponds to one of the six conjugates.

3.6 Isotopy

Definition 3.6.1. Given quasigroups (Q, \cdot) and $(R, *)$ *quasigroup homotopy* (θ, ϕ, ψ) is a triple of set maps $Q \rightarrow R$ with $x\theta * y\phi = (x \cdot y)\psi$. A *quasigroup isotopy* is a homotopy where each component bijects. In this case one says the quasigroups are *isotopic* and one writes $Q \sim R$. We say Q, R are *isotopes*. One says Q is *principally isotopic* to R if ψ is the identity.

Every quasigroup is isotopic to a loop.

Proposition 3.6.2. *Every quasigroup with element e is principally isotopic to a loop with element e .*

Theorem 3.6.1. *Every isotope of a quasigroup, Q , is isomorphic to a principal isotope of Q .*

Proof. Let θ, ϕ, ψ be the bijections of Q onto R which define the isotopism between (Q, \cdot) and $(R, *)$ so that $(x\theta) * (y\phi) = (x \cdot y)\psi$ for all x, y in Q . Then $\psi\theta^{-1}$ and $\psi\phi^{-1}$ are bijections from Q to Q , so the operation \otimes given by $(x\psi\theta^{-1}) \cdot (y\psi\phi^{-1})$ defines a principal isotope of Q . Now $(x\psi) * (y\psi) = (x\psi\theta^{-1})\theta * (y\psi\phi^{-1})\psi = (x\psi\theta^{-1} \cdot y\psi\phi^{-1})\psi = (x \otimes y)\psi$; so $(R, *)$ and (Q, \otimes) are isomorphic under $\psi : Q \rightarrow R$. \square

CHAPTER 4. Latin Squares

4.1 Introduction

This section gives the definition and some relevant results concerning latin squares. Latin squares are a combinatorial interpretation of quasigroups. This section provides background information culled from the literature so that the reader can understand results in the chapter on tri-quasigroups. These results serve as motivation and as a guide in the search for an algebraic interpretation of Wythoff quasigroups.

Definition 4.1.1. A *latin square* of order n is a square matrix with n^2 entries of n different elements, no element occurring twice in any row or column. The integer n is called the *order* of the latin square.

In this chapter, the elements of the latin square will be set to the integers $\{1, 2, \dots, n\}$. It is easy to see that a latin square forms the multiplication table for a quasigroup and any finite quasigroup gives rise to a latin square. A latin square, L , can be identified with a set of permutations (p_1, \dots, p_n) where p_i is the permutation that sends $(1, 2, \dots, n)$ to the i th row of L . Note that this is not necessarily a group.

Definition 4.1.2. The *quadrangle criterion* says that for any indices i, j, k, l it follows from $a_{jk} = a_{j_1k_1}, a_{ik} = a_{i_1k_1}, a_{il} = a_{i_1l_1}$ that $a_{il} = a_{i_1l_1}$.

Every group satisfies the quadrangle criterion since

$$\begin{aligned} a_{jl} &= a_j a_l = a_j (a_k a_k^{-1}) (a_i^{-1} a_i) a_l = (a_j a_k) (a_i a_k)^{-1} (a_i a_l) = a_{jk} a_{ik}^{-1} a_{il} \\ &= a_{j_1 k_1} a_{i_1 k_1}^{-1} a_{i_1 l_1} = (a_{j_1} a_{k_1}) (a_{i_1} a_{k_1})^{-1} (a_{i_1} a_{l_1}) = a_{j_1} a_{l_1} = a_{j_1 l_1} \end{aligned}$$

The converse is also true.

Lemma 4.1.3. *Any latin square satisfying the quadrangle criterion is a group.*

Proof. First, the identity element must be identified. If one labels the latin square by labeling the columns with the elements of the first row and similarly for the first column, the latin square is turned into the Caley table for a groupoid with an identity element, namely the element in the $(1,1)$ place. Since this element occurs exactly once in each row and column, invertibility of the operation is achieved. Now associativity must be shown.

It must be shown that $(ab)c = a(bc)$. Consider the subsquare determined by columns b, bc and e, a :

	b	bc
e	b	bc
a	ab	$a(bc)$.

Now consider the subsquare determined by rows b, ab and columns e, c :

	e	e
e	b	bc
b	ab	$(ab)c$.

By the quadrangle criterion, $a(bc) = (ab)c$. □

Definition 4.1.4. A *transversal* in a latin square of order n is a set of n elements, one in each row, one in each column, so that no element appears more than once. Two transversals that have no cells in common are said to be *parallel*.

Transversals are closely related to the concept of a complete mapping.

Definition 4.1.5. A *complete mapping* of a groupoid (G, \cdot) is a biunique mapping $x \mapsto x\theta$ of G so that the mapping $x \mapsto x\eta = x \cdot x\theta$ is also a biunique mapping of G onto G .

Proposition 4.1.6. *If quasigroup Q has a complete mapping if and only if the underlying latin square has a transversal.*

Proof. Let Q be a quasigroup on $1, 2, \dots, n$. Suppose Q has a complete mapping, say $\theta : i \mapsto a_i$ and $\eta : i \mapsto b_i$. Then Q has at least one transversal since $i \cdot a_i = b_i$ for all i , so the cell of the i th row and the a_i th column has b_i and for $i = 1, 2, \dots, n$ these are distinct.

Conversely, suppose L is a latin square with a transversal, b_1, b_2, \dots, b_n at cells $(1, a_1), \dots, (n, a_n)$. Then there is a quasigroup (Q, \cdot) with L as its Caley table for which: $1 \cdot a_1 = b_1, \dots, n \cdot a_n = b_n$ for which $\theta : i \mapsto a_i, \eta : i \mapsto b_i$ is a complete mapping. \square

Lemma 4.1.7. *If L is a latin square of order n with at least one transversal which also satisfies the quadrangle criterion, then L has a decomposition into n disjoint transversals.*

Proof. Since L satisfies the quadrangle criterion, it can be viewed as the Caley table for a group G . Suppose the transversal consists of the symbol c_1 from the first row, c_2 from the second down through the symbol c_n from the n th row. Now since G is a group, fix any $g \in G$. Now take c_1g from the first row, c_2g from the second, as before. As g varies through the n elements of the group, n disjoint transversals are formed.

To see this, suppose that $c_i = g_i g_{i'}$. That is, c_i is found in the i th row and the i' th column, since the c_i 's form a transversal i' is an injective function of i . Now since G is a group:

$$c_i g = (g_i g_{i'}) g = g_i (g_{i'} g) = g_i g_{i''} \quad (4.1)$$

where as $g_{i'}$ varies though G , so does $g_{i''}$. As a result $c_i g$ and $c_j g$ are always in distinct columns so the $g_i g$ form a transversal. Now (if $g \neq e$) $g_{i'} \neq g_{i''}$ for any i so the transversal formed by the c_i 's is disjoint from the $c_i g$'s. Similarly transversals corresponding to two different choices of g are disjoint. \square

4.2 Orthogonality

Definition 4.2.1. Two latin squares, $L_1 = (a_{ij})$ and $L_2 = (b_{ij})$ are said to be *orthogonal* if every ordered pair (a_{ij}, b_{ij}) occurs exactly once among the n^2 pairs for $i, j = 1, 2, \dots, n$. A set of pairwise orthogonal latin squares are said to be *mutually orthogonal*.

Proposition 4.2.2. *For a latin square of order n there exists at most $n-1$ mutually orthogonal latin squares.*

Proof. Without loss of generality, let the first row of each square be $1, 2, 3, \dots, n$. This accounts for the pairs $(1, 1), (2, 2), \dots, (n, n)$ in each pair of orthogonal squares. Now, consider the $(2, 1)$

position. There are at most $n - 1$ choices for this position, since 1 is in the $(1, 1)$ position and they must all be different in order for the squares to be orthogonal. \square

It is not always the case that there are $n - 1$ orthogonal squares.

It will be shown that the latin square of a cyclic group of even order has no orthogonal mate.

Lemma 4.2.3. *A latin square L has an orthogonal mate if and only if there are n pairwise parallel transversals.*

Proof. Suppose L_1 has n parallel transversals, $\ell_1, \ell_2, \dots, \ell_n$. Construct L_2 as follows. For $i = 1, 2, \dots, n$ at every cell represented by ℓ_i place an i . This is a latin square since ℓ_i is a transversal and therefore each i appears exactly once in each row and column. Moreover no ordered pair appears more than once since each i in L_2 is aligned with a transversal in L_1 so each (j, i) appears once for all i, j . \square

Theorem 4.2.4. *A latin square, L , based on a cyclic group, G , of order n has no transversal if n is even.*

Proof. Suppose L contains a transversal. Suppose this transversal is $\{g_i = p_i q_i\}$ for $i = 1, 2, \dots, n$ where $g_i, p_i, q_i \in G$. Let $G = \langle \sigma \rangle$. Now each element of G is a power of σ so write $g_i = \sigma^{a_i}$, $p_i = \sigma^{b_i}$, $q_i = \sigma^{c_i}$. So $g_i = p_i q_i$ may be written $\sigma^{a_i} = \sigma^{b_i} \sigma^{c_i}$. Note that since $\{g_i = p_i q_i\}$ is a transversal, $\{a_i\}$, $\{b_i\}$, $\{c_i\}$ are all $\{1, 2, \dots, n\}$. Therefore:

$$0 \equiv_n \sigma^{n(n+1)} = \sigma^{\sum b_i + \sum c_i} = \prod_{i=1}^n \sigma^{b_i} \sigma^{c_i} = \prod_{i=1}^n \sigma^{a_i} = \sigma^{\sum a_i} = \sigma^{\frac{1}{2}n(n+1)} \not\equiv_n 0$$

if n is even resulting in a contradiction. \square

Corollary 4.2.5. *If L is the latin square of a cyclic group of even order, it has no orthogonal mate.*

However, any group of odd order has an orthogonal mate.

Theorem 4.2.6. *The Cayley table of a group of odd order has a latin square that is its orthogonal mate.*

Two theorems regarding the existence of orthogonal mates for even order latin squares are given here. For proofs consult (17)

Theorem 4.2.7. *A finite group G of order n which has a cyclic Sylow 2-subgroup does not possess a complete mapping.*

This gives the following result.

Theorem 4.2.8. *If n is an odd multiple of two, no group of order n has an orthogonal mate.*

4.3 Pandiagonal latin squares

Definition 4.3.1. The *diagonals* of a latin square are: for each fixed i $1 \leq i \leq n$ let $0 \leq j \leq n - 1$)

- $(i, j + i)$ (Right diagonals)
- $(i, j - 1 - i)$ (Left diagonals).

The diagonals “wrap around” the latin square and are all the same length.

Definition 4.3.2. A *pandiagonal latin square* of order n is a latin square so that the every diagonal contains each element of the square exactly once.

A *left (right) semi pandiagonal latin square* is a latin square with only the left (right) diagonal criterion. Sometimes these are referred to simply as semi pandiagonal latin squares.

Pandiagonal latin squares are also referred to as: strongly diagonal latin squares (?), totally diagonal latin squares (8) and Knut Vik designs (3). Although latin squares of all orders exist, this is not the case for pandiagonal latin squares.

Theorem 4.3.3. *There is no pandiagonal latin square of even order.*

Proof. Assume there is a pandiagonal latin square, $K = (k_{ij})$, of order n for n even. Now consider the latin square of the group addition table for $Z_n = (z_{ij})$. The elements of the i th left diagonal of Z_n are all equal, while the elements on the i th left diagonal of K are distinct. Thus for every $j \in Z_n$ every (i, j) ordered pair appears exactly once in the set

(k_{ij}, z_{ij}) . Therefore K and Z_n are orthogonal. But Corollary 4.2.5 demonstrated that Z_n has no orthogonal mate for n even. This is a contradiction, so no even order K can exist \square

Corollary 4.3.4. *No semi pandiagonal latin square of even order can exist.*

Proof. The above technique shows that no left semi pandiagonal latin square exists and since every right semi pandiagonal latin square is simply the reflection of a left semi pandiagonal latin square, no right semi pandiagonal latin squares exist either. \square

Proposition 4.3.5. *There are semi pandiagonal latin squares of all odd orders.*

Proof. For n odd and $i, j = 0, 1, \dots, n-1$ let $a_{ij} = (n-2)i + j \pmod n$. This is a latin square since for fixed i , a_{ij} takes on all values as j varies. Similarly for fixed j , a_{ij} takes on all values as i varies since $(n-2)i$ takes on all such values since $(n-2), n$ are coprime.

Now, suppose the same element occurs on a diagonal, that is $(n-2)i + j \equiv_n (n-2)(i+a) + (j+a)$. Then $-a \equiv_n (n-2)a$ or $n-2 \equiv_n -1$ which is impossible. \square

Now it will be shown that no pandiagonal latin squares exist for orders divisible by 3. First some definitions and lemmas are in order.

Definition 4.3.6. A collection on n cells is a *super diagonal* if each row, column, left and right diagonal is represented exactly once in the collection. Super diagonals are *parallel* if they have no cell in common.

Lemma 4.3.7. *The necessary and sufficient conditions for a set of cells $S = \{(x_i, y_i) | i = 1, 2, \dots, n\}$ to be a super diagonal are:*

1. $\{x_i : i = 1, 2, \dots, n\} = \{1, 2, \dots, n\}$
2. $\{y_i : i = 1, 2, \dots, n\} = \{1, 2, \dots, n\}$
3. $\{y_i - x_i \pmod n : i = 1, 2, \dots, n\} = \{1, 2, \dots, n\}$
4. $\{y_i + x_i \pmod n : i = 1, 2, \dots, n\} = \{1, 2, \dots, n\}$

The proof is immediate. The first two conditions make the array a latin square, the third condition satisfies the conditions for right diagonals and the last condition satisfies the conditions for the left diagonals.

Lemma 4.3.8. *An $n \times n$ array has a super diagonal if and only if n is not divisible by 2 or 3.*

Proof. Suppose $D = \{(x_i, y_i) | i = 1, 2, \dots, n\}$ is a super diagonal, then by Lemma 4.3.7

$$\frac{n(n+1)}{2} = \sum_{i=1}^n i \equiv_n \sum_{i=1}^n (y_i - x_i) = \sum_{i=1}^n y_i - \sum_{i=1}^n x_i = 0 \quad (4.2)$$

which is impossible if n is even. Again from Lemma 4.3.7:

$$\begin{aligned} 2 \sum_{i=1}^n x_i y_i &= \sum_{i=1}^n (y_i + x_i)^2 - \sum_{i=1}^n y_i^2 + \sum_{i=1}^n x_i^2 \equiv_n - \sum_{i=1}^n i^2 \\ 2 \sum_{i=1}^n x_i y_i &= \sum_{i=1}^n x_i^2 + \sum_{i=1}^n y_i^2 - \sum_{i=1}^n (x_i - y_i)^2 \equiv_n \sum_{i=1}^n i^2 \end{aligned} \quad (4.3)$$

Thus

$$\frac{n(n+1)(2n+1)}{3} \equiv_n 0 \quad (4.4)$$

which is impossible if n is divisible by three. □

Theorem 4.3.9. *A pandiagonal square of order n exists only if and only if it is possible to find n parallel super diagonals.*

Proof. If there are n parallel super diagonals, fill each super diagonal with a fixed element, and the result is a pandiagonal latin square. If there is a pandiagonal latin square, the cells filled by each element form n parallel super diagonals. □

Corollary 4.3.10. *A pandiagonal latin square of order n exists only if n is not divisible by 2 or 3.*

CHAPTER 5. Greedy Quasigroups

5.1 Introduction

The addition tables for Nim and misère Nim as well as that of Digital Deletions were all generated by a greedy algorithm with certain initial conditions. This raises the question, what happens if one determines the initial state of the table and apply the mex-rule to generate the table? One will certainly get a quasigroup. What algebraic properties does this quasigroup have? Are new quasigroups generated when different initial conditions are specified? Are there subquasigroups? When do they appear? Finally, what initial conditions describe interesting games and can quasigroup theory be applied to already existing combinatorial games? This chapter describes the generation of greedy quasigroups and investigates various algebraic properties. The chapter concludes with generalizations of the initial concept.

5.2 Generation of greedy quasigroups

One can generate quasigroups using the mex-rule as follows. Place an element s in the multiplication table at $0 \cdot 0$. This element is called the *seed*. For each entry q_{ij} let $q_{ij} = \text{mex}(\{q_{kj}\}_{k=0}^{i-1} \cup \{q_{ik}\}_{k=0}^{j-1})$. Each quasigroup will be identified by its seed, since this seed determines the rest of the elements. So Q_s specifies the quasigroup generated with seed s . When necessary, I will specify operations in the same manner. Thus, \cdot_s is the multiplication on the quasigroup Q_s .

There are other initial conditions and restrictions that can be specified. For instance, Digital Deletions specifies that the first row cannot contain a 0. One has to be careful specifying restrictions. The table for Digital Deletions is not quite a quasigroup, since a 0 does not appear in the first column, so right multiplication by 0 is not bijective. This will be explored further.

Example 5.2.1. The first 11 rows and 11 columns of Q_2 :

2	0	1	3	4	5	6	7	8	9	10	11
0	1	2	4	3	6	5	8	7	10	9	12
1	2	0	5	6	3	4	9	10	7	8	13
3	4	5	0	1	2	7	6	9	8	11	10
4	3	6	1	0	7	2	5	11	12	13	8
5	6	3	2	7	0	1	4	12	11	14	9
6	5	4	7	2	1	0	3	13	14	12	15
7	8	9	6	5	4	3	0	1	2	15	14
8	7	10	9	11	12	13	1	0	3	2	4
9	10	7	8	12	11	14	2	3	0	1	5
10	9	8	11	13	14	12	15	2	1	0	3
11	12	13	19	8	9	15	14	4	4	3	0

Table 5.1 Part of the table for Q_2

By their construction, greedy quasigroups are commutative. However greedy quasigroups are not associative. In Q_s , $(0 \cdot 0) \cdot s + 1 = s \cdot s + 1 \neq 0 \cdot s + 1 = 0 \cdot (0 \cdot s + 1)$ (for $s \neq 0$). However, there are associating triples. Commutativity tells us that $(ab)a = a(ab) = a(ba)$. In fact, many triples are associating, and many are not.

5.3 Column structure of greedy quasigroups.

One can start analyzing these quasigroups by looking at their columns. Some interesting patterns are seen in the first few columns. In this thesis, the first column is the 0th column since it is the column representing right multiplication by 0.

Lemma 5.3.1. For $0 < x \leq s$, $s \cdot 0 = x - 1$. For $x > s$, $x \cdot 0 = x$.

Proof. Since $0 \cdot 0 = s$, $1 \cdot 0 = 0$, and applying the mex rule to each successive term, one has that $x \cdot 0 = \text{mex}\{s, 0, 1, \dots, 0 \cdot x - 1 = x - 2\} = x - 1$. For $x = s + 1$, $0 \cdot x = \text{mex}\{s, 0, 1, \dots, s - 1\} = s + 1$. Thus by induction, one can see that $0 \cdot x = x$ for $x > s$. \square

Lemma 5.3.2. For $0 \leq x \leq s$, $x \cdot 1 = x$. For $x > s$, $x \cdot 1 = \begin{cases} x + 1 & x - s \equiv_2 1 \\ x - 1 & x - s \equiv_2 0 \end{cases}$.

Proof. $0 \cdot 1 = 0$ (for $s \neq 0$). Then by induction, for $x \leq s$: $x \cdot 1 = \text{mex}\{0, 1, \dots, x-1, 0 \cdot x = x-1\} = x$. For $x > s$:

$$s+1 \cdot 1 = \text{mex}\{0, 1, \dots, s, (s+1) \cdot 0 = s+1\} = s+2.$$

$$s+2 \cdot 1 = \text{mex}\{0, 1, \dots, s, s+2, s+2 \cdot 0\} = s+1.$$

So, by induction, for $x-s \equiv_2 1$, $x \cdot 1 = \text{mex}\{0, 1, \dots, x-1, x \cdot 0\} = x+1$, and for $x-s \equiv_2 0$ $x \cdot 1 = \text{mex}\{0, 1, 2, \dots, x-3+1, x-2-1, x-1+1, x \cdot 0\} = x-1$. \square

Remark 5.3.3. From these lemmas one can see some sort of identity structure. While there is no identity element in Q_s , $x \cdot 1 = x$ for $x \leq s$ and $y \cdot 0 = y$ for $y > s$.

From these lemmas, the following conclusion can be drawn:

Theorem 5.3.4. For $x \geq 2$, $x \cdot x = 0$.

Proof. $0 \cdot 1 = 0 = 1 \cdot 0$. Thus the first place a 0 can appear in the second column is the second row, so it must appear there. Then the first place zero can and must appear in the third column is the third row. Fill in the first n columns by induction. The first place 0 can appear in the $n+1$ st column is in the $n+1$ st row. Thus, by induction $n \cdot n = 0$ for all $n \geq 2$. \square

Thus there is a unique element that is the square of infinitely many elements of any greedy quasigroup. This element is identified with zero. An element is said to be *nilpotent* if its square is 0. In fact, 0 is the only element that is the square of more than one element.

This fact is very important and plays a key role in most of the proofs in this paper.

Remark 5.3.5. It appears that at some point, the first $n+1$ elements in a column are precisely the numbers $0, 1, \dots, n$. When this happens, I say the column is *complete at entry n* .

For $x \cdot 2$, the structure is a bit less organized, since this column depends on the first two columns. Nevertheless, it can still be worked out. This column's structure allows us to discuss the possibility of subquasigroups. The structure of the second column depends on the congruence class of the seed mod 3.

Lemma 5.3.6. For $x < s$, $x \cdot 2 = \begin{cases} x + 1 & x \equiv_3 0, 1 : \\ x - 2 & x \equiv_3 2. \end{cases}$

Proof. First, one has that $0 \cdot 2 = \text{mex}\{s, 0\} = 1$; $1 \cdot 2 = \text{mex}\{0, 1, 1\} = 2$; $2 \cdot 2 = 0$.

Now, by induction,

$$\begin{aligned} 3n \cdot 2 &= \text{mex}(\{3n \cdot 0, 3n \cdot 1\} \cup \{(3n - i) \cdot 2\}_{i=1}^{3n}) \\ &= \text{mex}(\{3n - 1, 3n\} \cup \{(3n - 3i) \cdot 2\}_{i=1}^n \\ &\quad \cup \{(3n - 3i + 1) \cdot 2\}_{i=1}^n \cup \{(3n - 3i + 2) \cdot 2\}_{i=1}^n) \\ &= \text{mex}(\{3n - 1, 3n\} \cup \{3n - 3i + 1\}_{i=1}^n \cup \{3n - 3i + 2\}_{i=1}^n \cup \{3n - 3i\}_{i=1}^n) \\ &= 3n + 1. \end{aligned}$$

$$\begin{aligned} (3n + 1) \cdot 2 &= \text{mex}(\{3n, 3n + 1\} \cup \{(3n + 1 - i) \cdot 2\}_{i=1}^{3n+1}) \\ &= \text{mex}(\{3n, 3n + 1\} \cup \{(3n + 1 - (3i + 1)) \cdot 2\}_{i=0}^n \\ &\quad \cup \{(3n + 1 - (3i - 1)) \cdot 2\}_{i=1}^n \cup \{(3n + 1 - 3i) \cdot 2\}_{i=1}^n) \\ &= \text{mex}(\{3n, 3n + 1\} \cup \{3n - 3i + 1\}_{i=0}^n \cup \{3n - 3i\}_{i=1}^n \cup \{3n - 3i + 2\}_{i=1}^n) \\ &= 3n + 2. \end{aligned}$$

$$\begin{aligned} (3n + 2) \cdot 2 &= \text{mex}(\{3n + 1, 3n + 2\} \cup \{(3n + 2 - i) \cdot 2\}_{i=1}^{3n+2}) \\ &= \text{mex}(\{3n + 1, 3n + 2\} \cup \{(3n + 2 - (3i + 1)) \cdot 2\}_{i=1}^n \\ &\quad \cup \{(3n + 2 - (3i + 2)) \cdot 2\}_{i=0}^n \cup \{(3n + 2 - 3i) \cdot 2\}_{i=1}^n) \\ &= \text{mex}(\{3n + 1, 3n\} \cup \{3n - 3i + 2\}_{i=0}^n \cup \{3n - 3i + 1\}_{i=0}^n \cup \{3n - 3i\}_{i=1}^n) \\ &= 3n. \end{aligned}$$

□

Remark 5.3.7. For $3n + 2 < s$, $\{3n \cdot 2, 3n + 1 \cdot 2, 3n + 2 \cdot 2\} = \{3n + 1, 3n + 2, 3n\}$, so after each additional set of three terms, the column becomes complete again.

The post-seed behavior of the second column depends on the equivalence class of the seed mod 3. I will take each one in turn.

Lemma 5.3.8. *The structure of column 2 after the seed is as follows:*

For $s \equiv_3 0$ and $s \equiv_3 1$ and $x > s + 1$:

$$x \cdot 2 = \begin{cases} x + 1 & x - s \equiv_2 0 : \\ x - 1 & x - s \equiv_2 1. \end{cases}$$

Proof. For $s \equiv_3 0$:

$$s \cdot 2 = s + 1 \text{ from above;}$$

$$s + 1 \cdot 2 = \text{mex}(\{s + 1 \cdot 0, s + 1 \cdot 1\} \cup \{s \cdot 2\} \cup \{(s - i) \cdot 2\}_{i=1}^s)$$

$$= \text{mex}(\{s + 1, s + 2, s + 1\} \cup \{s - i\}_{i=1}^s) = s;$$

$$s + 2 \cdot 2 = \text{mex}(\{s + 2 \cdot 0, s + 2 \cdot 1\} \cup \{s + 1 \cdot 2, s \cdot 2\} \cup \{(s - i) \cdot 2\}_{i=1}^s)$$

$$= \text{mex}(\{s + 2, s + 1, s, s + 1\} \cup \{s - i\}_{i=1}^s) = s + 3;$$

$$s + 3 \cdot 2 = \text{mex}(\{s + 3 \cdot 0, s + 3 \cdot 1\} \cup \{s \cdot 2, s + 1 \cdot 2, s + 2 \cdot 2\} \cup \{(s - i) \cdot 2\}_{i=1}^s)$$

$$= \text{mex}(\{s + 3, s + 4, s + 1, s, s + 3\} \cup \{s - i\}_{i=1}^s) = s + 2.$$

At this point, the column is complete. Since this column depends on the 0th and 1st columns and the 1st column depends on the distance from the seed mod 2, one can replace $s + 2$ and $s + 3$ by the congruence classes of their distance from the seed mod 2 and repeat the argument.

For $s \equiv_3 1$:

$$s + 1 \cdot 2 = \text{mex}(\{s + 1, s + 2\} \cup \{s - i\}_{i=2}^s \cup \{s - 1 \cdot 2, s \cdot 2\})$$

$$= \text{mex}(\{s + 1, s + 2, s, s + 1\} \cup \{s - i\}_{i=2}^s) = s - 1.$$

Note that the column is complete at this point.

$$s + 2 \cdot 2 = \text{mex}(\{s + 2 \cdot 0, s + 2 \cdot 1\} \cup \{(s + 2 - i) \cdot 2\}_{i=1}^{s+2})$$

$$= \text{mex}(\{s + 2, s + 1\} \cup \{s + 2 - i\}_{i=1}^{s+2}) = s + 3.$$

$$s + 3 \cdot 2 = \text{mex}(\{s + 3 \cdot 0, s + 3 \cdot 1\} \cup \{(s + 3 - i) \cdot 2\}_{i=1}^{s+3})$$

$$= \text{mex}(\{s + 3, s + 4, s + 3\} \cup \{s + 2 - i\}_{i=1}^{s+2}) = s + 2.$$

Again, the column is complete at this point. One can replace $s + 2, s + 3$ with the congruence classes of their distance from the seed mod 2 and repeat the argument. \square

Lemma 5.3.9. For $s \equiv_3 2$, and $x > s$, $x \cdot 2 = \begin{cases} x + 2 & x - s \equiv_4 1, 2; \\ x - 2 & x - s \equiv_4 3, 0. \end{cases}$

Proof. First note that $\{(s - i) \cdot 2\}_{i=0}^2 = \{s - i\}_{i=0}^s$. Then:

$$\begin{aligned} s + 1 \cdot 2 &= \text{mex}(\{s + 1 \cdot 0, s + 1 \cdot 1\} \cup \{(s + 1 - i) \cdot 2\}_{i=1}^{s+1}) \\ &= \text{mex}(\{s + 1, s + 2\} \cup \{s - i\}_{i=0}^s) = s + 3; \\ s + 2 \cdot 2 &= \text{mex}(\{s + 2 \cdot 0, s + 1 \cdot 1\} \cup \{(s + 2 - i) \cdot 2\}_{i=1}^{s+2}) \\ &= \text{mex}(\{s + 2, s + 1, s + 3\} \cup \{s - i\}_{i=0}^s) = s + 4; \\ s + 3 \cdot 2 &= \text{mex}(\{s + 3 \cdot 0, s + 3 \cdot 1\} \cup \{s + 3 - i\}_{i=1}^{s+3}) \\ &= \text{mex}(\{s + 3, s + 4, s + 3, s + 4\} \cup \{s - i\}_{i=0}^s) = s + 1; \\ s + 4 \cdot 2 &= \text{mex}(\{s + 4 \cdot 0, s + 4 \cdot 1\} \cup \{s + 4 - i\}_{i=1}^{s+4}) \\ &= \text{mex}(\{s + 4, s + 3, s + 3, s + 4, s + 1\} \cup \{s - i\}_{i=0}^s) = s + 2. \end{aligned}$$

At this point, the column is complete. Now one can replace x with the congruence class of its distance from the seed mod 4 and repeat this argument. \square

Column 3 is the last column that I will analyze in this paper. Its structure is slightly more difficult than the previous columns. I am going to look at column 3 for $s \equiv_3 2$ only, since this is the only case I will need for future theorems. For each of these lemmas, suppose that the seed is large.

Lemma 5.3.10. *Some preliminary calculations:* $s \cdot 3 = 2; 1 \cdot 3 = 3, 2 \cdot 2 = 4, 3 \cdot 3 = 0, 4 \cdot 3 = 1.$

Proof.

$$\begin{aligned} 0 \cdot 3 &= \text{mex}(\{0 \cdot 0, 0 \cdot 1, 0 \cdot 2\}) = \text{mex}(\{s, 0, 1\}) = 2. \\ 1 \cdot 3 &= \text{mex}(\{1 \cdot 0, 1 \cdot 1, 1 \cdot 2, 0 \cdot 3\}) = \text{mex}(\{0, 1, 2, 2\}) = 3. \\ 2 \cdot 3 &= \text{mex}(\{2 \cdot 0, 2 \cdot 1, 2 \cdot 2, 0 \cdot 3, 1 \cdot 3\}) = \text{mex}(\{1, 2, 0, 2, 3\}) = 4. \end{aligned}$$

$$3 \cdot 3 = 0.$$

$$4 \cdot 3 = \text{mex}(\{4 \cdot 0, 4 \cdot 1, 4 \cdot 2, 0 \cdot 3, 1 \cdot 3, 2 \cdot 3, 3 \cdot 3\}) = \text{mex}(\{3, 4, 5, 2, 3, 4, 0\}) = 1.$$

□

Remark 5.3.11. At this point column 3 is complete.

Lemma 5.3.12. For $5 \leq x \leq s$:

$$x \cdot 3 = \begin{cases} x + 1 & x \equiv_9 5, 8; \\ x + 2 & x \equiv_9 6, 1, 2; \\ x - 2 & x \equiv_9 7, 0, 4; \\ x - 1 & x \equiv_9 3. \end{cases}$$

Remark 5.3.13. After each ninth step the column becomes complete.

Proof. Suppose this pattern holds up to x , where $x \equiv_9 5$ and $x < s$. (Suppose also that $x + 9 < s$.) Note that $x \equiv_3 2$.

$$x \cdot 3 = \text{mex}(\{x \cdot 0, x \cdot 1, x \cdot 2\} \cup \{x - i \cdot 3\}_{i=1}^x)$$

$$= \text{mex}(\{x - 1, x, x - 2\} \cup \{x - i\}_{i=1}^x) = x + 1;$$

$$x + 1 \cdot 3 = \text{mex}(\{x + 1 \cdot 0, x + 1 \cdot 1, x + 1 \cdot 2\} \cup \{x + 1 - i \cdot 3\}_{i=1}^{x+1})$$

$$= \text{mex}(\{x, x + 1, x + 2, x + 1\} \cup \{x - i\}_{i=1}^x) = x + 3;$$

$$x + 2 \cdot 3 = \text{mex}(\{x + 2 \cdot 0, x + 2 \cdot 1, x + 2 \cdot 2\} \cup \{x + 2 - i \cdot 3\}_{i=1}^{x+2})$$

$$= \text{mex}(\{x + 1, x + 2, x + 3, x + 1, x + 3\} \cup \{x - i\}_{i=1}^x) = x;$$

$$x + 3 \cdot 3 = \text{mex}(\{x + 3 \cdot 0, x + 3 \cdot 1, x + 3 \cdot 2\} \cup \{x + 3 - i \cdot 3\}_{i=1}^{x+3})$$

$$= \text{mex}(\{x + 2, x + 3, x + 1, x + 1, x + 3, x\} \cup \{x - i\}_{i=1}^x) = x + 4;$$

$$x + 4 \cdot 3 = \text{mex}(\{x + 4 \cdot 0, x + 4 \cdot 1, x + 4 \cdot 2\} \cup \{x + 4 - i \cdot 3\}_{i=1}^{x+4})$$

$$= \text{mex}(\{x + 3, x + 4, x + 5, x + 1, x + 3, x, x + 4\} \cup \{x - i\}_{i=1}^x) = x + 2.$$

At this point column 3 is complete again.

$$\begin{aligned}
x + 5 \cdot 3 &= \text{mex}(\{x + 5 \cdot 0, x + 5 \cdot 1, x + 5 \cdot 2\} \cup \{x + 5 - i \cdot 3\}_{i=1}^{x+5}) \\
&= \text{mex}(\{x + 4, x + 5, x + 6\} \cup \{x + 4 - i\}_{i=1}^{x+4}) = x + 7; \\
x + 6 \cdot 3 &= \text{mex}(\{x + 6 \cdot 0, x + 6 \cdot 1, x + 6 \cdot 2\} \cup \{x + 6 - i \cdot 3\}_{i=1}^{x+6}) \\
&= \text{mex}(\{x + 5, x + 6, x + 4, x + 7\} \cup \{x + 4 - i\}_{i=1}^{x+4}) = x + 8; \\
x + 7 \cdot 3 &= \text{mex}(\{x + 7 \cdot 0, x + 7 \cdot 1, x + 7 \cdot 2\} \cup \{x + 7 - i \cdot 3\}_{i=1}^{x+7}) \\
&= \text{mex}(\{x + 6, x + 7, x + 8, x + 7, x + 8\} \cup \{x + 4 - i\}_{i=1}^{x+4}) = x + 5; \\
x + 8 \cdot 3 &= \text{mex}(\{x + 8 \cdot 0, x + 8 \cdot 1, x + 8 \cdot 2\} \cup \{x + 8 - i \cdot 3\}_{i=1}^{x+8}) \\
&= \text{mex}(\{x + 7, x + 8, x + 9, x + 7, x + 8, x + 5\} \cup \{x + 4 - i\}_{i=1}^{x+4}) = x + 6.
\end{aligned}$$

At this point, column three is complete again. The next calculation to consider is $x + 9 \cdot 3$, and the pattern hold by induction. \square

Now, if one knows that $s \equiv_3 2$, then $s \equiv_9 2, 5, 8$. Each case yields a different pattern after the row containing the seed.

Lemma 5.3.14. *For $s \equiv_9 2$:*

$$x \cdot 3 = \begin{cases} x - 2 & x - s \equiv_4 1, 2; \\ x + 2 & x - s \equiv_4 3, 0. \end{cases}$$

Proof.

$$\begin{aligned}
s + 1 \cdot 3 &= \text{mex}(\{s + 1 \cdot 0, s + 1 \cdot 1, s + 1 \cdot 2\} \cup \{s + 1 - i \cdot 3\}_{i=1}^{s+1}) \\
&= \text{mex}(\{s + 1, s + 2, s + 3\} \cup \{s - i\}_{i=2}^s \cup \{s - 1 \cdot 3, s \cdot 3\}) \\
&= \text{mex}(\{s + 1, s + 2, s + 3\} \cup \{s - i\}_{i=2}^s \cup \{s + 1, s + 2\}) = s - 1; \\
s + 2 \cdot 3 &= \text{mex}(\{s + 2 \cdot 0, s + 2 \cdot 1, s + 2 \cdot 2\} \cup \{s + 2 - i \cdot 3\}_{i=1}^{s+2}) \\
&= \text{mex}(\{s + 2, s + 1, s + 4\} \cup \{s - i\}_{i=2}^s \cup \{s - 1 \cdot 3, s \cdot 3, s + 1 \cdot 3\}) \\
&= \text{mex}(\{s + 2, s + 1, s + 4\} \cup \{s - i\}_{i=2}^s \cup \{s + 1, s + 2, s - 1\}) = s.
\end{aligned}$$

At this point, the column is complete.

$$\begin{aligned}
s + 3 \cdot 3 &= \text{mex}(\{s + 3 \cdot 0, s + 3 \cdot 1, s + 3 \cdot 2\} \cup \{s + 3 - i \cdot 3\}_{i=1}^{s+3}) \\
&= \text{mex}(\{s + 3, s + 4, s + 1\} \cup \{s + 3 - i\}_{i=1}^{s+2}) = s + 5; \\
s + 4 \cdot 3 &= \text{mex}(\{s + 4 \cdot 0, s + 4 \cdot 1, s + 4 \cdot 2\} \cup \{s + 4 - i \cdot 3\}_{i=1}^{s+4}) \\
&= \text{mex}(\{s + 4, s + 3, s + 2\} \cup \{s + 2 - i\}_{i=1}^{s+2} \cup \{s + 3 \cdot 3\}) \\
&= \text{mex}(\{s + 4, s + 3, s + 1\} \cup \{s + 2 - i\}_{i=2}^s \cup \{s + 5\}) = s + 6; \\
s + 5 \cdot 3 &= \text{mex}(\{s + 5 \cdot 0, s + 5 \cdot 1, s + 5 \cdot 2\} \cup \{s + 5 - i \cdot 3\}_{i=1}^{s+5}) \\
&= \text{mex}(\{s + 5, s + 6, s + 7\} \cup \{s + 2 - i\}_{i=1}^{s+2} \cup \{s + 3 \cdot 3, s + 4 \cdot 3\}) \\
&= \text{mex}(\{s + 5, s + 6, s + 7\} \cup \{s + 2 - i\}_{i=2}^s \cup \{s + 5, s + 6\}) = s + 3; \\
s + 6 \cdot 3 &= \text{mex}(\{s + 6 \cdot 0, s + 6 \cdot 1, s + 6 \cdot 2\} \cup \{s + 6 - i \cdot 3\}_{i=1}^{s+6}) \\
&= \text{mex}(\{s + 6, s + 5, s + 8\} \cup \{s + 2 - i\}_{i=1}^{s+2} \cup \{s + 3 \cdot 3, s + 4 \cdot 3, s + 5 \cdot 3\}) \\
&= \text{mex}(\{s + 6, s + 5, s + 8\} \cup \{s + 2 - i\}_{i=2}^s \cup \{s + 5, s + 6, s + 3\}) = s + 4.
\end{aligned}$$

Column 3 is complete to this point. Since columns 0-2 depend on the distance from the seed mod 2 and 4, one can replace $s + 3$ through $s + 6$ above with any representative of the congruence classes of their distances from the seed mod 4 and get the same results. Thus the pattern repeats indefinitely. \square

Lemma 5.3.15. For $s \equiv_9 5$:

$$x \cdot 3 = \begin{cases} x - 1 & x - s \equiv_2 1; \\ x + 1 & x - s \equiv_2 0. \end{cases}$$

Proof.

$$\begin{aligned}
s + 1 \cdot 3 &= \text{mex}(\{s + 1 \cdot 0, s + 1 \cdot 1, s + 1 \cdot 2\} \cup \{s + 1 - i \cdot 3\}_{i=1}^{s+1}) \\
&= \text{mex}(\{s + 1, s + 2, s + 3\} \cup \{s - i \cdot 3\}_{i=1}^s \cup \{s \cdot 3\}) \\
&= \text{mex}(\{s + 1, s + 2, s + 3, s + 1\} \cup \{s - i\}_{i=1}^s) = s;
\end{aligned}$$

$$\begin{aligned}
s + 2 \cdot 3 &= \text{mex}(\{s + 2 \cdot 0, s + 2 \cdot 1, s + 2 \cdot 2\} \cup \{s + 2 - i \cdot 3\}_{i=1}^{s+2}) \\
&= \text{mex}(\{s + 2, s + 1, s + 4\} \cup \{s - i \cdot 3\}_{i=1}^s \cup \{s \cdot 3, s + 1 \cdot 3\}) \\
&= \text{mex}(\{s + 2, s + 1, s + 4, s + 1, s\} \cup \{s - i\}_{i=1}^s = s + 3); \\
s + 3 \cdot 3 &= \text{mex}(\{s + 3 \cdot 0, s + 3 \cdot 1, s + 3 \cdot 2\} \cup \{s + 3 - i \cdot 3\}_{i=1}^{s+3}) \\
&= \text{mex}(\{s + 3, s + 4, s + 1\} \cup \{s - i \cdot 3\}_{i=1}^s \cup \{s \cdot 3, s + 1 \cdot 3, s + 2 \cdot 3\}) \\
&= \text{mex}(\{s + 3, s + 4, s + 1, s + 1, s, s + 3\} \cup \{s - i\}_{i=1}^s = s + 2.
\end{aligned}$$

At this point, column 3 is complete. Look at the next four to establish that the pattern repeats.

$$\begin{aligned}
s + 4 \cdot 3 &= \text{mex}(\{s + 4 \cdot 0, s + 4 \cdot 1, s + 4 \cdot 2\} \cup \{s + 4 - i \cdot 3\}_{i=1}^{s+4}) \\
&= \text{mex}(\{s + 4, s + 3, s + 2\} \cup \{s + 4 - i \cdot 3\}_{i=1}^{s+4}) \\
&= \text{mex}(\{s + 4, s + 3, s + 2\} \cup \{s + 4 - i\}_{i=1}^{s+4} = s + 2) = s + 5; \\
s + 5 \cdot 3 &= \text{mex}(\{s + 5 \cdot 0, s + 5 \cdot 1, s + 5 \cdot 2\} \cup \{s + 5 - i \cdot 3\}_{i=1}^{s+5}) \\
&= \text{mex}(\{s + 5, s + 6, s + 7\} \cup \{s + 5 - i \cdot 3\}_{i=1}^{s+5}) \\
&= \text{mex}(\{s + 5, s + 6, s + 7, s + 5\} \cup \{s + 4 - i\}_{i=1}^{s+4} = s + 4;
\end{aligned}$$

$$\begin{aligned}
s + 6 \cdot 3 &= \text{mex}(\{s + 6 \cdot 0, s + 6 \cdot 1, s + 6 \cdot 2\} \cup \{s + 6 - i \cdot 3\}_{i=1}^{s+6}) \\
&= \text{mex}(\{s + 6, s + 5, s + 8\} \cup \{s + 6 - i \cdot 3\}_{i=1}^{s+6}) \\
&= \text{mex}(\{s + 6, s + 5, s + 8, s + 5, s + 4\} \cup \{s + 4 - i\}_{i=1}^{s+4} = s + 7; \\
s + 7 \cdot 3 &= \text{mex}(\{s + 7 \cdot 0, s + 7 \cdot 1, s + 7 \cdot 2\} \cup \{s + 7 - i \cdot 3\}_{i=1}^{s+7}) \\
&= \text{mex}(\{s + 7, s + 8, s + 5\} \cup \{s + 7 - i \cdot 3\}_{i=1}^{s+7}) \\
&= \text{mex}(\{s + 7, s + 8, s + 5, s + 5, s + 4, s + 7\} \cup \{s + 4 - i\}_{i=1}^{s+4} = s + 6.
\end{aligned}$$

Now, column 3 is complete. The elements in columns 0-2 depend on the distance from the seed mod 2 and 4, one can replace $s + 4$ through $s + 7$ by the congruence classes of their distances from the seed mod 4. □

Lemma 5.3.16. For $s \equiv_9 8$, $s + 1 \cdot 3 = s - 1$. For $x \geq s + 2$: $x \cdot 3 = \begin{cases} x + 1 & x - s \equiv_2 0; \\ x - 1 & x - s \equiv_2 1. \end{cases}$

Proof.

$$\begin{aligned} s + 1 \cdot 3 &= \text{mex}(\{s + 1 \cdot 0, s + 1 \cdot 1, s + 1 \cdot 2\} \cup \{s - 4 - i\}_{i=0}^{s-4} \cup \{s - 3 \cdot 3, s - 2 \cdot 2, s - 1 \cdot 3, s \cdot 3\}) \\ &= \text{mex}(\{s + 1, s + 2, s + 3, s - 2, s, s - 3, s + 1\} \cup \{s - 3 - i\}_{i=1}^{s-3}) = s - 1. \end{aligned}$$

At this point column 3 is complete.

$$s + 2 \cdot 3 = \text{mex}(\{s + 2 \cdot 0, s + 2 \cdot 1, s + 2 \cdot 2\} \cup \{s + 2 - i \cdot 3\}_{i=1}^{s+2})$$

$$= \text{mex}(\{s + 2, s + 1, s + 4\} \cup \{s + 2 - i\}_{i=1}^{s+2}) = s + 3;$$

$$s + 3 \cdot 3 = \text{mex}(\{s + 3 \cdot 0, s + 3 \cdot 1, s + 3 \cdot 2\} \cup \{s + 3 - i \cdot 3\}_{i=1}^{s+3})$$

$$= \text{mex}(\{s + 3, s + 4, s + 1, s + 3\} \cup \{s + 2 - i\}_{i=1}^{s+2}) = s + 2;$$

$$s + 4 \cdot 3 = \text{mex}(\{s + 4 \cdot 0, s + 4 \cdot 1, s + 4 \cdot 2\} \cup \{s + 4 - i \cdot 3\}_{i=1}^{s+4})$$

$$= \text{mex}(\{s + 4, s + 3, s + 2, s + 3, s + 2\} \cup \{s + 2 - i\}_{i=1}^{s+2}) = s + 5;$$

$$s + 5 \cdot 3 = \text{mex}(\{s + 5 \cdot 0, s + 5 \cdot 1, s + 5 \cdot 2\} \cup \{s + 5 - i \cdot 3\}_{i=1}^{s+5})$$

$$= \text{mex}(\{s + 5, s + 6, s + 7, s + 3, s + 2, s + 5\} \cup \{s + 2 - i\}_{i=1}^{s+2}) = s + 4.$$

Now since the elements in columns 0-2 depend on the distance from the seed mod 2 and 4, one can replace $s + 2$ through $s + 5$ by the congruence classes of their distances from the seed mod 4. □

5.4 Multiplication groups

The multiplication group for the greedy quasigroups is now considered. Since multiplication groups are permutation groups on Q , it is appropriate to briefly state some permutation group results.

5.4.1 Permutation Groups

The results here are not intended to be comprehensive, rather only the results and notation needed to understand and prove the following results are given. Readers are encouraged to consult (11) for more information.

Let G be a group acting on set Ω . For $\alpha \in \Omega$ the *orbit* of α is the set $\alpha^G := \{\alpha g | g \in G\}$. When $\alpha^G = \Omega$ one says G acts transitively on Ω . That is a transitive group has only one orbit. An equivalent characterization of a transitive group is one so that for all $\alpha, \beta \in \Omega$, there is a $g \in G$ so that $\alpha g = \beta$.

For each $\alpha \in \Omega$ define the stabilizer of α in G to be the set $G_\alpha := \{g \in G | \alpha g = \alpha\}$. A group G is said to act *regularly* on Ω if it is transitive and G_α is the identity for all $\alpha \in \Omega$.

A G -space is said to be *k-transitive* (or that G acts *k-transitively*) on Ω if for two sets of k distinct points in Ω , say $\{\alpha_i\}, \{\beta_i\}$ there is a $g \in G$ so that $\alpha_i g = \beta_i$ for $i = 1, 2, \dots, k$. If Ω is infinite and is *k-transitive* for all $k \in \mathbb{N}$, it is said to be *highly transitive*. The group is said to be *sharply k-transitive* if every such g is unique. It has been shown that there are no infinite sharply *k-transitive* groups for $k \geq 4$.

Similarly one can extend the idea of stabilizers. For $\Delta \subseteq \Omega$ the *setwise stabilizer* of Δ is $G_{\{\Delta\}} := \{g \in G | \Delta^g = \Delta\}$. The *pointwise stabilizer* of Δ is $G_{(\Delta)} := \{g \in G | \delta g = \delta \forall \delta \in \Delta\}$. Note that $G_{(\Delta)} \triangleleft G_{\{\Delta\}} \leq G$.

A *block* is a set Δ so that for all $g \in G$ either $\Delta^g = \Delta$ or $\Delta^g \cap \Delta = \emptyset$. If Δ is a non-trivial block and G acts transitively on Ω , then $\Sigma := \{\Delta^g | g \in G\}$ forms a partition of Ω and G acts on Σ . This new action can give insights into G . If G acts transitively on Ω and there are no non-trivial blocks, the action is said to be *primitive*. Primitivity is only discussed in reference to a transitive action.

Theorem 5.4.1. (Wielandt 1960) *If G is primitive and contains a 3-cycle, then $\text{Alt}(\Omega) \leq G$.*

Corollary 5.4.1. *Under the hypothesis of the above theorem, if Ω is infinite, then G is highly transitive.*

Jordan groups are now introduced. It turns out that Jordan blocks play a key role in a

central result below.

Definition 5.4.2. Let Ω be a G -space and let $\Omega = \Delta \cup \Gamma$ be a partition of Ω with $|\Gamma| > 1$. If there exists a subgroup H of G that fixes every point of Δ and is transitive on Γ the Γ is called a *Jordan set* for G in Ω and Δ is called a Jordan complement.

If G is k -transitive and $|\Delta| \leq k - 1$ then the set $\Gamma = \Omega \setminus \Delta$ is automatically a Jordan set. Such Jordan sets are said to be *improper*.

Definition 5.4.3. If G is transitive on Ω and there is a proper Jordan set for G in Ω then G is called a *Jordan group*.

Theorem 5.4.2. *Suppose that Ω is infinite and that G is primitive on Ω . Then:*

1. *if there is a finite Jordan set, then $\text{Alt}(\Omega) \leq G$;*
2. *if there are Jordan sets Γ_1, Γ_2 so that $\Gamma_1 \cap \Gamma_2$ is finite, but non-empty, then $\text{Alt}(\Omega) \leq G$.*

So, in particular, in both cases G is highly transitive. Also every subset Σ of Ω with more than two members is a Jordan set for G .

5.4.2 Basic results

Consider $\langle R(0), R(1), R(2) \rangle$ in $\text{Mlt}(Q_s)$.

$$R(0) = (0, s, s-1, s-2, \dots, 1)$$

$$R(1) = (s+1, s+2)(s+3, s+4)\dots(s+2n+1, s+2n+2)\dots$$

$$R(2) = (0, 2, 1)(3, 5, 4)\dots \text{ But one has to consider the seed mod } 3.$$

$$\text{For } s \equiv_3 0, \text{ one gets } (0, 2, 1)\dots(s-3, s-1, s-2) \cdot (s, s+1)(s+2, s+3)\dots$$

$$\text{For } s \equiv_3 1, \text{ one gets } (0, 2, 1)\dots(s, s+1, s-1) \cdot (s+2, s+3)\dots$$

$$\text{For } s \equiv_3 2, \text{ one gets } (0, 2, 1)\dots(s-1, s, s-2) \cdot (s+1, s+3)(s+2, s+4)(s+5, s+7)(s+6, s+8)\dots$$

Now let $\sigma_0 = R(0), \sigma_1 = R(1)$ and $\sigma_2 = R(2)$. Now $\sigma_0\sigma_1, \sigma_2 \in S_{\mathbb{N}}$.

A natural question is whether or not $\langle \sigma_0, \sigma_1, \sigma_2 \rangle$ form a transitive action on Q_s .

If so, is this group multiply transitive?

Definition 5.4.4. The orbit of 0 under $R(0) \in \text{Mlt}(Q_s)$ is called the *hub* and is denoted H_s .

The hub is an important structure in greedy quasigroups. For all Q_s the hub, $H_s = \{0, 1, \dots, s\}$.

Lemma 5.4.5. For all s , $\langle R(0) \rangle$ acts transitively on the hub.

Proof. By Lemma 5.3.1 $0 \cdot x = x - 1$ for $0 < x \leq s$ and $0 \cdot 0 = s$. Thus $xR(0)^x = 0$, and $0R(0)^{y+1} = s - y$. Therefore for $x, z = s - y \in H$, there is an n such that $xR(0)^n = z$. \square

Lemma 5.4.6. For $s \equiv_3 0, 1$, let $G = \langle R(0), R(1), R(2) \rangle$, $Q_s \setminus H_s$ is in one orbit of the action of G on Q_s . Moreover, one can choose $g \in G$ so that g stabilizes 1.

Proof. Let $x = s + 2n - i$, $y = s + 2m - j$, where $n, m \in \mathbb{N}$ and $i, j \in \{0, 1\}$.

Let $\tau = R(1)^i (R(2)R(1))^{m-n} R(1)^j$. I claim $x\tau = y$.

The initial multiplication by $R(1)^i$ sends both $s + 2n - i$ to $s + 2n$. Now an application of $R(2)R(1)$ sends $s + 2n$ to $s + 2n + 2$. So $(R(2)R(1))^t$ sends $s + 2n$ to $s + 2n + 2t$. Therefore $R(1)^i (R(2)R(1))^t$ sends $s + 2n - i$ to $s + 2n + 2t$. Finally $R(1)^j$ sends this to $s + 2n + 2t - j$. Therefore $(s + 2n - i)\tau = s + 2n + 2(m - n) - j = s + 2m - j$.

To stabilize 1, use $\tau = R(1)^i R_1(2, 0)^{m-n} R(1)^j$. Note that since $R_1(2, 0) = R(2)R(0)R(1)^{-1}$, on $Q_s \setminus H_s$, $R_1(2, 0)$ behaves like $R(2)R(1)$, since $xR(0) = x$ and $xR(1)^2 = x$ for $x \in Q_s \setminus H_s$. Thus $x\tau = xR(1)^i (R(2)R(1))^{n-m} R(1)^j = y$ as in the proof of Lemma 5.4.6. \square

Theorem 5.4.3. $\langle R(0), R(1), R(2) \rangle$ acts transitively on Q_s for $s \equiv_3 0, 1$.

Proof. Using Lemmas 5.4.5 and 5.4.6, it remains to show a hub element can be sent to a non hub element, since the inverse operation will send a non-hub element to a hub element. Note that $s \cdot 2 = s + 1$ in this case. So to send a hub element h to a non hub element $s + 2n - j$, use $\sigma = R(0)^{h+1} R(2)R(1) (R(2)R(1))^{n-1} R(1)^j$. \square

For $s \equiv_3 2$ the situation is more complex.

Lemma 5.4.7. Let $\sigma_{k,i} = R(2)^k R(1)^i$, $k, i \in \{0, 1\}$. Then in Q_s for $s \equiv_3 2$, $\sigma_{k,i}$ sends $s + 4n - 2k - i$ to $s + 4n$.

Proof. Since multiplication by 2 adds or subtracts 2, $R(2)^k$ sends $s + 4n - 2k - i$ to $s + 4n - i$. Now multiplication by 1 adds or subtracts 1. So $R(1)^i$ sends $s + 4n - i$ to $s + 4n$. \square

Lemma 5.4.8. *For $s \equiv_3 2$, $\tau = R(3)R(2)$ sends $s + 4n$ to $s + 4n + 4$.*

Proof. First, $(s + 4n)R(3) = s + 4n + 2$ by Lemma 5.3.14. Then $(s + 4n + 2)R(2) = s + 4n + 4$ by Lemma 5.3.9. Thus $(4n)\tau = (4n)R(3)R(2) = 4n + 4$. \square

Lemma 5.4.9. *For $s \equiv_3 2$, $\tau = R(3)R(2)R(1)$ sends $s + 4n$ to $s + 4n + 4$.*

Proof. First, $(s + 4n)R(3) = s + 4n + 1$ by Lemmas 5.3.15 and 5.3.16. Then $(s + 4n + 1)R(2) = s + 4n + 3$ by Lemma 5.3.9 and $(s + 4n + 3)R(1) = s + 4n + 4$ by Lemma 5.3.2. Thus $(4n)\tau = (4n)R(3)R(2)R(1) = 4n + 4$. \square

Lemma 5.4.10. *For $s \equiv_3 2$, $G = \langle R(1), R(2), R(3) \rangle$ $Q_s \setminus H_s$ is in one orbit of the action of G on Q_s . Moreover, one can choose $g \in G$ so that g stabilizes 1.*

Proof. Show that any $x \in Q_s \setminus H_s$ can be sent to $y \in Q_s \setminus H_s$. Let $x = 4n - 2k - i$ and $y = 4m - 2k' - i'$, where $k, k', i, i' \in \{0, 1\}$. Then for $\phi = \sigma_{k,i}\tau^{m-n}\sigma_{k',i'}$, $x\phi = y$:

$$(s + 4n - 2k - i)\phi = (s + 4n - 2k - i)\sigma_{k,i}\tau^{m-n}\sigma_{k',i'} \quad (5.1)$$

$$= (s + 4n)\tau^{m-n}\sigma_{k',i'} \quad (5.2)$$

$$= (s + 4m)\sigma_{k',i'} \quad (5.3)$$

$$= s + 4m - k' - i' \quad (5.4)$$

Thus $x\phi = y$.

Note that outside the hub $R(0)$ stabilizes x . So $\alpha := R_1(3, 0)R_1(2, 0)R(1)$ behaves like $R(3)$ and stabilizes 1 while $\beta := R_1(2, 0)R(1)$ behaves like $R(2)$ and stabilizes 1. Now apply Lemma 5.4.10 with α in place of $R(3)$ and β in place of $R(2)$ \square

Theorem 5.4.4. *For $s \equiv_3 2$, $\langle R(0), R(1), R(2), R(3) \rangle$ acts transitively on Q_s .*

Proof. One only needs to show one can send a hub element to a non-hub element as before.

Let $h \in H_s$ and $x = s + 4n - 2k - i$.

First, let $\psi = R(0)^{h+1}R(3)\sigma_{1,1}\tau^{n-1}\sigma_{k,i}$. Then $h\psi = x$ by the above lemmas. \square

5.4.3 2-transitivity

The goal of this section is to prove that $\text{Mlt}(Q_s)$ is 2-transitive.

In this section $G = \langle R(0), R(1), R(2), R(3) \rangle$.

Lemma 5.4.11. *Let $H = \langle R(0), R(2) \rangle$. Then H_s is in one orbital of the action of H on Q_s for $s \equiv_3 0, 1$.*

Proof. Given $h_1, h_2, x_1, x_2 \in H_s$, there is an n so that $h_1 R(0)^n = s$ (by Lemma 5.4.5). So $h_1 R(0)^n R(2) = s+1$. Let $h_2 R(0)^n R(2) = k$. Now choose m so that $k R(0)^m = x_2 R(0)^{-(s-x_1)} R(2)^{-1}$. Thus for $\sigma = R(0)^n R(2) R(0)^m R(2)^{-1} R(0)^{s-x_1}$ $h_1 \sigma = x_1$ and $h_2 \sigma = x_2$. \square

Lemma 5.4.12. *Let $H = \langle R(0), R(3) \rangle$. Then H_s is in one orbital of the action of H on Q_s for $s \equiv_3 2$.*

Proof. Given $h_1, h_2, x_1, x_2 \in H_s$, there is an n so that $h_1 R(0)^n = s$ (by Lemma 5.4.5). So $h_1 R(0)^n R(3) = s+1$. Let $h_2 R(0)^n R(3) = k$. Now choose m so that $k R(0)^m = x_2 R(0)^{-(s-x_1)} R(3)^{-1}$. Thus for $\sigma = R(0)^n R(3) R(0)^m R(3)^{-1} R(0)^{s-x_1}$ $h_1 \sigma = x_1$ and $h_2 \sigma = x_2$. \square

Remark 5.4.13. The above two lemmas, along with the fact that $hR(1) = h \forall h \in H_s$ show that the hub is in one orbital of the action of G .

Lemma 5.4.14. *For $x_1 \in Q_s \setminus H_s$ and h_1, h_2, h_3 there is a σ so that $x_1 \sigma = h_2$ and $h_1 \sigma = h_3$.*

Proof. Use $R(0)^n$ for some n so send h_1 to 1. By Lemmas 5.4.6 and 5.4.10, there is a β so that $1\beta = 1$ and $x_1\beta = s+1$. Then for $s \equiv_3 0, 1$ $\gamma = R(0)^n \beta R(2)^{-1}$ is such that $x_1 \gamma, h_1 \gamma \in H_s$. For $s \equiv_3 2$ use $\gamma = R(0)^n \beta R(3)^{-1}$.

Now since H_s is in one orbital of the action of $\langle R(0), R(2), R(3) \rangle$ (Remark 5.4.13), the proof is complete. \square

Lemma 5.4.15. *For $x_1, x_2 \in Q_s \setminus H_s$ and $h_1, h_2 \in H_s$, there is a σ so that $x_i \sigma = h_i$.*

Proof. Let α be so that $x_1 \alpha = 1$. Then perhaps $x_2 \alpha = h \in H_s$. Then by Lemma 5.4.11, there is a β , so that $1\beta = h_1, h\beta = h_2$. Thus $\sigma = \alpha\beta$.

If $x_2 \alpha = x \notin H_s$ apply Lemma 5.4.14. \square

Theorem 5.4.5. *G acts 2-transitively on Q_s .*

Proof. Find a σ that sends $(x_1, x_2) \in Q_s^2$ to (y_1, y_2) . First by the above three lemmas, there is a map α so that $(x_1, x_2)\alpha = (0, 1)$, and a map β so that $(y_1, y_2)\beta = (0, 1)$. Then $(x_1, x_2)\alpha\beta^{-1} = (y_1, y_2)$ \square

5.4.4 High transitivity

It has been shown how to construct permutations in $G \leq \text{Mlt}(Q_s)$ that are 2-transitive. The question is whether one can go farther.

First note that since G is 2-transitive it is primitive (Lemma 4.10 in (11)). Therefore one can apply Lemma 10.8 in (11) with the hub as the Jordan set. This theorem says that if a permutation group on Ω is primitive on an infinite set with a subgroup H that is transitive on a set, X , and fixes the complement of X , the multiplication group is highly transitive. Moreover, if X is finite, $\text{Alt}(\Omega) \leq F$. Thus $\text{Alt}(\mathbb{N}) \leq F \leq \text{Mlt}(Q_s)$.

5.5 Subquasigroups

Each greedy quasigroup has a unique singleton subquasigroup: $\{0\}$ in the elementary 2-group Q_0 , and $\{1\}$ in Q_s for $s > 0$. The singleton subquasigroup and the empty subquasigroup are referred to as the *trivial* subquasigroups of the greedy quasigroups. The group Q_0 has uncountably many subquasigroups, since for each of the uncountably many subsets S of \mathbb{N} , the vector

$$(0\chi_S, 1\chi_S, \dots, n\chi_S, \dots) \tag{5.5}$$

of values of the characteristic function of S generates a distinct subgroup of the isomorphic copy $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ of Q_0 .

Proposition 5.5.1. *The greedy quasigroup Q_1 has uncountably many subquasigroups.*

Proof. Outside the hub $\{0, 1\}$, the multiplication on Q_1 is constructed exactly as in Q_0 . Thus for each subgroup P of Q_0 with $\{0, 1\} \leq P$, the subset P of \mathbb{N} forms a subquasigroup of Q_1 . But Q_0 has uncountably many such subgroups P . \square

The respective hubs H_1 and H_2 of Q_1 and Q_2 form cyclic groups, with 1 as the identity element. These cases are exceptional.

Proposition 5.5.2. *For $s > 2$, the hub H_s does not form a subquasigroup of Q_s .*

Proof. It was shown that $\langle R(0), R(1), R(2), R(3) \rangle$ is transitive for all $s \geq 3$. Thus there is a subquasigroup, H , contains $0, 1, 2$ then $H = Q_s$. Let H be a subquasigroup. If $0 \in H$, then $H_s \subset H$. In particular for $s \geq 3$, $0, 1, 2, 3 \in H$ and $H = Q_s$. Suppose $x \neq 0, 1 \in H$, then $x \cdot x = 0 \in H$, so as above $H = Q_s$. \square

Proposition 5.5.3. *For $s \geq 2$, Q_s is simple.*

Proof. This follows immediately since $\text{Mlt}(Q_s)$ is 2-transitive. \square

5.6 Homomorphisms

One natural question is whether any of the quasigroups are isomorphic.

Since Q_0 is the addition table for nim, it is a group. I have already remarked that Q_i is non-associative for $i \neq 0$. Thus Q_0 is not isomorphic to any Q_i . Suppose there is a homomorphism $\phi : Q_i \rightarrow Q_j$. What properties does it have?

Theorem 5.6.1. *For $i \neq j$, $Q_i \not\cong Q_j$.*

Proof. In both Q_i, Q_j , 0 is the unique element that fixes infinitely many elements. So for any isomorphism ϕ , $\phi : 0 \mapsto 0$. In $\text{Mlt}(Q_i)$, $R(0)$ is an $i + 1$ -cycle, but in Q_j $R(0)$ is a $j + 1$ -cycle. Therefore $\text{Mlt}(Q_i) \not\cong \text{Mlt}(Q_j)$. Thus $Q_i \not\cong Q_j$. \square

One can actually prove stronger results.

Lemma 5.6.1 (Nilpotence Lemma).

- (a) *If ϕ is injective then there is a $k \in Q_i$ such that $k, k\phi$ are both nilpotent.*
- (b) *If ϕ is surjective then there is a $k \in Q_i$ such that $k, k\phi$ are both nilpotent.*

Proof. For $i \neq 0$, there are only two elements $k \in Q_i$ such that $k \cdot k \neq 0$, namely $0, 1$, and similarly for Q_j .

- (a) Let ϕ be injective. Let $x, y \in Q_i$. Suppose that $x\phi, y\phi$ are not nilpotent. Let $z \in Q_i$ be nilpotent, then $z\phi$ is not $x\phi, y\phi$ and these are the only non-nilpotent elements in Q_j . Thus both $z, z\phi$ are nilpotent.
- (b) Since ϕ is surjective, at most two of the nilpotent elements of Q_j can be the image of non-nilpotent elements of Q_i . There must be nilpotent elements on Q_i that are mapped to nilpotent elements of Q_j .

□

Lemma 5.6.2. *Let $\phi : Q_i \rightarrow Q_j$ be a homomorphism and $0_i\phi = 0_j$. If $x \cdot x = 0_i$, then $x\phi \cdot x\phi = 0_j$.*

Proof. $0_j = 0_i\phi = (x \cdot x)\phi = x\phi \cdot x\phi$.

□

Lemma 5.6.3. *If there is an element $x \in Q_i$ such that $x \cdot x = 0$ and $x\phi \cdot x\phi = 0$, then $0_i\phi = 0_j$.*

Proof. Let k be one such element. Then $0_i\phi = (k \cdot k)\phi = k\phi \cdot k\phi = 0_j$

□

Remark 5.6.4. In particular, Lemma 5.6.2 and Lemma 5.6.3 are true for surjective and injective homomorphisms.

Lemma 5.6.5. *For any homomorphism $\phi : Q_i \rightarrow Q_j$ and $i, j \neq 0, 1$, $1_i\phi = 1_j$.*

Proof. This follows from the fact that 1_i is the only idempotent element of Q_i . (Everything else other than 0_i is nilpotent).

□

Lemma 5.6.6. *For any surjective (injective) homomorphism $\phi : Q_i \rightarrow Q_j$, $s_i\phi = s_j$.*

Proof. $s_i\phi = (0_i \cdot 0_i)\phi = 0_i\phi \cdot 0_i\phi = 0_j \cdot 0_j = s_j$.

□

Remark 5.6.7. In fact, this is true if $0_i\phi = 0_j$.

Theorem 5.6.8 (Homomorphism Theorem).

(a) *There is no injective homomorphism $\phi : Q_i \rightarrow Q_j$.*

(b) *There is no surjective homomorphism $\phi : Q_i \rightarrow Q_j$.*

Proof. Note, by looking at the multiplication table for Q_j , that $s_j L(0_j)^{j+1} = s_j$ and $s_j L(0_i)^{i+1} \neq s_j$ for $i < j$. Since $s_i \phi = s_j$, then $s_j = s_i \phi = s_i R(0_i)^{i+1} \phi = s_i \phi R(0_i \phi)^{i+1} = s_j R(0_j)^{i+1}$. Thus the hub gets mapped to the hub. Thus $j+1 | i+1$. Perhaps one can “loop” several times, but one must always complete the loop. Thus there is no injective or surjective homomorphism $\phi : Q_i \rightarrow Q_j$, if $i < j$.

So, suppose that $j+1 | i+1$, but $j \neq i$. Note that $s_i R(0)^{j-1}$ is nilpotent. Then $s_i R(0)^{j-1} \phi = s_i \phi R(0 \phi)^{j-1} = s_j R(0_j)^{j-1} = 1_j$. This contradicts Lemma 5.6.2, since a nilpotent must be mapped to a nilpotent and 1_j is idempotent. □

Corollary 5.6.9. $Q_i \not\cong Q_j$ for $i \neq j$.

Not only are the Q_i 's not isomorphic, there is no injective or surjective homomorphism between them. It is natural to ask whether there is any non-trivial homomorphism between them. Of course, there is the trivial homomorphism $x\phi = 1, \forall x \in Q_i$ for any Q_i, Q_j . It turns out that this is the only homomorphism $\phi : Q_i \rightarrow Q_j$ for $i \neq j$, for $j > 0$. If $j = 0$, then $x\phi = 0$ is the trivial homomorphism.

Theorem 5.6.10. *The only homomorphism $\phi : Q_i \rightarrow Q_j$ for $i \neq j$ is the trivial homomorphism.*

Proof. If there is a nilpotent element x such that $x\phi$ is also nilpotent, by Lemma 5.6.3 $0_i \phi = 0_j$, so then by Lemma 5.6.6 $s_i \phi = s_j$. Then the homomorphism fails as in Theorem 5.6.8.

Thus for any nilpotent x , $x\phi$ is either 0 or 1. If $x \neq 0$ and $x\phi = 0$, then $0\phi = (x \cdot x)\phi = x\phi x\phi = 0_j \cdot 0_j = s_j$. Then for any nilpotent y , $s_j = 0\phi = (y \cdot y)\phi = y\phi \cdot y\phi$. So s_j is the square of $y\phi$. Thus $y\phi = 0_j$ for any nilpotent y . Now, $s_i \phi = (0_i \cdot 0_i)\phi = 0_i \phi 0_i \phi = s_j \cdot s_j = 0_j$.

However, in any Q_i there are nilpotent elements x, y such that $xy = s_i$. Then $s_i \phi = (xy)\phi = x\phi y\phi = 0_j \cdot 0_j = s_j$. This is a contradiction, so one can't have that $x\phi = 0_j$. Thus $x\phi = 1_j$ for all nilpotent x . In particular $s_i \phi = 1$, so $0_i \phi = (s_i \cdot s_i)\phi = s_i \phi \cdot s_i \phi = 1_j \cdot 1_j = 1$. Thus ϕ is trivial. □

The driving force behind the algebraic properties seems to be the definition $0 \cdot 0 = s$, where 0 is the unique element that is the square of infinitely many elements. This curious property has been the key idea in most of the above proofs. It is remarkable that such a simple property is so powerful.

Greedy quasigroups are generated by a very simple algorithm. Only one quasigroup multiplication is defined, and the rest of the table is filled in with a natural rule. Nevertheless, greedy quasigroups have a very interesting algebraic structure.

5.7 Generalized greedy quasigroups

A natural extension of greedy quasigroups is the change the location of the seed. Instead of placing the seed at $(0, 0)$ the seed can be placed at (i, j) . The quasigroup generated this way will be denoted $Q_s^{i,j}$, where s is the seed, i is the row, and j is the column. In this notation, Q_s is denoted $Q_s^{(0,0)}$. These are not necessarily commutative.

Definition 5.7.1. Denote a typical *option* of a by a' . Unless stated otherwise, the options of a are all non-negative integers less than a .

Definition 5.7.2. A product ab is *greedy* if $ab = \text{mex}\{a'b, ab'\}$.

Proposition 5.7.3. *Suppose that $ab = c$ where all products ab' are greedy. Then $b = \text{mex}\{a' \setminus c, a \setminus c'\}$.*

Proof. Suppose $\text{mex}\{a' \setminus c, a \setminus c'\} = d$ for some fixed $d < b$. Then $a'd \neq c$ for all a' and $ad \neq c'$ for all c' . Thus $ad \geq c$. It cannot be that $ad = c$ since $ab = c$. Now if $ad > c$, $ad = \text{mex}\{a'd, ad'\}$ since ad is greedy because $d = b'$ for some b' . There exists an $f < d$ such that $af = c$ since no $a'd = c$. Now there is a contradiction, since $ab = c$ and $f < d < b$. Thus $d \in \{a' \setminus c, a \setminus c'\}$ for all $d < b$, so $\text{mex}\{a' \setminus c, a \setminus c'\} \geq b$.

If $\text{mex}\{a' \setminus c, a \setminus c'\} > b$, then either $a' \setminus c = b$ or $a \setminus c' = b$. So either $a'b = c$ or $ab = c'$. These are both false. So $\text{mex}\{a' \setminus c, a \setminus c'\} = b$. □

Proposition 5.7.4. *Let $lk = x \neq s$ be greedy, and $l < i, k > j$. Then $\text{mex}\{l' \setminus x, l \setminus x'\} = k$.*

Proof. Since $lk = x$ is greedy, $x = \text{mex}\{l'k, lk'\}$. Then $\text{mex}\{l' \setminus x, l \setminus x'\} \leq \text{mex}\{l' \setminus x, l \setminus l'k, l \setminus lk'\} = \text{mex}\{l' \setminus x, l \setminus l'k, k'\} = k$. Now suppose $\text{mex}\{l' \setminus x, l \setminus x'\} = a < k$. Then $l'a \neq x, la \neq x'$ for all x' . Then either la is greedy or $a = j$ since the only non greedy entries occur in column j or row i . If la is greedy then $la = x$ or $lb = x$ for $b < a$. Either case is false since $lk = x$ and $a < k$. The only remaining option is $a = j$ where $la \neq x', l'a, la', la \neq x$. Thus apparently $la = x$, but since $lk = x$, x must be excluded from the options of la . The only remaining possibility is that $s = ij = x$ which is excluded by assumption. \square

That is x is the greedy answer, but lj isn't x . So x must be "below" lj , so x must be s .

Remark 5.7.5. The only non greedy elements appear in the set $\{i'j, ij', ij\}$.

Proposition 5.7.6. *Let $lk = s$ for $l < i, k > j$. Then $k = \text{mex}\{l' \setminus s, l \setminus s', i \setminus s\}$.*

(This is the one exception to the above Proposition.)

Proof. $\text{mex}\{l' \setminus s, l \setminus s', i \setminus s\} \leq \text{mex}\{l' \setminus s, l \setminus l'k, l \setminus lk', i \setminus s\} = \text{mex}\{l' \setminus s, l \setminus l'k, k', j\} = k$. Suppose $\text{mex}\{l' \setminus s, l \setminus s', i \setminus s\} = n < k$. Then $l'n \neq s, ln \neq s', in \neq s$. Thus $n \neq j$ since $ij = s$. Thus ln is greedy by the remark, so either $ln = s$ or $mn = s$ for $m < l < i$. These are false since $lk = s$ and $l'n \neq s$ for all l' . \square

Proposition 5.7.7. *Let $x = \text{mex}\{i'l, il', ij\}$ for $l < j$. Then $\text{mex}\{i' \setminus x, i \setminus x', i \setminus s\} = l$.*

Proof. First: $\text{mex}\{i' \setminus x, i \setminus x', i \setminus s\} \leq \text{mex}\{i' \setminus x, i \setminus i'l, i \setminus il', i \setminus ij\} = \text{mex}\{i' \setminus x, i \setminus i'l, l', j\} = l$. Suppose $\text{mex}\{i' \setminus x, i \setminus x' \setminus i \setminus s\} = k < l$.

Then $i'k \neq x, ik \neq x', ik \neq s$. Consider the product ik and note that $ik \neq x$. Now either there is $n < k < l$ so that $in = x$ which is false since no $il' = x$ or ik is not greedy since x is the mex of the options. The only remaining reason to exclude x as a possibility for ik is if $x = s$. But $k \neq j$ so $ik \neq s$. Thus no such k exists. \square

Theorem 5.7.8. *Conjugates Theorem.*

1. $(Q_s^{i,j}, \cdot)^{op} = (Q_s^{j,i}, \cdot)$.
2. $(Q_s^{i,j}, \setminus) = (Q_j^{i,s}, \cdot)$.

$$3. (Q_s^{i,j}, /) = (Q_i^{s,j}, \cdot).$$

$$4. (Q_s^{i,j}, \backslash)^{op} = (Q_j^{s,i}, \cdot).$$

$$5. (Q_s^{i,j}, /)^{op} = (Q_i^{j,s}, \cdot)$$

Proof. Remark: if s is in row a or column b , it needs to be considered it as well, otherwise it does not need to be specifically considered. The notation (s) to indicate that this is the case. First note that $ij = s$ in $Q_s^{i,j}$ means that $ji = s$ in $Q_s^{i,j\ op}$. Let \cdot be the multiplication in Q and \circ be the multiplication in Q^{op} .

By induction (1) $ab = \text{mex}\{a' \cdot b, a \cdot b', (s)\} = \text{mex}\{b \circ a', b' \circ a, (s)\} = \text{mex}\{b' \circ a, b \circ a', (s)\} = b \circ a$.

To show (2): first note that $i \backslash s = j$ by calculation.

Let $ab = x$. If either $a < i, b < j$ or $a > i$. Then $a \backslash x = \text{mex}\{a \backslash x', a' \backslash x\}$ by Proposition 5.7.3.

If $a < i, b > j, \neq x$ then Then $a \backslash x = \text{mex}\{a \backslash x', a' \backslash x\}$ by Proposition 5.7.4.

If $a = i$ then $a \backslash x = \text{mex}\{a \backslash x', a' \backslash x, a \backslash s\}$ by Proposition 5.7.7.

Finally, if $ab = s$ for $a < i, b > j$, one has $b = \text{mex}\{a' \backslash s, a \backslash s', j\}$ by Proposition 5.7.6.

If one replaces \backslash with \cdot one gets $Q_j^{i,s}$.

To show that $(Q_s^{i,j}, /) = (Q_i^{s,j}, \cdot)$, recognize that $(Q_s^{i,j}, /) = (Q_s^{i,j\ op}, \backslash)$.

To show that $(Q_s^{i,j}, \backslash)^{op} = (Q_j^{s,i}, \cdot)$, apply statement 1 to statement 2. Likewise, to prove $(Q_s^{i,j}, /)^{op} = (Q_i^{j,s}, \cdot)$, apply statement 1 to statement 3. \square

Remarkably, the six permutations of s, i, j correspond to the six conjugates of $Q_s^{i,j}$.

5.8 Transfinite extensions of greedy quasigroups

Each greedy quasigroup consists of only finite entries. Thus \mathbb{N} is the set of entries for each Q_i . What if one extends the entries by adding $\omega, \omega + 1, \dots, \omega + n, \dots$? The following table is the result for Q_0 :

Table 5.2 Transfinite extension of Q_0

	0	1	2	3			ω	$\omega + 1$	$\omega + 2$	$\omega + 3$
0	0	1	2	3			ω	$\omega + 1$	$\omega + 2$	$\omega + 3$
1	1	0	3	2	...		$\omega + 1$	ω	$\omega + 3$	$\omega + 2$
2	2	3	0	1			$\omega + 2$	$\omega + 3$	ω	$\omega + 1$
3	3	2	1	0			$\omega + 3$	$\omega + 2$	$\omega + 1$	ω
ω	ω	$\omega + 1$	$\omega + 2$	$\omega + 3$			0	1	2	3
$\omega + 1$	$\omega + 1$	ω	$\omega + 3$	$\omega + 2$			1	0	3	2
$\omega + 2$	$\omega + 2$	$\omega + 3$	ω	$\omega + 1$...		2	3	0	1
$\omega + 3$	$\omega + 3$	$\omega + 2$	$\omega + 1$	ω			3	2	1	0
$\omega 2$	$\omega 2$	$\omega 2 + 1$	$\omega 2 + 2$	$\omega 2 + 3$			$\omega 3$	$\omega 3 + 1$	$\omega 3 + 2$	$\omega 3 + 3$
$\omega 2 + 1$	$\omega 2 + 1$	$\omega 2$	$\omega 2 + 3$	$\omega 2 + 2$			$\omega 3 + 1$	$\omega 3$	$\omega 3 + 3$	$\omega 3 + 2$
$\omega 2 + 2$	$\omega 2 + 2$	$\omega 2 + 3$	$\omega 2$	$\omega 2 + 1$...		$\omega 3 + 2$	$\omega 3 + 3$	$\omega 3$	$\omega 3 + 1$
$\omega 2 + 3$	$\omega 2 + 3$	$\omega 2 + 2$	$\omega 2 + 1$	$\omega 2$			$\omega 3 + 3$	$\omega 3 + 2$	$\omega 3 + 1$	$\omega 3$
$\omega 3$	$\omega 3$	$\omega 3 + 1$	$\omega 3 + 2$	$\omega 3 + 3$						
$\omega 3 + 1$	$\omega 3 + 1$	$\omega 3$	$\omega 3 + 3$	$\omega 3 + 2$						
$\omega 3 + 2$	$\omega 3 + 2$	$\omega 3 + 3$	$\omega 3$	$\omega 3 + 1$						
$\omega 3 + 3$	$\omega 3 + 3$	$\omega 3 + 2$	$\omega 3 + 1$	$\omega 3$						

Looking at the cosets one arrives at the following:

$$\begin{array}{cccc}
 Q_0 & \omega + Q_0 & \omega \cdot 2 + Q_0 & \omega \cdot 3 + Q_0 \\
 \omega + Q_0 & Q_0 & \omega \cdot 3 + Q_0 & \omega \cdot 2 + Q_0 \\
 \omega \cdot 2 + Q_0 & \omega \cdot 3 + Q_0 & Q_0 & \omega + Q_0 \\
 \omega \cdot 3 + Q_0 & \omega \cdot 2 + Q_0 & \omega + Q_0 & Q_0
 \end{array}$$

This pattern actually continues for all powers of ω .

5.8.1 Infinite seeds

Alternatively, once can start with a transfinite seed. The following is the resulting table if one starts with ω as the seed.

\cdot	0	1	2	3	4	5	6	7	8	9
0	ω	0	1	2	3	4	5	6	7	8
1	0	1	2	3	3	5	6	7	8	9
2	1	2	0	4	5	3	7	8	6	10
3	2	3	4	0	1	6	8	5	9	7
4	3	4	5	1	0	2	9	10	11	6
5	4	5	3	6	2	0	1	9	10	11
6	5	6	7	8	9	1	0	2	3	4
7	6	7	8	5	10	9	2	0	1	3

Table 5.3 Q_ω

This is not a quasigroup, if the underlying set is the finite natural numbers, but if the underlying set is set to $Q = \{0, 1, 2, \dots, \omega, 1 + \omega, \dots, \omega 2\}$, then (Q, \cdot) is a quasigroup. In this quasigroup, the hub is the set of natural numbers. This can be thought of as the “all-hub” greedy quasigroup.

5.9 The greedy idempotent quasigroup

Although one can impose any algebraic restriction and apply the greedy algorithm, it is not immediately clear that any such algebra would be interesting. However, requiring the quasigroup to be idempotent is interesting and the quasigroup is related to the already generated quasigroups. First, look at the Cayley table:

\cdot	1	2	3	4	5	6	7
1	1	3	2	5	4	7	6
2	3	2	1	6	7	4	5
3	2	1	3	7	6	5	4
4	5	6	7	4	1	2	3
5	4	7	6	1	5	3	2
6	7	4	5	2	3	6	1
7	6	5	4	3	2	1	7

Table 5.4 Q_I : the greedy idempotent quasigroup

A quick glance indicates that there are subquasigroups of orders 1 and 3. This quasigroup is actually directly related to Q_0 . If one turns this into a loop using the construction given in

(37):

$$x + y = \begin{cases} 0 & \text{if } x = y \\ x \cdot y & \text{otherwise} \end{cases} \quad (5.6)$$

one gets the loop Q_0 .

Since any commutative idempotent quasigroup gives rise to a Steiner triple system, Q_I allows one to quickly construct a Steiner system of order $2^n - 1$ for any n .

5.10 Conclusion

Greedy quasigroups have several interesting algebraic properties. It seems remarkable that a simple algorithm for generating quasigroups would lead to such results. Research into other properties of these quasigroups will continue. Hopefully a complete characterization of the multiplication will soon be found which will greatly expedite other results.

CHAPTER 6. Wythoff Quasigroups

6.1 Introduction

In the same way that greedy quasigroups arise by generalizing nim, Wythoff quasigroups arise by generalizing Wythoff's game. As it might be expected, many of the results that hold for greedy quasigroups also hold for Wythoff quasigroups, but sometimes the proof has to be different. Sometimes the structure of Wythoff quasigroups precludes using the methods used previously and new methods need to be sought.

6.2 Definition and basic properties

A Wythoff quasigroup is generated by selecting a natural number s , called the *seed* and defining $0 \cdot 0 = s$. Then the remaining table is filled in using the definition:

$$l \cdot m = \text{mex}(\{l' \cdot m \mid l' < l\} \cup \{l \cdot m' \mid m' < m\} \cup \{(l - c) \cdot (m - c) \mid c \leq \min(l, m)\}) \quad (6.1)$$

Call the resulting quasigroup W_s .

Notice each entry is the smallest entry that does not appear above, to the left or “northwest” of it.

It is not immediately clear from the definition that Wythoff quasigroups are indeed quasigroups. The following proposition justifies the name.

Proposition 6.2.1. W_s is a quasigroup for all s .

Proof. It must be shown that each element appears exactly once in each column. Clearly, no element can appear twice in a column. It remains to be shown that each element appears in a

	0	1	2	3	4	5	6	7	
0	5	0	1	2	3	4	6	7	...
1	0	1	2	3	4	5	7	8	...
2	1	2	0	4	5	3	8	6	...
3	2	3	4	6	1	0	5	9	...
4	3	4	5	1	2	6	9	0	...
5	4	5	3	0	6	7	10	2	...
6	6	7	8	5	9	10	3	11	...
7	7	8	6	9	0	2	11	4	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Example 6.2.1.

Table 6.1 Part of the multiplication table for W_5

column. If not then for some n , there is a column, j such that n does not appear in column j . In order not to place n at $i \cdot j$, either n is already in row column j , i , or the diagonal or ij or there is an $m < n$ not already in column j . Since it is necessary to avoid placing n in column, j , look at the later three cases. The most times one can avoid n using the fact it has already appeared is j times, since there j columns before column j , but perhaps, they are aligned so that the next j entries of column j are excluded by the fact that their diagonal contains an n . Next, one can exhaust each element less than n . Thus, after $2j + n$ entries in column j , one must have n as an entry. Thus n appears in each column, for all n , so the multiplication is surjective and W_s is a quasigroup. \square

Theorem 6.2.2. *The W_s 's are commutative.*

Proof. By induction using (6.1)

$$\begin{aligned}
 l \cdot m &= \text{mex}(\{i \cdot m \mid i < l\} \cup \{l \cdot j \mid j < m\} \cup \{(l - c) \cdot (m - c) \mid c \leq \min(l, m)\}) \\
 &= \text{mex}(\{i \cdot m \mid i < l\} \cup \{l \cdot j \mid j < m\} \cup \{(m - c) \cdot (l - c) \mid c \leq \min(l, m)\}) \\
 &= \text{mex}(\{i \cdot l \mid i < m\} \cup \{m \cdot j \mid j < l\}) \cup \{(m - c) \cdot (l - c) \mid c \leq \min(l, m)\}) = m \cdot l.
 \end{aligned}$$

(The induction hypothesis is used for the second equality.) \square

Theorem 6.2.3. *The W_s 's are non-associative.*

Proof. The W_i 's do not have a unique identity element; ($1 \cdot h = h$ for h in the hub, and $0 \cdot x = x$ for x not in the hub.) Therefore the W_i 's cannot be associative \square

Remark 6.2.2. One can exhibit non-associative triples. For $s \geq 1$, one has that $2 \cdot s = 0$, $2 \cdot 0 = 1$, $2 \cdot 1 = 2$. Thus $(2 \cdot 2) \cdot 0 = 0 \cdot 0 = s$, but $2 \cdot (2 \cdot 0) = 2 \cdot 1 = 2$. So $(2 \cdot 2) \cdot 0 \neq 2 \cdot (2 \cdot 0)$.

6.3 Some calculations on columns

In this section I investigate the patterns resulting from multiplication by quasigroup elements. Suppose that $s > 0$. Later, it will be implicitly assumed that s is sufficiently large.

Lemma 6.3.1. *For $0 < x \leq s$, $0 \cdot x = x - 1$. For $x > s$, $x0 = x$.*

Proof. $0 \cdot 0 = s$ be definition. Then for $x \leq s$, $0x = \text{mex}(\{0 \cdot 0, 0 \cdot 1, m, \dots, 0(x-1)\}) = \text{mex}(\{s, 0, \dots, x-2\}) = x-1$.

For $x > s$, $0x = \text{mex}(\{0 \cdot 0, 0 \cdot 1, m, \dots, 0s, 0(s+1), \dots, 0(x-1)\}) = \text{mex}(\{s, 0, \dots, s-1, s+1, \dots, x-1\}) = x$. \square

Corollary 6.3.2. *0 is the unique element such that $0x = x0 = x$ for infinitely many x .*

Lemma 6.3.3. *For $x \leq s$, $1x = x$.*

Proof. For $x \leq s$: $1x = \text{mex}(\{1 \cdot 0, \dots, 1 \cdot (x-1), 0 \cdot x, 0 \cdot (x-1)\}) = \text{mex}(\{0, \dots, x-1, x-2, x-1\}) = x$. \square

Lemma 6.3.4. *There is exactly one idempotent element in each W_s .*

Proof. For W_0 , $0 \cdot 0 = 0$, and $0 \cdot x = x$ for all x , by Lemma 6.3.1, so there is no other x such that $x^2 = x$ by the construction of W_0 . For W_s , $s \geq 1$, $1x = x$, and $0 \cdot 0 = s \neq 0$. Now by Lemma 6.3.3 $x0 = x$ for $x > s$ and $x1 = x$ for $x \leq s$, thus the only x such that $x^2 = x$ is 1. \square

Lemma 6.3.5. *For $x > s$:*

$$x \cdot 1 = \begin{cases} x+1 & x-s \equiv_3 1, 2 \\ x-2 & x-s \equiv_3 0 \end{cases}$$

Proof.

$$\begin{aligned}
1 \cdot (s + 1) &= \text{mex}(\{1 \cdot 0, \dots, 1 \cdot s, 0 \cdot s, 0 \cdot (s - 1)\}) \\
&= \text{mex}(\{0, \dots, s, s + 1, s - 2\}) \\
&= s + 2 = x + 1
\end{aligned}$$

$$\begin{aligned}
1 \cdot (s + 2) &= \text{mex}(\{1 \cdot 0, \dots, 1 \cdot s, 0 \cdot s + 2, 0 \cdot s + 1\}) \\
&= \text{mex}(\{0, \dots, s, s + 2, s + 1\}) \\
&= s + 3 = x + 2
\end{aligned}$$

$$\begin{aligned}
1 \cdot (s + 3) &= \text{mex}(\{1 \cdot 0, \dots, 1 \cdot s, 0 \cdot s + 3, 0 \cdot s + 2\}) \\
&= \text{mex}(\{0, \dots, s, s + 3, s + 2\}) \\
&= s + 1 = x - 2
\end{aligned}$$

At this point the column is complete. Suppose this holds for all entries up to $s + 3n$. In particular, suppose that $\{0 \cdot 1, \dots, 0 \cdot s + 3n\} = \{y | y \leq s + 3n\}$.

$$\begin{aligned}
1 \cdot (s + 3n + 1) &= \text{mex}(\{1 \cdot 0, \dots, 1 \cdot s + 3n\} \cup \{0 \cdot s + 3n + 1, 0 \cdot s + 3n\}) \\
&= \text{mex}(\{0, \dots, s + 3n + 1\} \cup \{s + 3n + 1, s + 3n\}) \\
&= s + 3n + 2 = x + 1
\end{aligned}$$

$$\begin{aligned}
1 \cdot (s + 3n + 2) &= \text{mex}(\{1 \cdot 0, \dots, 1 \cdot s + 3n, s + 3n + 1\} \cup \{0 \cdot s + 3n + 2, 0 \cdot s + 3n + 1\}) \\
&= \text{mex}(\{0, \dots, s + 3n, s + 3n + 2\} \cup \{s + 3n + 2, s + 3n + 1\}) \\
&= s + 3n + 3 = x + 1
\end{aligned}$$

$$\begin{aligned}
1 \cdot (s + 3n + 3) &= \text{mex}(\{1 \cdot 0, \dots, 1 \cdot s + 3n, s + 3n + 1, s + 3n + 2\} \\
&\quad \cup \{0 \cdot s + 3n + 3, 0 \cdot s + 3n + 2\}) \\
&= \text{mex}(\{0, \dots, s + 3n, s + 3n + 2, s + 3n + 3\} \cup \{s + 3n + 3, s + 3n + 2\}) \\
&= s + 3n + 1 = x - 2
\end{aligned}$$

□

Lemma 6.3.6. For $x \leq s$,

$$2 \cdot x = \begin{cases} x + 1 & x \equiv_3 0, 1 \\ x - 2 & x \equiv_3 2 \end{cases}$$

Proof.

$$2 \cdot 0 = \text{mex}(\{s, 0\}) = 1$$

$$2 \cdot 1 = \text{mex}(\{0, 1, 0\}) = 2$$

$$2 \cdot 2 = \text{mex}(\{1, 2, 1, s\}) = 0$$

Suppose this pattern continues through $3n - 1$. Note that the column is complete after each three entries.

$$\begin{aligned} 2 \cdot 3n &= \text{mex}(\{0, 1, \dots, 3n - 1\} \cup \{0 \cdot 3n, 1 \cdot 3n, 1 \cdot 3n - 1, 0 \cdot 3n - 2\}) \\ &= \text{mex}(\{0, 1, \dots, 3n - 1\} \cup \{3n - 1, 3n, 3n - 1, 3n - 2\}) \\ &= 3n + 1 \end{aligned}$$

$$\begin{aligned} 2 \cdot 3n + 1 &= \text{mex}(\{0, 1, \dots, 3n - 1\} \cup \{2 \cdot 3n, 0 \cdot 3n + 1, 1 \cdot 3n + 1, 1 \cdot 3n, 0 \cdot 3n - 1\}) \\ &= \text{mex}(\{0, 1, \dots, 3n - 1\} \cup \{3n + 1, 3n, 3n + 1, 3n, 3n + 1\}) \\ &= 3n + 2 \end{aligned}$$

$$\begin{aligned} 2 \cdot 3n + 2 &= \text{mex}(\{0, 1, \dots, 3n - 1\} \cup \{2 \cdot 3n, 2 \cdot 3n + 2, 0 \cdot 3n + 2, 1 \cdot 3n + 2, 1 \cdot 3n + 1, 0 \cdot 3n\}) \\ &= \text{mex}(\{0, 1, \dots, 3n - 1\} \cup \{3n + 1, 3n + 2, 3n + 1, 3n + 2, 3n + 1, 3n + 2\}) \\ &= 3n \end{aligned}$$

The column is again complete, thus completing the proof by induction. \square

Corollary 6.3.7. For $s \equiv_5 6$, column 3 is complete at s .

The next pattern outside the hub depends on where the above pattern stops. There are three cases.

Lemma 6.3.8. *For $s \equiv_3 0$, one has the following:*

$$2 \cdot x = \begin{cases} x + 2 & x = s + 1 \\ x - 2 & x = s + 2 \\ x - 1 & x = s + 3 \\ x + 2 & x - s \equiv_3 1 \\ x - 1 & x - s \equiv_3 0, 2 \end{cases}$$

Proof. Note that $s = 3n$ for some n , so the column becomes complete at $s - 1$.

$$\begin{aligned} 2 \cdot s + 1 &= \text{mex}(\{2 \cdot 0, \dots, 2 \cdot s - 1, 2 \cdot s\} \cup \{1 \cdot s + 1, 0 \cdot s + 10, 1 \cdot s, 0 \cdot s - 1\}) \\ &= \text{mex}(\{0, \dots, s - 1, s + 1\} \cup \{s + 2, s + 1, s, s - 2\}) \\ &= s + 3 \end{aligned}$$

$$\begin{aligned} 2 \cdot s + 2 &= \text{mex}(\{2 \cdot 0, \dots, 2 \cdot s - 1, 2 \cdot s, 2 \cdot s + 12\} \cup \{1 \cdot s + 2, 0 \cdot s + 1, 1 \cdot s + 1, 0 \cdot s\}) \\ &= \text{mex}(\{0, \dots, s - 1, s + 1, s + 3\} \cup \{s + 3, s + 2, s + 2, s - 1\}) \\ &= s \end{aligned}$$

$$\begin{aligned} 2 \cdot s + 3 &= \text{mex}(\{2 \cdot 0, \dots, 2 \cdot s - 1, 2 \cdot 2 \cdot s + 1, 2 \cdot s + 2\} \cup \{1 \cdot s + 3, 0 \cdot s + 3, 1 \cdot s + 2, 0 \cdot s + 1\}) \\ &= \text{mex}(\{0, \dots, s - 1, s + 1, s + 3, s\} \cup \{s + 1, s + 3, s + 3, s + 1\}) \\ &= s + 2 \end{aligned}$$

This concludes the initial calculation part. Note that the column is complete.

$$\begin{aligned} 2 \cdot s + 4 &= \text{mex}(\{2 \cdot 0, \dots, 2 \cdot s + 4\} \cup \{0 \cdot s + 4, 1 \cdot s + 4, 0 \cdot s + 2, 1 \cdot s + 3\}) \\ &= \text{mex}(\{0, \dots, s + 3\} \cup \{s + 4, s + 5, s + 2, s + 1\}) \\ &= s + 6 \end{aligned}$$

$$\begin{aligned}
2 \cdot s + 5 &= \text{mex}(\{2 \cdot 0, \dots, 2 \cdot s + 3, 2 \cdot s + 4\} \cup \{0 \cdot s + 5, 1 \cdot s + 5, 0 \cdot s + 3, 1 \cdot s + 4\}) \\
&= \text{mex}(\{0, \dots, s + 3, s + 6\} \cup \{s + 5, s + 6, s + 3, s + 5\}) \\
&= s + 4
\end{aligned}$$

$$\begin{aligned}
2 \cdot s + 6 &= \text{mex}(\{2 \cdot 0, \dots, 2 \cdot s + 3, 2 \cdot s + 4, 2 \cdot s + 5\} \cup \{0 \cdot s + 6, 1 \cdot s + 6, 0 \cdot s + 4, 1 \cdot s + 5\}) \\
&= \text{mex}(\{0, \dots, s + 3, s + 6, s + 4\} \cup \{s + 6, s + 4, s + 4, s + 6\}) \\
&= s + 5
\end{aligned}$$

At this point the column is complete. By induction, suppose this pattern holds up to $s + 3n$.

$$\begin{aligned}
2 \cdot s + 3n + 1 &= \text{mex}(\{0 \cdot 2, \dots, s + 3n \cdot 2\} \\
&\quad \cup \{0 \cdot s + 3n + 1, 1 \cdot s + 3n + 1, 0 \cdot s + 3n - 1, 1 \cdot s + 3n\}) \\
&= \text{mex}(\{0, \dots, s + 3n\} \cup \{s + 3n + 1, s + 3n + 2, s + 3n - 1, s + 3n - 2\}) \\
&= s + 3n + 3
\end{aligned}$$

$$\begin{aligned}
2 \cdot s + 3n + 2 &= \text{mex}(\{0 \cdot 2, \dots, s + 3n \cdot 2, s + 3n + 1 \cdot 2\} \\
&\quad \cup \{0 \cdot s + 3n + 2, 1 \cdot s + 3n + 2, 0 \cdot s + 3n, 1 \cdot s + 3n + 1\}) \\
&= \text{mex}(\{0, \dots, s + 3n, s + 3n + 3\} \cup \{s + 3n + 2, s + 3n + 3, s + 3n, s + 3n + 2\}) \\
&= s + 3n + 1
\end{aligned}$$

$$\begin{aligned}
2 \cdot s + 3n + 3 &= \text{mex}(\{0 \cdot 2, \dots, s + 3n \cdot 2, s + 3n + 1 \cdot 2, s + 3n + 2 \cdot 2\} \\
&\quad \cup \{0 \cdot s + 3n + 3, 1 \cdot s + 3n + 3, 0 \cdot s + 3n + 1, 1 \cdot s + 3n + 2\}) \\
&= \text{mex}(\{0, \dots, s + 3n, s + 3n + 3, s + 3n + 1\} \\
&\quad \cup \{s + 3n + 3, s + 3n + 1, s + 3n + 1, s + 3n, s + 3n + 3\}) \\
&= s + 3n + 2
\end{aligned}$$

□

Lemma 6.3.9. *For $s \equiv_3 2$, one has the following:*

$$2 \cdot x = \begin{cases} x + 2 & x - s \equiv_3 1 \\ x - 1 & x - s \equiv_3 0, 2 \end{cases}$$

Proof. This pattern is exactly the same as for $s \equiv_3 0$. The proof is likewise the same. Ignore the $s + 1, s + 2, s + 3$ special cases, and start the induction at $s + 1$ after first noting that the column is complete at s . \square

Lemma 6.3.10. *For $s \equiv_3 1$ one has the following:*

$$2 \cdot x = \begin{cases} x - 2 & x = s + 1 \\ x + 2 & x - s \equiv_3 2 \\ x - 1 & x - s \equiv_3 0, 1 \end{cases}$$

Proof. First note that the column is complete at $s - 2$.

$$\begin{aligned} 2 \cdot s + 1 &= \text{mex}(\{2 \cdot 0, \dots, 2 \cdot s\} \cup \{0 \cdot s + 1, 1 \cdot s + 1, 1 \cdot s, 0 \cdot s - 1\}) \\ &= \text{mex}(\{0, \dots, s - 2, s, s + 1\} \cup \{s + 1, s + 2, s - 2, s - 2\}) \\ &= s - 1 \end{aligned}$$

And now the column is complete, since $s \cdot 2 = s + 1, s - 1 \cdot s = s - 1$.

$$\begin{aligned} 2 \cdot s + 2 &= \text{mex}(\{2 \cdot 0, \dots, 2 \cdot s + 1\} \cup \{0 \cdot s + 2, 1 \cdot s + 2, 1 \cdot s + 1, 0 \cdot s\}) \\ &= \text{mex}(\{0, \dots, s + 1\} \cup \{s + 2, s + 3, s + 2, s - 1\}) \\ &= s + 4 \end{aligned}$$

$$\begin{aligned} 2 \cdot s + 3 &= \text{mex}(\{2 \cdot 0, \dots, 2 \cdot s + 1, 2 \cdot s + 2\} \cup \{0 \cdot s + 3, 1 \cdot s + 3, 1 \cdot s + 2, 0 \cdot s + 1\}) \\ &= \text{mex}(\{0, \dots, s + 1, s + 4\} \cup \{s + 3, s + 1, s + 3, s + 1\}) \\ &= s + 2 \end{aligned}$$

$$\begin{aligned} 2 \cdot s + 4 &= \text{mex}(\{2 \cdot 0, \dots, 2 \cdot s + 1, 2 \cdot s + 2, 2 \cdot s + 3\} \cup \{0 \cdot s + 4, 1 \cdot s + 4, 1 \cdot s + 3, 0 \cdot s + 2\}) \\ &= \text{mex}(\{0, \dots, s + 1, s + 4, s + 2\} \cup \{s + 4, s + 5, s + 1, s + 2\}) \\ &= s + 3 \end{aligned}$$

Suppose by induction this holds up to $s + 3n + 1$.

$$\begin{aligned}
2 \cdot s + 3n + 2 &= \text{mex}(\{2 \cdot 0, \dots, 2 \cdot s + 3n + 1\}) \\
&\quad \cup \{0 \cdot s + 3n + 2, 1 \cdot s + 3n + 2, 1 \cdot s + 3n + 1, 0 \cdot s + 3n\}) \\
&= \text{mex}(\{0, \dots, s + 3n + 1\} \cup \{s + 3n + 2, s + 3n + 3, s + 3n + 2, s + 3n - 1\}) \\
&= s + 3n + 4
\end{aligned}$$

$$\begin{aligned}
2 \cdot s + 3n + 3 &= \text{mex}(\{2 \cdot 0, \dots, 2 \cdot s + 3n + 1, 2 \cdot s + 3n + 2\}) \\
&\quad \cup \{0 \cdot s + 3n + 3, 1 \cdot s + 3n + 3, 1 \cdot s + 3n + 2, 0 \cdot s + 3n + 1\}) \\
&= \text{mex}(\{0, \dots, s + 3n + 1, s + 3n + 4\}) \\
&\quad \cup \{s + 3n + 3, s + 3n + 1, s + 3n + 3, s + 3n + 1\}) \\
&= s + 3n + 2
\end{aligned}$$

$$\begin{aligned}
2 \cdot s + 3n + 4 &= \text{mex}(\{2 \cdot 0, \dots, 2 \cdot s + 3n + 1, 2 \cdot s + 3n + 2, 2 \cdot s + 3n + 3\}) \\
&\quad \cup \{0 \cdot s + 3n + 4, 1 \cdot s + 3n + 4, 1 \cdot s + 3n + 3, 0 \cdot s + 3n + 2\}) \\
&= \text{mex}(\{0, \dots, s + 3n + 1, s + 3n + 4, s + 3n + 2\}) \\
&\quad \cup \{s + 3n + 4, s + 3n + 5, s + 3n + 1, s + 3n + 2\}) \\
&= 3 + 3n + 3
\end{aligned}$$

□

Lemma 6.3.11. For $0 \leq x \leq s$

$$3 \cdot x = \begin{cases} x + 2 & x \equiv_6 0, 1, 2, 3 \\ x - 3 & x \equiv_6 4 \\ x - 5 & x \equiv_6 5 \end{cases}$$

Proof.

$$\begin{aligned}
3 \cdot 0 &= \text{mex}(\{0 \cdot 0, 1 \cdot 0, 2 \cdot 0\}) \\
&= \text{mex}(\{s, 0, 1\}) \\
&= 2
\end{aligned}$$

$$\begin{aligned}
3 \cdot 1 &= \text{mex}(\{3 \cdot 0\} \cup \{0 \cdot 1, 1 \cdot 1, 2 \cdot 1\} \cup \{2 \cdot 0\}) \\
&= \text{mex}(\{2\} \cup \{0, 1, 2\} \cup \{1\}) \\
&= 3
\end{aligned}$$

$$\begin{aligned}
3 \cdot 2 &= \text{mex}(\{3 \cdot 0, 3 \cdot 1\} \cup \{0 \cdot 2, 1 \cdot 2, 2 \cdot 2\} \cup \{2 \cdot 1, 1 \cdot 0\}) \\
&= \text{mex}(\{2, 3\} \cup \{1, 2, 0\} \cup \{2, 0\}) \\
&= 4
\end{aligned}$$

$$\begin{aligned}
3 \cdot 3 &= \text{mex}(\{3 \cdot 0, 33 \cdot 1, 3 \cdot 2\} \cup \{0 \cdot 3, 1 \cdot 3, 2 \cdot 3\} \cup \{2 \cdot 2, 1 \cdot 1, 0 \cdot 0\}) \\
&= \text{mex}(\{2, 3, 4\} \cup \{2, 3, 4\} \cup \{0, 1, s\}) \\
&= 5
\end{aligned}$$

$$\begin{aligned}
3 \cdot 4 &= \text{mex}(\{3 \cdot 0, 33 \cdot 1, 3 \cdot 2, 3 \cdot 3\} \cup \{0 \cdot 4, 1 \cdot 4, 2 \cdot 4\} \cup \{2 \cdot 3, 1 \cdot 2, 0 \cdot 1\}) \\
&= \text{mex}(\{2, 3, 4, 5\} \cup \{3, 4, 5\} \cup \{4, 2, 0\}) \\
&= 1
\end{aligned}$$

$$\begin{aligned}
3 \cdot 5 &= \text{mex}(\{3 \cdot 0, 33 \cdot 1, 3 \cdot 2, 3 \cdot 3, 3 \cdot 4\} \cup \{0 \cdot 5, 1 \cdot 5, 2 \cdot 5\} \cup \{2 \cdot 4, 1 \cdot 3, 0 \cdot 2\}) \\
&= \text{mex}(\{2, 3, 4, 5, 1\} \cup \{4, 5, 3\} \cup \{5, 3, 2\}) \\
&= 0
\end{aligned}$$

At this point both column 3 and 2 are complete. Since columns 0,1 have period one, each column is at the same point in its cycle and one can replace x by its congruence class mod 6.

□

Remark 6.3.12. This theorem only holds for $s > 6$, since if $s = 5$, $3 \cdot 3$ cannot be 5 since it is on the same diagonal as $0 \cdot 0 = 5$.

Corollary 6.3.13. For $s \equiv_6 5$, the column is complete in the hub.

Definition 6.3.14. A column is said to be *semi-complete* at x if the column contains all the elements less than x and contains $x + 1$ (but not x itself).

A column is said to be *2-semi-complete* at x if the column contains all the elements less than $x - 1$ and contains $x + 1$ and $x + 2$.

Lemma 6.3.15. For $s \equiv_6 0, 2, 3$ column 3 is semi-complete at $s + 3$

Proof. Look at each case individually.

For $s \equiv_6 0$:

$$\begin{aligned} 3 \cdot s + 1 &= \text{mex}(\{3 \cdot 0, \dots, 3 \cdot s - 1, 3 \cdot s\} \\ &\quad \cup \{0 \cdot s + 1, 1 \cdot s + 1, 2 \cdot s + 1\} \cup \{0 \cdot s - 2, 1 \cdot s - 1, 2 \cdot s\}) \\ &= \text{mex}(\{0, \dots, s - 1, s + 2\} \cup \{s + 1, s + 2, s + 3\} \cup \{s - 3, s - 1, s + 1\}) \\ &= s \end{aligned}$$

$$\begin{aligned} 3 \cdot s + 2 &= \text{mex}(\{3 \cdot 0, \dots, 3 \cdot s - 1, 3 \cdot s, 3 \cdot s + 1\} \\ &\quad \cup \{0 \cdot s + 2, 1 \cdot s + 2, 2 \cdot s + 2\} \cup \{0 \cdot s - 1, 1 \cdot s, 2 \cdot s + 1\}) \\ &= \text{mex}(\{0, \dots, s - 1, s + 2, s\} \cup \{s + 2, s + 3, s\} \cup \{s - 2, s, s + 3\}) \\ &= s + 1 \end{aligned}$$

$$\begin{aligned} 3 \cdot s + 3 &= \text{mex}(\{3 \cdot 0, \dots, 3 \cdot s - 1, 3 \cdot s, 3 \cdot s + 1, 3 \cdot s + 2\} \\ &\quad \cup \{0 \cdot s + 3, 1 \cdot s + 3, 2 \cdot s + 3\} \cup \{0 \cdot s, 1 \cdot s + 1, 2 \cdot s + 2\}) \\ &= \text{mex}(\{0, \dots, s - 1, s + 2, s, s + 1\} \cup \{s + 3, s + 1, s + 2\} \cup \{s - 1, s + 2, s\}) \\ &= s + 4 \end{aligned}$$

So at $s + 3$ column 3 contains $0, \dots, s + 2$ and $s + 4$. Thus it is semi-complete there.

Now, for $s \equiv_6 2$:

$$\begin{aligned} 3 \cdot s + 1 &= \text{mex}(\{3 \cdot 0, \dots, 3 \cdot s - 3, 3 \cdot s - 2, 3 \cdot s - 1, 3 \cdot s\} \\ &\quad \cup \{0 \cdot s + 1, 1 \cdot s + 1, 2 \cdot s + 1\} \cup \{0 \cdot s - 2, 1 \cdot s - 1, 2 \cdot s\}) \\ &= \text{mex}(\{0, \dots, s - 3, s, s + 1, s + 2\} \cup \{s + 1, s + 2, s + 3\} \cup \{s - 3, s - 1, s - 2\}) \\ &= s + 4 \end{aligned}$$

$$\begin{aligned}
3 \cdot s + 2 &= \text{mex}(\{3 \cdot 0, \dots, 3 \cdot s - 3, 3 \cdot s - 2, 3 \cdot s - 1, 3 \cdot s, 3 \cdot s + 1\} \\
&\quad \cup \{0 \cdot s + 2, 1 \cdot s + 2, 2 \cdot s + 2\} \cup \{0 \cdot s - 1, 1 \cdot s, 2 \cdot s + 1\}) \\
&= \text{mex}(\{0, \dots, s - 3, s, s + 1, s + 2, s + 4\} \cup \{s + 2, s + 3, s + 1\} \cup \{s - 2, s, s + 3\}) \\
&= s - 1
\end{aligned}$$

$$\begin{aligned}
3 \cdot s + 2 &= \text{mex}(\{3 \cdot 0, \dots, 3 \cdot s - 3, 3 \cdot s - 2, 3 \cdot s - 1, 3 \cdot s, 3 \cdot s + 1, 3 \cdot s + 2\} \\
&\quad \cup \{0 \cdot s + 3, 1 \cdot s + 3, 2 \cdot s + 3\} \cup \{0 \cdot s, 1 \cdot s + 1, 2 \cdot s + 2\}) \\
&= \text{mex}(\{0, \dots, s - 3, s, s + 1, s + 2, s + 4, s - 1\} \cup \{s + 3, s + 1, s + 2\} \\
&\quad \cup \{s - 1, s + 2, s + 1\}) \\
&= s - 2
\end{aligned}$$

At this point the column is $\{0, \dots, s - 3, s, s + 1, s + 2, s + 4, s - 1, s - 2\}$. Thus it is semi-complete at $s + 3$.

$$\begin{aligned}
3 \cdot s + 1 &= \text{mex}(\{3 \cdot 0, \dots, 3 \cdot s\} \\
&\quad \cup \{0 \cdot s + 1, 1 \cdot s + 1, 2 \cdot s + 1\} \cup \{0 \cdot s - 2, 1 \cdot s - 1, 2 \cdot s\}) \\
&= \text{mex}\{0, \dots, s - 4, s - 1, s, s + 1, s + 2\} \cup \{s + 1, s + 2, s + 3\} \cup \{s - 3, s - 1, s + 1\}) \\
&= s - 2
\end{aligned}$$

$$\begin{aligned}
3 \cdot s + 2 &= \text{mex}(\{3 \cdot 0, \dots, 3 \cdot s + 1\} \\
&\quad \cup \{0 \cdot s + 2, 1 \cdot s + 2, 2 \cdot s + 2\} \cup \{0 \cdot s - 1, 1 \cdot s, 0 \cdot s + 1\}) \\
&= \text{mex}(\{0, \dots, s - 4, s - 1, s, s + 1, s + 2, s - 2\} \cup \{s + 2, s + 3, s\} \cup \{s - 2, s, s + 3\}) \\
&= s - 3
\end{aligned}$$

$$\begin{aligned}
3 \cdot s + 3 &= \text{mex}(\{3 \cdot 0, \dots, 3 \cdot s + 2\} \\
&\quad \cup \{0 \cdot s + 3, 1 \cdot s + 3, 2 \cdot s + 3\} \cup \{0 \cdot s, 1 \cdot s + 1, 0 \cdot s + 2\}) \\
&= \text{mex}(\{0, \dots, s - 4, s - 1, s, s + 1, s + 2, s - 2, s - 3\} \cup \{s + 3, s + 1, s + 2\} \\
&\quad \cup \{s - 1, s + 2, s\}) \\
&= s + 4
\end{aligned}$$

At this point the column is $\{0, \dots, s-4, s-1, s, s+1, s+2, s-2, s-3, s+4\}$, so the column is semi-complete at $s+3$. \square

Lemma 6.3.16. *For $s \equiv_6 0, 2, 3$*

$$3 \cdot x = \begin{cases} x+3 & x \equiv_6 4 \\ x-2 & x \equiv_6 5 \\ x+2 & x \equiv_6 0 \\ x+3 & x \equiv_6 1 \\ x-2 & x \equiv_6 2 \\ x-4 & x \equiv_6 3 \end{cases}$$

Additionally the column is semi-complete every six terms.

Proof. Note that after $s+3$ the pattern for $2 \cdot x$ is identical for $2 \equiv_3 0$ and $s \equiv_3 2$.

$$\begin{aligned} 3 \cdot s+4 &= \text{mex}(\{3 \cdot 0, \dots, 3 \cdot s+3\} \cup \{0 \cdot s+4, 1 \cdot s+4, 2 \cdot s+4\} \cup \{0 \cdot s+1, 1 \cdot s+2, 2 \cdot s+3\}) \\ &= \text{mex}(\{0, \dots, s+2, s+4\} \cup \{s+4, s+5, s+6\} \cup \{s+1, s+3, s+2\}) \\ &= s+7 \end{aligned}$$

$$\begin{aligned} 3 \cdot s+5 &= \text{mex}(\{3 \cdot 0, \dots, 3 \cdot s+3, 3 \cdot s+4\} \cup \{0 \cdot s+5, 1 \cdot s+5, 2 \cdot s+5\} \\ &\quad \cup \{0 \cdot s+2, 1 \cdot s+3, 2 \cdot s+4\}) \\ &= \text{mex}(\{0, \dots, s+2, s+4, s+7\} \cup \{s+5, s+6, s+4\} \cup \{s+2, s+1, s+6\}) \\ &= s+3 \end{aligned}$$

$$\begin{aligned} 3 \cdot s+6 &= \text{mex}(\{3 \cdot 0, \dots, 3 \cdot s+3, 3 \cdot s+4, 3 \cdot s+5\} \\ &\quad \cup \{0 \cdot s+6, 1 \cdot s+6, 2 \cdot s+6\} \cup \{0 \cdot s+3, 1 \cdot s+4, 2 \cdot s+5\}) \\ &= \text{mex}(\{0, \dots, s+2, s+4, s+7, s+3\} \cup \{s+6, s+4, s+5\} \cup \{s+3, s+5, s+4\}) \\ &= s+8 \end{aligned}$$

$$\begin{aligned}
3 \cdot s + 7 &= \text{mex}(\{3 \cdot 0, \dots, 3 \cdot s + 3, 3 \cdot s + 4, 3 \cdot s + 5, 3 \cdot s + 6\}) \\
&\cup \{0 \cdot s + 7, 1 \cdot s + 7, 2 \cdot s + 7\} \cup \{0 \cdot s + 4, 1 \cdot s + 5, 2 \cdot s + 6\}) \\
&= \text{mex}(\{0, \dots, s + 2, s + 4, s + 7, s + 3\} \cup \{s + 7, s + 8, s + 9\} \cup \{s + 4, s + 6, s + 5\}) \\
&= s + 10
\end{aligned}$$

$$\begin{aligned}
3 \cdot s + 8 &= \text{mex}(\{3 \cdot 0, \dots, 3 \cdot s + 3, \dots, 3 \cdot s + 7\}) \\
&\cup \{0 \cdot s + 8, 1 \cdot s + 8, 2 \cdot s + 8\} \cup \{0 \cdot s + 5, 1 \cdot s + 6, 2 \cdot s + 7\}) \\
&= \text{mex}(\{0, \dots, s + 2, s + 4, s + 7, s + 3, s + 8, s + 10\} \cup \{s + 8, s + 9, s + 7\} \\
&\cup \{s + 5, s + 4, s + 9\}) \\
&= s + 6
\end{aligned}$$

$$\begin{aligned}
3 \cdot s + 9 &= \text{mex}(\{3 \cdot 0, \dots, 3 \cdot s + 3, \dots, 3 \cdot s + 8\}) \\
&\cup \{0 \cdot s + 9, 1 \cdot s + 9, 2 \cdot s + 9\} \cup \{0 \cdot s + 6, 1 \cdot s + 7, 2 \cdot s + 8\}) \\
&= \text{mex}(\{0, \dots, s + 2, s + 4, s + 7, s + 3, s + 8, s + 10, s + 6\} \\
&\cup \{s + 9, s + 7, s + 8\} \cup \{s + 6, s + 8, s + 7\}) \\
&= s + 5
\end{aligned}$$

At this point the column is $\{0, \dots, s + 8, s + 10\}$ so it is semi-complete at $s + 9$. Now since column 2 has a period of three and columns 0,1 have a period of 1, the same point in their periods is reached as was at $s + 3$, so one can replace each term by its distance from the seed modulo 6 and do the same proof. \square

Lemma 6.3.17. *For $s \equiv_6 5$ (and $s \neq 5$):*

$$x \cdot 3 = \begin{cases} x + 3 & x = s + 1, s + 2, s + 3, s + 4 \\ x - 2 & x = s + 5 \\ x - 5 & x = s + 6, s + 7 \\ x + 2 & x = s + 8, s + 9 \end{cases}$$

Proof.

$$\begin{aligned}
3 \cdot s + 1 &= \text{mex}(\{3 \cdot 0, \dots, 3 \cdot s\} \cup \{0 \cdot s + 1, 1 \cdot s + 1, 2 \cdot s + 1\} \cup \{0 \cdot s - 2, 1 \cdot s - 1, 2 \cdot s\}) \\
&= \text{mex}(\{0, \dots, s\} \cup \{s + 1, s + 2, s + 3\} \cup \{s - 3, s - 1, s - 2\}) \\
&= s + 4
\end{aligned}$$

$$\begin{aligned}
3 \cdot s + 2 &= \text{mex}(\{3 \cdot 0, \dots, 3 \cdot s, 3 \cdot s + 1\} \cup \{0 \cdot s + 2, 1 \cdot s + 2, 2 \cdot s + 2\} \cup \{0 \cdot s - 1, 1 \cdot s, 2 \cdot s + 1\}) \\
&= \text{mex}(\{0, \dots, s, s + 4\} \cup \{s + 2, s + 3, s + 1\} \cup \{s - 2, s, s - 2, s + 3\}) \\
&= s + 5
\end{aligned}$$

$$\begin{aligned}
3 \cdot s + 3 &= \text{mex}(\{3 \cdot 0, \dots, 3 \cdot s, 3 \cdot s + 1, 3 \cdot s + 2\} \\
&\quad \cup \{0 \cdot s + 3, 1 \cdot s + 3, 2 \cdot s + 3\} \cup \{0 \cdot s, 1 \cdot s + 1, 2 \cdot s + 2\}) \\
&= \text{mex}(\{0, \dots, s, s + 4, s + 5\} \cup \{s + 3, s + 1, s + 2\} \cup \{s - 1, s + 2, s + 1\}) \\
&= s + 6
\end{aligned}$$

$$\begin{aligned}
3 \cdot s + 4 &= \text{mex}(\{3 \cdot 0, \dots, 3 \cdot s, 3 \cdot s + 1, 3 \cdot s + 2, 3 \cdot s + 3\} \\
&\quad \cup \{0 \cdot s + 4, 1 \cdot s + 4, 2 \cdot s + 4\} \cup \{0 \cdot s + 1, 1 \cdot s + 2, 2 \cdot s + 3\}) \\
&= \text{mex}(\{0, \dots, s, s + 4, s + 5, s + 6\} \cup \{s + 4, s + 5, s + 6\} \cup \{s + 1, s + 3, s + 2\}) \\
&= s + 7
\end{aligned}$$

$$\begin{aligned}
3 \cdot s + 5 &= \text{mex}(\{3 \cdot 0, \dots, 3 \cdot s, 3 \cdot s + 1, 3 \cdot s + 2, 3 \cdot s + 3, 3 \cdot s + 4\} \\
&\quad \cup \{0 \cdot s + 5, 1 \cdot s + 5, 2 \cdot s + 5\} \cup \{0 \cdot s + 2, 1 \cdot s + 3, 2 \cdot s + 4\}) \\
&= \text{mex}(\{0, \dots, s, s + 4, s + 5, s + 6, s + 7\} \cup \{s + 5, s + 6, s + 4\} \cup \{s + 2, s + 1, s + 6\}) \\
&= s + 3
\end{aligned}$$

$$\begin{aligned}
3 \cdot s + 6 &= \text{mex}(\{3 \cdot 0, \dots, 3 \cdot s, 3 \cdot 1, \dots, 3 \cdot s + 4, 3 \cdot s + 5\} \\
&\quad \cup \{0 \cdot s + 6, 1 \cdot s + 6, 2 \cdot s + 6\} \cup \{0 \cdot s + 3, 1 \cdot s + 4, 2 \cdot s + 5\}) \\
&= \text{mex}(\{0, \dots, s, s + 4, s + 5, s + 6, s + 7, s + 3\} \cup \{s + 6, s + 4, s + 5\} \cup \{s + 3, s + 5, s + 4\}) \\
&= s + 1
\end{aligned}$$

$$\begin{aligned}
3 \cdot s + 7 &= \text{mex}(\{3 \cdot 0, \dots, 3 \cdot s, 3 \cdot 1, \dots, 3 \cdot s + 4, 3 \cdot s + 5, 3 \cdot s + 6\} \\
&\quad \cup \{0 \cdot s + 7, 1 \cdot s + 7, 2 \cdot s + 7\} \cup \{0 \cdot s + 4, 1 \cdot s + 5, 2 \cdot s + 6\}) \\
&= \text{mex}(\{0, \dots, s, s + 4, s + 5, s + 6, s + 7, s + 3, s + 1\} \cup \{s + 7, s + 8, s + 9\} \\
&\quad \cup \{s + 4, s + 6, s + 5\}) \\
&= s + 2
\end{aligned}$$

At this point the column is actually complete.

$$\begin{aligned}
3 \cdot s + 8 &= \text{mex}(\{3 \cdot 0, \dots, 3 \cdot s + 7\} \cup \{0 \cdot s + 8, 1 \cdot s + 8, 2 \cdot s + 8\} \cup \{0 \cdot s + 5, 1 \cdot s + 6, 2 \cdot s + 7\}) \\
&= \text{mex}(\{0, \dots, s + 7\} \cup \{s + 8, s + 9, s + 7\} \cup \{s + 5, s + 4, s + 9\}) \\
&= s + 10
\end{aligned}$$

$$\begin{aligned}
3 \cdot s + 9 &= \text{mex}(\{3 \cdot 0, \dots, 3 \cdot s + 7, 3 \cdot s + 8\} \cup \{0 \cdot s + 9, 1 \cdot s + 9, 2 \cdot s + 9\} \\
&\quad \cup \{0 \cdot s + 6, 1 \cdot s + 7, 2 \cdot s + 8\}) \\
&= \text{mex}(\{0, \dots, s + 7, s + 10\} \cup \{s + 9, s + 7, s + 8\} \cup \{s + 6, s + 8, s + 7\}) \\
&= s + 11
\end{aligned}$$

□

Remark 6.3.18. At this point the column is 2-semi-complete.

Lemma 6.3.19. *Let $s \equiv_6 5$. For $x \geq s + 10$*

$$3 \cdot x = \begin{cases} x + 3 & x - s \equiv_6 4 \\ x - 2 & x - s \equiv_6 5 \\ x - 4 & x - s \equiv_6 0 \\ x + 3 & x - s \equiv_6 1 \\ x - 2 & x - s \equiv_6 2 \\ x + 2 & x - s \equiv_6 3 \end{cases}$$

Furthermore, the column is 2-semi-complete at x when $x \equiv_6 3$.

Proof. Note that the column is 2-semi-complete at $s + 9$.

$$\begin{aligned}
3 \cdot s + 10 &= \text{mex}(\{3 \cdot 0, \dots, 3 \cdot s + 9\} \cup \{0 \cdot s + 10, 1 \cdot s + 10, 2 \cdot s + 10\} \cup \{0 \cdot s + 7, 1 \cdot s + 8, 2 \cdot s + 9\}) \\
&= \text{mex}(\{0, \dots, s + 7, s + 10, s + 11\} \cup \{s + 10, s + 11, s + 12\} \cup \{s + 7, s + 9, s + 8\}) \\
&= s + 13
\end{aligned}$$

$$\begin{aligned}
3 \cdot s + 11 &= \text{mex}(\{3 \cdot 0, \dots, 3 \cdot s + 9, 3 \cdot s + 10\} \\
&\quad \cup \{0 \cdot s + 11, 1 \cdot s + 11, 2 \cdot s + 11\} \cup \{0 \cdot s + 8, 1 \cdot s + 9, 2 \cdot s + 10\}) \\
&= \text{mex}(\{0, \dots, s + 7, s + 10, s + 11, s + 13\} \cup \{s + 11, s + 12, s + 10\} \cup \{s + 8, s + 7, s + 12\}) \\
&= s + 9
\end{aligned}$$

$$\begin{aligned}
3 \cdot s + 12 &= \text{mex}(\{3 \cdot 0, \dots, 3 \cdot s + 9, 3 \cdot s + 10, 3 \cdot s + 11\} \\
&\quad \cup \{0 \cdot s + 12, 1 \cdot s + 12, 2 \cdot s + 12\} \cup \{0 \cdot s + 9, 1 \cdot s + 10, 2 \cdot s + 11\}) \\
&= \text{mex}(\{0, \dots, s + 7, s + 10, s + 11, s + 13, s + 9\} \\
&\quad \cup \{s + 12, s + 10, s + 11\} \cup \{s + 9, s + 11, s + 10\}) \\
&= s + 8
\end{aligned}$$

$$\begin{aligned}
3 \cdot s + 13 &= \text{mex}(\{3 \cdot 0, \dots, 3 \cdot s + 9, 3 \cdot s + 10, 3 \cdot s + 11, 3 \cdot s + 12\} \\
&\quad \cup \{0 \cdot s + 13, 1 \cdot s + 13, 2 \cdot s + 13\} \cup \{0 \cdot s + 10, 1 \cdot s + 11, 2 \cdot s + 12\}) \\
&= \text{mex}(\{0, \dots, s + 7, s + 10, s + 11, s + 13, s + 9, s + 8\} \\
&\quad \cup \{s + 13, s + 14, s + 15\} \cup \{s + 10, s + 12, s + 11\}) \\
&= s + 16
\end{aligned}$$

$$\begin{aligned}
3 \cdot s + 14 &= \text{mex}(\{3 \cdot 0, \dots, 3 \cdot s + 9, 3 \cdot s + 10, 3 \cdot s + 11, 3 \cdot s + 12, 3 \cdot s + 13\} \\
&\quad \cup \{0 \cdot s + 14, 1 \cdot s + 14, 2 \cdot s + 14\} \cup \{0 \cdot s + 11, 1 \cdot s + 12, 2 \cdot s + 13\}) \\
&= \text{mex}(\{0, \dots, s + 7, s + 10, s + 11, s + 13, s + 9, s + 8, s + 16\} \\
&\quad \cup \{s + 14, s + 15, s + 13\} \cup \{s + 11, s + 10, s + 15\}) \\
&= s + 12
\end{aligned}$$

$$\begin{aligned}
3 \cdot s + 15 &= \text{mex}(\{3 \cdot 0, \dots, 3 \cdot s + 9, 3 \cdot s + 10, 3 \cdot s + 11, 3 \cdot s + 12, 3 \cdot s + 13, 3 \cdot 14\} \\
&\quad \cup \{0 \cdot s + 15, 1 \cdot s + 15, 2 \cdot s + 15\} \cup \{0 \cdot s + 12, 1 \cdot s + 13, 2 \cdot s + 14\}) \\
&= \text{mex}(\{0, \dots, s + 7, s + 10, s + 11, s + 13, s + 9, s + 8, s + 16, s + 12\} \\
&\quad \cup \{s + 15, s + 13, s + 14\} \cup \{s + 12, s + 14, s + 13\}) \\
&= s + 17
\end{aligned}$$

At this point column 3 is $\{0, \dots, s + 13, s + 16, s + 17\}$ so it is 2-semi-complete at $s + 17$. Since column 2 has period 3, it has completed two complete periods and it at the same place in its period as it was 6 steps ago. Thus one can replace x by its congruence class mod 6 in this proof. \square

Corollary 6.3.20. *For $s \equiv_6 5$ column 3 is never complete outside the hub.*

Lemma 6.3.21. *Inside the hub, column 4 is complete at $x \equiv_{18} 12$ and only at those places.*

Proof.

$$4 \cdot 0 = \text{mex}(\{3 \cdot 0, 2 \cdot 0, 1 \cdot 0, 0 \cdot 0\})$$

$$= \text{mex}(\{2, 1, 0, s\}) = 3$$

$$4 \cdot 1 = \text{mex}(\{4 \cdot 0, 3 \cdot 1, 2 \cdot 1, 1 \cdot 1, 0 \cdot 1, 3 \cdot 0\})$$

$$= \text{mex}(\{3, 3, 2, 1, 0, 3\}) = 4$$

$$4 \cdot 2 = \text{mex}(\{4 \cdot 0, 4 \cdot 1, 3 \cdot 2, 2 \cdot 2, 1 \cdot 2, 0 \cdot 2, 3 \cdot 1, 2 \cdot 0\})$$

$$= \text{mex}(\{3, 4, 4, 0, 2, 1, 3, 1\}) = 5$$

$$4 \cdot 3 = \text{mex}(\{4 \cdot 0, 4 \cdot 1, 4 \cdot 2, 3 \cdot 3, 2 \cdot 3, 1 \cdot 3, 0 \cdot 3, 3 \cdot 2, 2 \cdot 1, 1 \cdot 0\})$$

$$= \text{mex}(\{3, 4, 5, 5, 4, 3, 2, 4, 2, 0\}) = 1$$

$$4 \cdot 4 = \text{mex}(\{4 \cdot 0, 4 \cdot 1, 4 \cdot 2, 4 \cdot 3, 3 \cdot 4, 2 \cdot 4, 1 \cdot 4, 0 \cdot 4, 3 \cdot 3, 2 \cdot 2, 1 \cdot 1, 0 \cdot 0\})$$

$$= \text{mex}(\{3, 4, 5, 1, 1, 5, 4, 3, 5, 0, 1, s\}) = 2$$

$$4 \cdot 5 = \text{mex}(\{4 \cdot 0, 4 \cdot 1, 4 \cdot 2, 4 \cdot 3, 4 \cdot 4, 3 \cdot 5, 2 \cdot 5, 1 \cdot 5, 0 \cdot 5, 3 \cdot 4, 2 \cdot 3, 1 \cdot 2, 0 \cdot 1\})$$

$$= \text{mex}(\{3, 4, 5, 1, 2, 0, 3, 5, 4, 1, 4, 2, 0\}) = 6$$

$$4 \cdot 6 = \text{mex}(\{4 \cdot 0, 4 \cdot 1, 4 \cdot 2, 4 \cdot 3, 4 \cdot 4, 4 \cdot 5, 3 \cdot 6, 2 \cdot 6, 1 \cdot 6, 0 \cdot 6, 3 \cdot 5, 2 \cdot 4, 1 \cdot 3, 0 \cdot 2\})$$

$$= \text{mex}(\{3, 4, 5, 1, 2, 6, 8, 7, 6, 5, 0, 5, 3, 1\}) = 9$$

$$4 \cdot 7 = \text{mex}(\{4 \cdot 0, 4 \cdot 1, 4 \cdot 2, 4 \cdot 3, 4 \cdot 4, 4 \cdot 5, 4 \cdot 6, 3 \cdot 7, 2 \cdot 7, 1 \cdot 7, 0 \cdot 7, 3 \cdot 6, 2 \cdot 5, 1 \cdot 4, 0 \cdot 3\})$$

$$= \text{mex}(\{3, 4, 5, 1, 2, 6, 9, 9, 8, 7, 6, 8, 3, 4, 2\}) = 0$$

$$4 \cdot 8 = \text{mex}(\{4 \cdot 0, 4 \cdot 1, 4 \cdot 2, 4 \cdot 3, 4 \cdot 4, 4 \cdot 5, 4 \cdot 6, 4 \cdot 7, 3 \cdot 8, 2 \cdot 8, 1 \cdot 8, 0 \cdot 8, 3 \cdot 7, 2 \cdot 6, 1 \cdot 5, 0 \cdot 4\})$$

$$= \text{mex}(\{3, 4, 5, 1, 2, 6, 9, 0, 10, 6, 8, 7, 9, 7, 5, 3\}) = 11$$

$$4 \cdot 9 = \text{mex}(\{4 \cdot 0, 4 \cdot 1, 4 \cdot 2, 4 \cdot 3, 4 \cdot 4, 4 \cdot 5, 4 \cdot 6, 4 \cdot 7, 4 \cdot 8, 3 \cdot 9, 2 \cdot 9, 1 \cdot 9, 0 \cdot 9,$$

$$3 \cdot 8, 2 \cdot 7, 1 \cdot 6, 0 \cdot 5\})$$

$$= \text{mex}(\{3, 4, 5, 1, 2, 6, 9, 0, 11, 11, 10, 9, 8, 10, 8, 6, 4\}) = 7$$

$$4 \cdot 10 = \text{mex}(\{4 \cdot 0, 4 \cdot 1, 4 \cdot 2, 4 \cdot 3, 4 \cdot 4, 4 \cdot 5, 4 \cdot 6, 4 \cdot 7, 4 \cdot 8, 4 \cdot 9, 3 \cdot 10, 2 \cdot 10, 1 \cdot 10, 0 \cdot 10, \\ 3 \cdot 9, 2 \cdot 8, 1 \cdot 7, 0 \cdot 6\})$$

$$= \text{mex}(\{3, 4, 5, 1, 2, 6, 9, 0, 11, 7, 7, 11, 10, 9, 11, 6, 7, 5\}) = 8$$

$$4 \cdot 11 = \text{mex}(\{4 \cdot 0, 4 \cdot 1, 4 \cdot 2, 4 \cdot 3, 4 \cdot 4, 4 \cdot 5, 4 \cdot 6, 4 \cdot 7, 4 \cdot 8, 4 \cdot 9, 4 \cdot 10, 3 \cdot 11, 2 \cdot 11, 1 \cdot 11, 0 \cdot 11, \\ 3 \cdot 10, 2 \cdot 9, 1 \cdot 8, 0 \cdot 7\})$$

$$= \text{mex}(\{3, 4, 5, 1, 2, 6, 9, 0, 11, 7, 8, 6, 9, 11, 10, 7, 10, 8, 6\}) = 12$$

$$4 \cdot 12 = \text{mex}(\{4 \cdot 0, 4 \cdot 1, 4 \cdot 2, 4 \cdot 3, 4 \cdot 4, 4 \cdot 5, 4 \cdot 6, 4 \cdot 7, 4 \cdot 8, 4 \cdot 9, 4 \cdot 10, 4 \cdot 11, 3 \cdot 12, \\ 2 \cdot 12, 1 \cdot 12, 0 \cdot 12,$$

$$3 \cdot 11, 2 \cdot 10, 1 \cdot 9, 0 \cdot 8\})$$

$$= \text{mex}(\{3, 4, 5, 1, 2, 6, 9, 0, 11, 7, 8, 12, 14, 13, 12, 11, 6, 11, 9, 7\}) = 10$$

And column 4 is complete. Now suppose that column 4 is complete for at $12 + 18m$. Show that it is complete at $12 + 18(m + 1)$.

$$4 \cdot (12 + 18m + 1) = \text{mex}(\{4 \cdot 0, \dots, 4 \cdot (12 + 18m),$$

$$3 \cdot (12 + 18m + 1), 2 \cdot (12 + 18m + 1), 1 \cdot (12 + 18m + 1), 0 \cdot (12 + 18m + 1),$$

$$3 \cdot (12 + 18m), 2 \cdot (12 + 18m - 1), 1 \cdot (12 + 18m - 2), 0 \cdot (12 + 18m - 3)\})$$

$$= \text{mex}(\{0, \dots, 12 + 18m,$$

$$18m + 15, 18m + 14, 18m + 13, 18m + 12,$$

$$18m + 14, 18m + 9, 18m + 10, 18m + 8\}) = 18m + 16$$

$$4 \cdot (12 + 18m + 2) = \text{mex}(\{4 \cdot 0, \dots, 4 \cdot (12 + 18m), 4 \cdot (12 + 18m + 1),$$

$$3 \cdot (12 + 18m + 2), 2 \cdot (12 + 18m + 2), 1 \cdot (12 + 18m + 2), 0 \cdot (12 + 18m + 2),$$

$$3 \cdot (12 + 18m + 1), 2 \cdot (12 + 18m), 1 \cdot (12 + 18m - 1), 0 \cdot (12 + 18m - 2)\})$$

$$= \text{mex}(\{0, \dots, 18m + 12, 18m + 16,$$

$$18m + 16, 18m + 12, 18m + 14, 18m + 13,$$

$$18 + 15, 18m + 13, 18m + 11, 18m + 9\}) = 18m + 17$$

$$\begin{aligned}
4 \cdot (12 + 18m + 3) &= \text{mex}(\{4 \cdot 0, \dots, 4 \cdot (12 + 18m), 4 \cdot (12 + 18m + 1), 4 \cdot (12 + 18m + 2), \\
&\quad 3 \cdot (12 + 18m + 3), 2 \cdot (12 + 18m + 3), 1 \cdot (12 + 18m + 3), 0 \cdot (12 + 18m + 3), \\
&\quad 3 \cdot (12 + 18m + 2), 2 \cdot (12 + 18m + 1), 1 \cdot (12 + 18m), 0 \cdot (12 + 18m - 1)\}) \\
&= \text{mex}(\{0, \dots, 18m + 12, 18m + 16, 18m + 17, \\
&\quad 18m + 16, 18m + 16, 18m + 15, 18m + 14, \\
&\quad 18m + 16, 18m + 14, 18m + 12, 18m + 10\}) = 18m + 13 \\
4 \cdot (12 + 18m + 4) &= \text{mex}(\{4 \cdot 0, \dots, 4 \cdot (12 + 18m + 2), 4 \cdot (12 + 18m + 3) \\
&\quad 3 \cdot (12 + 18m + 4), 2 \cdot (12 + 18m + 4), 1 \cdot (12 + 18m + 4), 0 \cdot (12 + 18m + 4), \\
&\quad 3 \cdot (12 + 18m + 3), 2 \cdot (12 + 18m + 2), 1 \cdot (12 + 18m + 1), 0 \cdot (12 + 18m)\}) \\
&= \text{mex}(\{0, \dots, 18m + 12, 18m + 16, 18m + 17, 18m + 13 \\
&\quad 18m + 13, 18m + 17, 18m + 16, 18m + 15, \\
&\quad 18m + 17, 18m + 12, 18m + 13, 18m + 11\}) = 18m + 14 \\
4 \cdot (12 + 18m + 5) &= \text{mex}(\{4 \cdot 0, \dots, 4 \cdot (12 + 18m + 3), 4 \cdot (12 + 18m + 4) \\
&\quad 3 \cdot (12 + 18m + 5), 2 \cdot (12 + 18m + 5), 1 \cdot (12 + 18m + 5), 0 \cdot (12 + 18m + 5), \\
&\quad 3 \cdot (12 + 18m + 4), 2 \cdot (12 + 18m + 3), 1 \cdot (12 + 18m + 2), 0 \cdot (12 + 18m + 1)\}) \\
&= \text{mex}(\{0, \dots, 18m + 12, 18m + 16, 18m + 17, 18m + 13, 18m + 14 \\
&\quad 18m + 12, 18m + 15, 18m + 17, 18m + 16, \\
&\quad 18m + 13, 18m + 16, 18m + 14, 18m + 12\}) = 18m + 18 \\
4 \cdot (12 + 18m + 6) &= \text{mex}(\{4 \cdot 0, \dots, 4 \cdot (12 + 18m + 4), 4 \cdot (12 + 18m + 5) \\
&\quad 3 \cdot (12 + 18m + 6), 2 \cdot (12 + 18m + 6), 1 \cdot (12 + 18m + 6), 0 \cdot (12 + 18m + 6), \\
&\quad 3 \cdot (12 + 18m + 5), 2 \cdot (12 + 18m + 4), 1 \cdot (12 + 18m + 3), 0 \cdot (12 + 18m + 2)\}) \\
&= \text{mex}(\{0, \dots, 18m + 14, 18m + 16, 18m + 17, 18m + 18 \\
&\quad 18m + 20, 18m + 19, 18m + 18, 18m + 17, \\
&\quad 18m + 12, 18m + 17, 18m + 15, 18m + 13\}) = 18m + 21
\end{aligned}$$

$$\begin{aligned}
4 \cdot (12 + 18m + 7) &= \text{mex}(\{4 \cdot 0, \dots, 4 \cdot (12 + 18m + 5), 4 \cdot (12 + 18m + 6) \\
&\quad 3 \cdot (12 + 18m + 7), 2 \cdot (12 + 18m + 7), 1 \cdot (12 + 18m + 7), 0 \cdot (12 + 18m + 7), \\
&\quad 3 \cdot (12 + 18m + 6), 2 \cdot (12 + 18m + 5), 1 \cdot (12 + 18m + 4), 0 \cdot (12 + 18m + 3)\}) \\
&= \text{mex}(\{0, \dots, 18m + 14, 18m + 16, 18m + 17, 18m + 18, 18m + 21 \\
&\quad 18m + 21, 18m + 20, 18m + 19, 18m + 18, \\
&\quad 18m + 20, 18m + 15, 18m + 16, 18m + 14\}) = 18m + 22
\end{aligned}$$

$$\begin{aligned}
4 \cdot (12 + 18m + 8) &= \text{mex}(\{4 \cdot 0, \dots, 4 \cdot (12 + 18m + 6), 4 \cdot (12 + 18m + 7) \\
&\quad 3 \cdot (12 + 18m + 8), 2 \cdot (12 + 18m + 8), 1 \cdot (12 + 18m + 8), 0 \cdot (12 + 18m + 8), \\
&\quad 3 \cdot (12 + 18m + 7), 2 \cdot (12 + 18m + 6), 1 \cdot (12 + 18m + 5), 0 \cdot (12 + 18m + 4)\}) \\
&= \text{mex}(\{0, \dots, 18m + 14, 18m + 16, 18m + 17, 18m + 18, 18m + 21, 18m + 22 \\
&\quad 18m + 22, 18m + 18, 18m + 20, 18m + 19, \\
&\quad 18m + 21, 18m + 19, 18m + 17, 18m + 15\}) = 18m + 23
\end{aligned}$$

$$\begin{aligned}
4 \cdot (12 + 18m + 9) &= \text{mex}(\{4 \cdot 0, \dots, 4 \cdot (12 + 18m + 7), 4 \cdot (12 + 18m + 8) \\
&\quad 3 \cdot (12 + 18m + 9), 2 \cdot (12 + 18m + 9), 1 \cdot (12 + 18m + 9), 0 \cdot (12 + 18m + 9), \\
&\quad 3 \cdot (12 + 18m + 8), 2 \cdot (12 + 18m + 7), 1 \cdot (12 + 18m + 6), 0 \cdot (12 + 18m + 5)\}) \\
&= \text{mex}(\{0, \dots, 18m + 14, 18m + 16, 18m + 17, 18m + 18, 18m + 21, 18m + 22, \\
&\quad 18m + 23, 18m + 23, 18m + 22, 18m + 21, 18m + 20 \\
&\quad 18m + 22, 18m + 20, 18m + 18, 18m + 16\}) = 18m + 15
\end{aligned}$$

$$\begin{aligned}
4 \cdot (12 + 18m + 10) &= \text{mex}(\{4 \cdot 0, \dots, 4 \cdot (12 + 18m + 8), 4 \cdot (12 + 18m + 9) \\
&\quad 3 \cdot (12 + 18m + 10), 2 \cdot (12 + 18m + 10), 1 \cdot (12 + 18m + 10), \\
&\quad 0 \cdot (12 + 18m + 10), 3 \cdot (12 + 18m + 9), 2 \cdot (12 + 18m + 8), \\
&\quad 1 \cdot (12 + 18m + 7), 0 \cdot (12 + 18m + 6)\}) \\
&= \text{mex}(\{0, \dots, 18m + 18, 18m + 21, 18m + 22, 18m + 23 \\
&\quad 18m + 19, 18m + 23, 18m + 22, 18m + 21 \\
&\quad 18m + 23, 18m + 18m + 19, 18m + 17\}) = 18m + 20
\end{aligned}$$

$$\begin{aligned}
4 \cdot (12 + 18m + 11) &= \text{mex}(\{4 \cdot 0, \dots, 4 \cdot (12 + 18m + 9), 4 \cdot (12 + 18m + 10) \\
&\quad 3 \cdot (12 + 18m + 11), 2 \cdot (12 + 18m + 11), 1 \cdot (12 + 18m + 11), \\
&\quad 0 \cdot (12 + 18m + 11), 3 \cdot (12 + 18m + 10), 2 \cdot (12 + 18m + 9), \\
&\quad 1 \cdot (12 + 18m + 8), 0 \cdot (12 + 18m + 7)\}) \\
&= \text{mex}(\{0, \dots, 18m + 18, 18m + 20, \dots, 18m + 23 \\
&\quad 18m + 18, 18m + 21, 18m + 23, 18m + 22, \\
&\quad 18m + 19, 18m + 22, 18m + 20, 18m + 18\}) = 18m + 24
\end{aligned}$$

$$\begin{aligned}
4 \cdot (12 + 18m + 12) &= \text{mex}(\{4 \cdot 0, \dots, 4 \cdot (12 + 18m + 10), 4 \cdot (12 + 18m + 11) \\
&\quad 3 \cdot (12 + 18m + 12), 2 \cdot (12 + 18m + 12), 1 \cdot (12 + 18m + 12), \\
&\quad 0 \cdot (12 + 18m + 12), 3 \cdot (12 + 18m + 11), 2 \cdot (12 + 18m + 10), \\
&\quad 1 \cdot (12 + 18m + 9), 0 \cdot (12 + 18m + 8)\}) \\
&= \text{mex}(\{0, \dots, 18m + 18, 18m + 20, \dots, 18m + 24 \\
&\quad 18m + 26, 18m + 25, 18m + 24, 18m + 23, \\
&\quad 18m + 18, 18m + 23, 18m + 21, 18m + 19\}) = 18m + 27
\end{aligned}$$

$$\begin{aligned}
4 \cdot (12 + 18m + 13) &= \text{mex}(\{4 \cdot 0, \dots, 4 \cdot (12 + 18m + 11), 4 \cdot (12 + 18m + 12) \\
&\quad 3 \cdot (12 + 18m + 13), 2 \cdot (12 + 18m + 13), 1 \cdot (12 + 18m + 13), \\
&\quad 0 \cdot (12 + 18m + 13), 3 \cdot (12 + 18m + 12), 2 \cdot (12 + 18m + 11), \\
&\quad 1 \cdot (12 + 18m + 10), 0 \cdot (12 + 18m + 9)\}) \\
&= \text{mex}(\{0, \dots, 18m + 18, 18m + 20, \dots, 18m + 24, 18m + 27 \\
&\quad 18m + 27, 18m + 26, 18m + 25, 18m + 24, \\
&\quad 18m + 26, 18m + 21, 18m + 22, 18m + 20\}) = 18m + 19
\end{aligned}$$

$$\begin{aligned}
4 \cdot (12 + 18m + 14) &= \text{mex}(\{4 \cdot 0, \dots, 4 \cdot (12 + 18m + 12), 4 \cdot (12 + 18m + 13) \\
&\quad 3 \cdot (12 + 18m + 14), 2 \cdot (12 + 18m + 14), 1 \cdot (12 + 18m + 14) \\
&\quad 0 \cdot (12 + 18m + 14), 3 \cdot (12 + 18m + 13), 2 \cdot (12 + 18m + 12), \\
&\quad 1 \cdot (12 + 18m + 11), 0 \cdot (12 + 18m + 10)\}) \\
&= \text{mex}(\{0, \dots, 18m + 24, 18m + 27 \\
&\quad 18m + 28, 18m + 24, 18m + 26, 18m + 25, \\
&\quad 18m + 27, 18m + 25, 18m + 23, 18m + 21\}) = 18m + 29
\end{aligned}$$

$$\begin{aligned}
4 \cdot (12 + 18m + 15) &= \text{mex}(\{4 \cdot 0, \dots, 4 \cdot (12 + 18m + 13), 4 \cdot (12 + 18m + 14) \\
&\quad 3 \cdot (12 + 18m + 15), 2 \cdot (12 + 18m + 15), 1 \cdot (12 + 18m + 15), \\
&\quad 0 \cdot (12 + 18m + 15), 3 \cdot (12 + 18m + 14), 2 \cdot (12 + 18m + 13), \\
&\quad 1 \cdot (12 + 18m + 12), 0 \cdot (12 + 18m + 11)\}) \\
&= \text{mex}(\{0, \dots, 18m + 24, 18m + 27, 18m + 29 \\
&\quad 18m + 29, 18m + 28, 18m + 27, 18m + 26, \\
&\quad 18m + 28, 18m + 26, 18m + 24, 18m + 22\}) = 18m + 25
\end{aligned}$$

$$\begin{aligned}
4 \cdot (12 + 18m + 16) &= \text{mex}(\{4 \cdot 0, \dots, 4 \cdot (12 + 18m + 14), 4 \cdot (12 + 18m + 15) \\
&\quad 3 \cdot (12 + 18m + 16), 2 \cdot (12 + 18m + 16), 1 \cdot (12 + 18m + 16), \\
&\quad 0 \cdot (12 + 18m + 16), 3 \cdot (12 + 18m + 15), 2 \cdot (12 + 18m + 14), \\
&\quad 1 \cdot (12 + 18m + 13), 0 \cdot (12 + 18m + 12)\}) \\
&= \text{mex}(\{0, \dots, 18m + 25, 18m + 27, 18m + 29 \\
&\quad 18m + 25, 18m + 29, 18m + 28, 18m + 27, \\
&\quad 18m + 29, 18m + 24, 18m + 25, 18m + 23\}) = 18m + 26
\end{aligned}$$

$$\begin{aligned}
4 \cdot (12 + 18m + 17) &= \text{mex}(\{4 \cdot 0, \dots, 4 \cdot (12 + 18m + 15), \\
&\quad 4 \cdot (12 + 18m + 16), 3 \cdot (12 + 18m + 17), 2 \cdot (12 + 18m + 17), \\
&\quad 1 \cdot (12 + 18m + 17), 0 \cdot (12 + 18m + 17), 3 \cdot (12 + 18m + 16), \\
&\quad 2 \cdot (12 + 18m + 15), 1 \cdot (12 + 18m + 14), 0 \cdot (12 + 18m + 13)\}) \\
&= \text{mex}(\{0, \dots, 18m + 27, 18m + 29 \\
&\quad 18m + 24, 18m + 27, 18m + 29, 18m + 28, \\
&\quad 18m + 25, 18m + 28, 18m + 26, 18m + 24\}) = 18m + 30
\end{aligned}$$

$$\begin{aligned}
4 \cdot (12 + 18m + 18) &= \text{mex}(\{4 \cdot 0, \dots, 4 \cdot (12 + 18m + 16), 4 \cdot (12 + 18m + 17) \\
&\quad 3 \cdot (12 + 18m + 18), 2 \cdot (12 + 18m + 18), 1 \cdot (12 + 18m + 18), \\
&\quad 0 \cdot (12 + 18m + 18), 3 \cdot (12 + 18m + 17), 2 \cdot (12 + 18m + 16), \\
&\quad 1 \cdot (12 + 18m + 15), 0 \cdot (12 + 18m + 14)\}) \\
&= \text{mex}(\{0, \dots, 18m + 27, 18m + 29, 18m + 30 \\
&\quad 18m + 32, 18m + 31, 18m + 30, 18m + 29, \\
&\quad 18m + 24, 18m + 29, 18m + 27, 18m + 25\}) = 18m + 28
\end{aligned}$$

At this point, column 4 is complete. □

6.4 Subquasigroups

Note that since $0 \cdot 0 = s$ and $sR(0)^n = s - n$ (for $n \leq s$). Thus if s or 0 is in a subquasigroup, S , then $\{0, 1, \dots, s\} \subset S$. This subset is called the **hub** as with the Q_i 's. Note that there may be other subquasigroup, since one cannot say that a particular element's inclusion in S forces 0 to be in S as with the Q_i 's.

Remark 6.4.1. For $s = 0, 1, 2$ there is a subquasigroup, namely $\{0, 1, 2\}$.

Theorem 6.4.1. *No subquasigroups of W_i contain the hub for $s \geq 3$.*

Proof. Let S be a subquasigroup of W_s . First consider $s \equiv_3 1$. First note that $H_s \subset S$ by the above remarks. One has that $s \cdot 2 = s + 1 \in S$. Thus $s + 1 \cdot 1 = s + 2 \in S$. Now $s + 2 \cdot 2 = s + 3 \in S$. So $\{0, \dots, s, s + 1, s + 2, s + 3\} \subset S$. Repeat the same argument starting at $s + 3$. Each time one adds the next three elements. Thus any subquasigroup that contains the hub is all of W_s for $s \equiv_3 1$.

For $s \equiv_3 0, 2$ note that it was proven that column 3 is never complete outside the hub. Now, $s \cdot 2 = s + 1$ for $s \equiv_3 0$. If $s \equiv_6 2$, $s \cdot 3 = s + 2$ and $s + 2 \cdot 2 = s + 1$. So in both of these cases, the hub generates an element outside the hub and since column 3 is never complete outside the hub, there are no non-trivial subquasigroups. So $S = W_s$.

The last case is $s \equiv_6 5$. By Lemma 6.3.21 column 4 is not complete at s for $s \equiv_6 5$. Moreover, as seen in the proof, $4 \cdot x = 1$ for $x \equiv_5 6$. Thus in this case, $4 \cdot s = s + 1$, and since column 3 is never complete, $S = W_s$. \square

Remark 6.4.2. It remains to be shown whether or not there is a subquasigroup contained inside the hub.

6.5 Non-isomorphism

Changing the seed does, indeed, give a new quasigroup.

Theorem 6.5.1. *For all $i \neq j$ there is no surjective quasigroup homomorphism $\phi : W_i \rightarrow W_j$.*

Proof. Note that each W_i , has exactly one element, 0, such that $0x = x0 = x$ for infinitely many x . Also, each W_i has exactly one idempotent element, with the exception of W_1 . For W_0 , this element is 0. For W_i $i \neq 0, 1$ this element is 1 which does not act as an identity for infinitely many elements. Now any homomorphism $\phi : W_i \rightarrow W_j$ must send $0 \mapsto 0$, since 0 is the unique element that acts as an identity for infinitely many elements. Thus $i\phi = 0 \cdot 0\phi = 0\phi 0\phi = 0 \cdot 0 = j$. Thus the seed gets mapped to the seed. Now, $j = i\phi = iR(0)^i\phi = iR(0)^i$. Thus $j + 1 | i + 1$. But if $i + 1 = k(j + 1)$, for $k \neq 1$, then there is some element, e , other than 1 such that $e\phi = 1$. But this is impossible, since 1 is idempotent, but e is not. Thus $k = 1$ and $i = j$. Thus for $i \neq j$, there is no such ϕ for $i, j \geq 2$. Since W_1 has no idempotent, $i, j \neq 1$. \square

Corollary 6.5.1. *For $i \neq j$, $W_i \not\cong W_j$.*

6.6 Conclusion

Wythoff quasigroups appear to be harder to analyze than greedy quasigroups. Since it is not the case that almost every element squares to the same element, some of the methods previously used do not apply. Since Wythoff's game is in some sense "hard" (see the chapter on Wythoff's game) it is only reasonable to expect the same from these quasigroups. Finding a complete characterization of the multiplication seems difficult. Research into this interesting class of quasigroups will be ongoing.

CHAPTER 7. Game Theory Applications

7.1 Introduction

Greedy quasigroups arose out of a desire to better understand certain combinatorial games, particularly Digital Deletions. In this chapter I will discuss the relevance of greedy quasigroups to combinatorial games and analyze Digital Deletions.

Since greedy and Wythoff quasigroups are generalizations of combinatorial games it is reasonable to seek a combinatorial game interpretation. Such an interpretation is given and this interpretation is connected to other work in the literature on games.

7.2 Playing greedy quasigroups as games

The game of Nim has Q_0 as its addition table, and values in (two pile) Wythoff's game can be computed using W_0 . The natural question is whether Q_s or W_s has any game theoretical application. In particular is there a game with Q_s or W_s as its table of values.

In fact, for each Q_s and W_s there is a fairly natural corresponding game. Imagine playing a game of nim. Suppose (without loss of generality) that there are two piles. When both piles are exhausted, the table is removed leaving a single nim heap of size s . This game corresponds exactly to Q_s . In the language of (39) this is a sequential compound of games. At first, it appears this game is not very interesting since the last person to play in the nim game loses since a sensible opponent will remove the entire nim-heap and win the whole game. If played as a single game, the game is no more interesting than nim, but when played as a component in a sum of games, calculating values becomes important. Of course, even the sum of games reduces to a game of nim, but as seen in Chapter 5 calculating values seems to be difficult. A slightly different characterization is the following: place a nim heap on a sufficiently large

chess board. Allow the nim heap to move “north” and “west” on the board, as a Rook moves in chess. When the nim-heap reaches the upper left corner, players move remove some or all of its counters. If several nim-heaps of different sizes are scattered about the board, one has an example of a sum of such games. In a similar fashion, one can realize W_s as a game. The first characterization is identical, play Wythoff’s game to its conclusion, and remove the table and play in the nim-heap. The only adjustment needed for the second characterization is to allow the nim-heaps on the board to move as chess Queens, moving north, west and directly northwest.

In reality, these games are simply another example of nim-in-disguise. So, in that light they are not very interesting. However, this characterization raises the question of the relative difficulty performing calculations in the quasigroups.

7.3 Analysis of Digital Deletions.

In ONAG, Conway says ”The inductive definitions of f_n tell us that each entry in the table is the mex of the numbers above and to the left of it, except that 0 is not allowed in the f_0 line. One can deduce that the entries in each line are ultimately arithmetico-periodic, so that the game has in principle a complete theory.” Then goes on to say that while some columns can be analyzed, ”there seem to be no easy answers.” (ONAG 192).

It turns out that greedy quasigroups play a role in Digital Deletions. If one treats the Digital Deletions table as a quasigroup and find its left and right division tables, one can possibly gain insight into its structure. The only problem is that Digital Deletions is not quite a quasigroup. There is no 0 in the first row. Thus there is no x in the table such that $0x = 0$. That is to say that $0 \setminus 0$ is undefined. Notice that $0/0 = 1$. In fact the only undefined division is $0 \setminus 0$. This is similar to division in a field, where division by 0 is undefined. However, Digital Deletions one can always right divide by 0, and one can left divide anything but 0 by 0. So in some sense it is easier to divide by 0 in Digital Deletions. The left division table for Digital Deletions is:

?	0	1	2	3	4	5	6	7	8
0	1	2	3	3	5	6	7	8	9
1	2	0	4	5	3	7	8	6	10
2	3	4	0	1	6	8	5	9	7
3	4	5	1	0	2	9	10	11	6
4	5	3	6	2	0	1	9	10	11
5	6	7	8	9	1	0	2	3	4
6	7	8	5	10	9	2	0	1	3
7	8	6	9	11	10	3	1	0	2
8	9	10	7	6	11	4	3	2	0

This looks a lot like a greedy quasigroup. The only problem is that $0 \setminus 0$ is undefined in the Digital Deletions table. If one can define it properly, one can make a greedy quasigroup out of its left division table. Two choices come to mind. The first is -1 . Certainly that fits the above pattern. The top row descends as one moves from right to left. The other alternative is ω . If one sets $0 \setminus 0 = \omega$ one can then fill in the greedy quasigroup as before. In this interpretation, one can imagine that 0 is in the 0 row at the ω position. This interpretation allows us to use the transfinite extensions mentioned earlier.

Since Digital Deletions is not commutative, one should not expect the right division table to resemble the left division table. In fact, the right division table is quite different from the left division table. It is given below:

1	2	3	4	5	6	7	8	9	10
0	1	2	3	4	5	6	7	8	9
2	0	1	5	3	4	8	6	7	11
3	4	0	1	2	7	5	9	6	8
4	3	5	0	1	2	9	10	11	6
5	6	4	2	0	1	3	11	10	7
6	5	7	8	9	0	1	2	3	4
7	8	6	9	10	3	0	1	2	5
8	7	9	6	11	10	2	0	1	3
9	10	8	7	6	11	4	3	9	1

This is exactly the table from Digital Deletions! Even though the structure of Digital Deletions may be hard to nail down directly, using the left and right division tables may help us get a better understanding of its structure. Since the right division table of Digital Deletions is itself, the right division table of the right division table of Digital Deletions is still Digital Deletions. It turns out that if one looks at the left division table for the left division table for Digital Deletions, he get Digital Deletions back again.

7.4 Conclusion

It is not surprising that although these quasigroups can be interpreted as games, the games are not radically different from nim. Nim modifications are quite common, but usually the change in strategy is minimal. What is remarkable is that greedy quasigroups are connected to Digital Deletions in such a simple way. If a simple characterization of multiplication can be found for the hub of a greedy quasigroup, a complete division table for Digital Deletions would be readily available and perhaps an “easy answer” for the rows of Digital Deletions can be found after all.

CHAPTER 8. Pandiagonal Latin Squares as Algebras

8.1 Introduction

Wythoff quasigroups appear to be examples of infinite pandiagonal latin squares. The question is whether Wythoff quasigroups are examples of a greedy algebra of a certain type. In other words, can the diagonal uniqueness be captured algebraically? This chapter will attempt to answer that question. Although complete results are not given, partial progress is made. An algebraic interpretation of latin squares with a complete set of transversals is given and interesting identities are derived.

8.2 Latin squares with transversals

Suppose the diagonal criterion is relaxed and instead of having diagonals containing unique elements, one only requires the latin square to have a complete set of transversals. Index the transversals by the set of elements in the latin square. Since each transversal intersects each row exactly once, a binary operation can be imposed on the set of rows and transversals, say $r \rightarrow t = x$ means that x is the element in row r and transversal t . Similarly a binary operation can be imposed on the columns and transversals, $t \downarrow c = y$ where y is in transversal t and column c . It is apparent that each of these are quasigroups. Given a row and transversal, there is one intersection. Given a transversal (or row) and an element, this element appears exactly once, in a particular row (or transversal).

This new criterion admits latin squares that are not pandiagonal or even isotopic to a pandiagonal latin square. For example:

1	2	3	4
4	3	2	1
2	1	4	3
3	4	1	2

Table 8.1 A latin square with 4 transversals

is a latin square of order four with four transversals. However from Corollary 4.3.4 there are no semi-pandiagonal latin squares of order 4.

The above conditions are not quite enough. It must be assured that the operations agree on the latin square. Let \cdot be the usual quasigroup operation on the rows and columns of the latin square and \rightarrow, \downarrow be as above. Then it must be the case that:

$$r \cdot c = r \rightarrow t \Leftrightarrow r \cdot c = t \downarrow c \quad (8.1)$$

In other words, suppose x is at the intersection of row r and column c and this is in transversal t . This arrangement determines all three of the quasigroup operations. This can be expressed as an identity:

$$a \cdot b = (a \leftarrow (a \cdot b)) \downarrow b \quad (8.2)$$

Therefore, latin squares with a complete set of transversals are realizable as a variety. From this point on, $a \cdot b$ will be written as juxtaposition, ab , and will take precedence in order of operations.

Theorem 8.2.1. *Let $(Q, \cdot, /, \backslash, \rightarrow, \nearrow, \leftarrow, \downarrow, \uparrow, \nwarrow)$ be an algebra of type $(2, 2, 2, 2, 2, 2, 2, 2, 2, 2)$, so that the reducts, $(Q, \cdot, /, \backslash)$, $(Q, \rightarrow, \nearrow, \leftarrow)$, $(Q, \downarrow, \uparrow, \nwarrow)$ are quasigroups. Furthermore, suppose that $ab \uparrow b = a \leftarrow ab$. The Cayley table for (Q, \cdot) is a latin square with a complete set of transversals. Any such latin square can be realized this way.*

Proof. Let L be a latin square with a complete set of transversals. Index the transversals by the element at the intersection of the first row and the transversal. Since transversal t contains all the elements of L , and contains them exactly once, $r \rightarrow t = x$ is uniquely defined. Also, since every element appears exactly once in transversal t , both $r \leftarrow x = t$ and $x \nearrow t$ are uniquely defined. The above discussion shows that $a \cdot b = (a \leftarrow (a \cdot b)) \downarrow b$ which is equivalent to $ab \uparrow b = a \leftarrow ab$.

Now suppose the algebraic conditions are satisfied. Show that the Cayley table for (Q, \cdot) is a latin square with a complete set of transversals.

Claim: $\{(a_i, b_i) : a_i \leftarrow a_i b_i = j\} = T_j$ for fixed j is a transversal, the j th transversal.

Proof: Suppose there are a_1, a_2 with $(a_1, b), (a_2, b) \in T_j$ i.e. $a_1 \leftarrow a_1 b = j = a_2 \leftarrow a_2 b$. Then $a_1 b = j \downarrow b = a_2 b$ by (8.2) and therefore $a_1 = a_2$ since $(Q, \downarrow, \swarrow, \uparrow)$ is a quasigroup. Thus an entry does not appear twice in the same column.

Suppose there are b_1, b_2 with $(a, b_1), (a, b_2) \in T_j$ so that $a \leftarrow a \cdot b_1 = j = a \leftarrow a b_2$. Then $a b_1 = a b_2$ since $(Q, \rightarrow, \nearrow, \leftarrow)$ is a quasigroup; so $b_1 = b_2$ since (Q, \cdot) is a quasigroup. Thus an entry does not appear twice in the same row.

Let $a_1 b_1 = a_2 b_2$ with $a_1 \neq a_2$. Show that $(a_1, b_1), (a_2, b_2)$ cannot both be in T_j . Let, $j = (a_1 \leftarrow a_1 b_1)$ and $k = (a_2 \leftarrow a_2 b_2)$. Now $a_1 \rightarrow j = a_2 \rightarrow k$. Since $a_1 \neq a_2, j \neq k$. Thus T_j does not contain the same element twice. Therefore, each T_j is a transversal.

Finally, it must be shown that the transversals really are a complete set. This can be done, by letting $a_i = 1$ and letting b_i range over the values in Q . In this way the transversal intersecting the b_i th entry in the first row is identified. \square

Such an algebra will be referred to as a *tri-quasigroup*.

Definition 8.2.1. A *tri-quasigroup* is an algebra with 9 binary operations, $(Q, \cdot, /, \backslash, \rightarrow, \nearrow, \leftarrow, \downarrow, \uparrow, \nwarrow)$ so that the reducts, $(Q, \cdot, /, \backslash)$, $(Q, \rightarrow, \nearrow, \leftarrow)$, $(Q, \downarrow, \uparrow, \nwarrow)$ are quasigroups with the additional identity $ab \uparrow b = a \leftarrow ab$ for all $a, b \in Q$.

8.3 Identities in tri-quasigroups

Tri-quasigroups possess several interesting identities. This section will discuss various identities. Most of the proofs were done using Prover9, an automated identity prover. A couple proofs are given below to give the reader a sense of the flavor of how the proofs work. The output files are included in an appendix.

Proposition 8.3.1.

$$(a \leftarrow ab) \downarrow b = a \rightarrow (ab \uparrow b) \tag{8.3}$$

(“Reverse arrows and reassociate.”)

Proof. Since by assumption $ab \uparrow b = a \leftarrow ab$: $a \rightarrow (ab \uparrow b) = a \rightarrow (a \leftarrow ab) = ab = a \rightarrow (ab \uparrow b)$. Also $ab = (ab \uparrow b) \downarrow b = (a \leftarrow ab)$. \square

This identity is the part of the motivation behind the naming of the operations $\rightarrow, \leftarrow, \uparrow, \downarrow$. (The remaining two (\nearrow, \nwarrow) were chosen since they resemble the divisions they represent in their respective quasigroup reducts.)

Proposition 8.3.2. *For any tri-quasigroup: $(x \downarrow y) \nearrow x = (x \downarrow y)/y$*

(Simply change right divisions and denominators).

Proof. In the equation $(x \rightarrow y) \nearrow y = x$ replace y with $(x \leftarrow y)$:

$$(x \rightarrow (x \leftarrow y)) \nearrow (x \leftarrow y) = x \quad (8.4)$$

$$y \nearrow (x \leftarrow y) = x. \quad (8.5)$$

In the equation $x \leftarrow xy = xy \uparrow y$ replace x with x/y :

$$(x/y) \cdot y \uparrow y = (x/y) \leftarrow (xy/y \cdot y) \quad (8.6)$$

$$x \uparrow y = (x/y) \leftarrow x. \quad (8.7)$$

Replace x in (8.5) with y/x and apply (8.7):

$$y \nearrow ((y/x) \leftarrow y) = y/x \quad (8.8)$$

$$y \nearrow (y \uparrow x) = y/x. \quad (8.9)$$

Finally, replace y in (8.9) with $y \downarrow x$:

$$(y \downarrow x) \nearrow ((y \downarrow x) \uparrow x) = (y \downarrow x)/x \quad (8.10)$$

$$(y \downarrow x) \nearrow y = (y \downarrow x)/x. \quad (8.11)$$

\square

There is a similar identity for left divisions:

Proposition 8.3.3. *For any tri-quasigroup:*

$$x \nearrow (y \rightarrow x) = y \setminus (y \rightarrow x) \quad (8.12)$$

Proof. In the equation $x \nearrow (x \downarrow y) = y$ replace x with $x \uparrow y$:

$$(x \uparrow y) \nearrow ((x \uparrow y) \downarrow y) = y \quad (8.13)$$

$$(x \uparrow y) \nearrow x = x. \quad (8.14)$$

In the equation $x \leftarrow xy = xy \uparrow y$ replace y with $x \setminus y$:

$$x \leftarrow (x \cdot (x \setminus y)) = (x \cdot (x \setminus y)) \uparrow (x \setminus y) \quad (8.15)$$

$$(x \leftarrow y) = y \uparrow (x \setminus y) \quad (8.16)$$

In (8.14) replace y with $y \setminus x$ and apply (8.16):

$$(x \uparrow (y \setminus x)) \nearrow x = y \setminus x \quad (8.17)$$

$$(y \leftarrow x) \nearrow x = y \setminus x \quad (8.18)$$

Replace x in (8.18) with $x \rightarrow y$:

$$y \leftarrow (x \rightarrow y) \nearrow (x \rightarrow y) = y \setminus (x \rightarrow y) \quad (8.19)$$

$$x \nearrow (x \rightarrow y) = y \setminus (x \rightarrow y) \quad (8.20)$$

□

Proposition 8.3.4. *Each of the following hold in any tri-quasigroup:*

$$(x \leftarrow y) \downarrow (x \setminus y) = y \quad (8.21)$$

$$(x/y) \rightarrow (x \uparrow y) = x \quad (8.22)$$

See Appendix 2 for proofs.

8.4 Restriction to isotopy classes

When writing a latin square, it makes sense to write the first row in lexicographic order, since one could always rename the elements of any square so that the first elements are in lexicographic order and have an identical latin square. This corresponds to making the quasi-group reduct (Q, \cdot) of a tri-quasigroup a left loop. It also makes sense to order the rows so that the first column is in numerical order. This restriction has no effect on the existence of transversals, since if an entry is in a particular transversal rearranging the order of the rows does not put it in the same row or column of another element of the same transversal. Essentially this process reduces the reduct (Q, \cdot) to isotopy classes. Furthermore, one can index the transversals by the element of the transversal in the first row of the latin square. This makes (Q, \rightarrow) into a left-loop.

Imposing such identity structure allows for more interesting identities. The following proposition is not surprising and is rather obvious:

Proposition 8.4.1. *For any tri-quasigroup such that $(Q, \cdot, \backslash, /)$ is a left-loop with identity 0:*

$$x \uparrow x = 0 \leftarrow x \tag{8.23}$$

$$(0 \leftarrow x) \downarrow x = x \tag{8.24}$$

$$(0 \leftarrow x) \nearrow x = x \tag{8.25}$$

Proposition 8.4.2. *For any tri-quasigroup such that $(Q, \cdot, \backslash, /)$ is a left-loop with identity 0:*

$$x \downarrow (0 \rightarrow x) = 0 \rightarrow x \tag{8.26}$$

$$x \nwarrow (0 \rightarrow x) = 0 \rightarrow x \tag{8.27}$$

As expected, if (Q, \cdot) is made into a right-loop instead, similar identities arise.

Proposition 8.4.3. *For any tri-quasigroup such that $(Q, \cdot, \backslash, /)$ is a right-loop with identity 0*

$$x \leftarrow x = x \uparrow 0 \tag{8.28}$$

$$x \rightarrow (x \uparrow 0) = x \tag{8.29}$$

$$x \nearrow (x \uparrow 0) = x \tag{8.30}$$

Proposition 8.4.4. *For any tri-quasigroup such that $(Q, \cdot, \backslash, /)$ is a right-loop with identity 0*

$$(x \downarrow 0) \rightarrow x = x \downarrow 0 \tag{8.31}$$

$$(x \downarrow 0) \nearrow x = x \downarrow 0 \tag{8.32}$$

8.5 Conclusion

Tri-quasigroups appear to have very interesting identities. Certainly there are more identities waiting to be discovered. Perhaps tri-quasigroups can be transformed into pandiagonal squares by the addition of additional identities or conditions and perhaps not. Either result would be interesting.

CHAPTER 9. Greedy Rings

Quasigroups are not the only algebras that can be generated in a greedy fashion. Conway elegantly generates the field On_2 in (14). This chapter will slightly generalize his results and will detail the creation of a ring in the same spirit he used to create a field. The field obtained is in some sense a “greedy field.” Just as one can create a quasigroup using the minimal-excluded element function, one can also create other algebras. With other algebraic structures, one has to verify that when each element is placed in the table, the algebra still has all desired properties.

To create a ring, one has to create an addition table and a multiplication table. The addition table must be a group, and the multiplication table must be associative and distribute over addition. Each of these properties must, in principle, be checked at each step.

9.1 Greedy ring table

Start by defining the table for \otimes in the ring. (The addition, denoted \oplus , is nim-addition.) First $0 \otimes 0$ can and so must be 0, so 0 is the 0 of the ring. Next $1 \otimes 1$ shouldn't be 0, so it can be 1, thus 1 is the 1 of the ring. So one now has the table given below.

Now $2 \otimes 2$ needs to be computed. It has not been specified that this is an integral domain, so it is fine to set $2 \otimes 2 = 0$. Since multiplication is required to be distributive, it must be that $2 \otimes 3 = 2 \otimes (1 \oplus 2) = 2 \otimes 1 \oplus 2 \otimes 2 = 2 \oplus 0 = 2$. Then one can calculate $3 \otimes 3 = (1 \oplus 2) \otimes 3 = 1 \otimes 3 \oplus 2 \otimes 3 = 3 \oplus 2 = 1$. One now has:

\otimes	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2						
3	0	3						
4	0	4						
5	0	5						
6	0	6						
7	0	7						

Table 9.1 Step 1 for the greedy ring

\otimes	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	0	2				
3	0	3	2	1				
4	0	4						
5	0	5						
6	0	6						
7	0	7						

Table 9.2 Step 2 for the greedy ring

Now one needs to specify $2 \otimes 4$. Nothing rules out 0, so it is tried. Now one can find

$$2 \otimes 5 = 2 \otimes (4 \oplus 1) = 2 \otimes 4 \oplus 2 \otimes 1 = 0 \oplus 2 = 2$$

$$2 \otimes 6 = 2 \otimes (4 \oplus 2) = 2 \otimes 4 \oplus 2 \otimes 2 = 0 \oplus 0 = 0$$

$$2 \otimes 7 = 2 \otimes (5 \oplus 2) = 2 \otimes 5 \oplus 2 \otimes 2 = 2 \oplus 0 = 2$$

$$4 \otimes 3 = 4 \otimes (2 \oplus 1) = 4 \otimes 2 \oplus 4 \otimes 1 = 0 \oplus 4 = 4$$

$$5 \otimes 3 = 5 \otimes 2 \oplus 5 \otimes 1 = 2 \oplus 5 = 7$$

$$6 \otimes 3 = 6 \otimes 2 \oplus 6 \otimes 1 = 0 \oplus 6 = 6$$

$$7 \otimes 3 = 7 \otimes 2 \oplus 7 \otimes 1 = 2 \oplus 7 = 5$$

This takes us to $4 \otimes 4$ which one can set to 0. One now has:

Now one specifies $4 \otimes 4 = 0$ and the rest of the table is filled in as above:

\otimes	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	0	2	0	2	0	2
3	0	3	2	1	4	7	6	5
4	0	4	0	4				
5	0	5	2	7				
6	0	6	0	6				
7	0	7	2	5				

Table 9.3 Step 3 for the greedy ring

\otimes	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	0	2	0	2	0	2
3	0	3	2	1	4	7	6	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	0	6	0	6	0	6
7	0	7	2	5	4	3	6	1

Table 9.4 Step 4 for the greedy ring

This can be characterized in the following manner: first define $2^n \otimes 2^m = 0$ (for $m, n \neq 0$).

Then write each factor as a sum of powers of 2:

$$a = \sum_{k=1}^n a_k 2^k + a_0, \quad b = \sum_{k=1}^n b_k 2^k + b_0 \quad (9.1)$$

where $a_k = 0, 1$ and n is the largest exponent necessary. Now since the product of two powers of 2 is 0:

$$a \otimes b = a_0 \otimes \sum_{k=0}^n b_k 2^k \oplus b_0 \otimes \sum_{k=0}^n a_k 2^k \oplus a_0 \cdot b_0 \quad (9.2)$$

Now in order for this to be a ring, \otimes must be associative.

Proposition 9.1.1. \otimes as defined above is associative.

Proof. In this proof $+$ binds more strongly than \otimes .

$$(a \otimes b) \otimes c = \left(a_0 \otimes \sum_{k=0}^n b_k 2^k \oplus b_0 \otimes \sum_{k=0}^n a_k 2^k \oplus a_0 \cdot b_0 \right) \otimes \left(\sum_{k=1}^n c_k 2^k + c_0 \right) \quad (9.3)$$

$$= a_0 c_0 \sum_{k=1}^n b_k 2^k \oplus b_0 c_0 \sum_{k=1}^n a_k 2^k \oplus a_0 b_0 \sum_{k=1}^n c_k 2^k \oplus a_0 b_0 c_0 \quad (9.4)$$

$$= \left(\sum_{k=1}^n a_k 2^k + a_0 \right) \otimes \left(c_0 \sum_{k=1}^n b_k 2^k \oplus b_0 \sum_{k=1}^n c_k 2^k \oplus b_0 c_0 \right) \quad (9.5)$$

$$= a \otimes (b \otimes c) \quad (9.6)$$

□

CHAPTER 10. Summary

Greedy and Wythoff quasigroups are generated using a simple algorithm, but give us some surprising results. Each one is not isomorphic to each of the others and most don't have any (non-trivial) subquasigroups. Although they are generated using a very systematic algorithm, their structure becomes less and less ordered as one moves deeper into the table. Since they are generalizations on nim, it is not surprising that they can be realized as a combinatorial game. Greedy quasigroups appear in the analysis of Digital Deletions as a left division table and quite possibly lead to an easy characterization. Perhaps they appear in the analysis of other combinatorial games.

The following questions arise:

- How many error terms are in the hub for any given s ? That is how many products of hub elements do not produce an element of the hub.
- Greedy quasigroups seem to be totally symmetric for entries greater than the seed. Is this really the case?
- How badly non-associative are greedy quasigroups? In particular, if $x, y, z > s \in Q_s$ is it true that $(xy)z = x(yz)$? What percentage of triples are associative?

Similar questions may be asked about Wythoff quasigroups. Wythoff quasigroups are harder to analyze than greedy quasigroups. A useful fact about greedy quasigroups, namely that almost all elements square to the same element, is not true for Wythoff quasigroups. In fact $x^2 \neq y^2$ for $x \neq y$ by their construction. The following questions are still open:

- Find an exact characterization of subquasigroups. My conjecture is that there are none, but this is not proven yet.

- Are there non-trivial homomorphisms $\phi : W_i \rightarrow W_j$?
- Complete an analysis of the multiplication group for W_s .
- Can the techniques used to resolve the above be applied to greedy quasigroups, and does doing so lead to new insights.

Generalized greedy quasigroups are interesting extensions of the greedy algebra concept. The conjugate theorem seems to imply that the “greediness” is fundamental to the quasigroups so generated since all the conjugates display the same structure. These quasigroups have not been investigated very fully. The following are some of the questions that seem to be the most important for future research.

- When are two generalized greedy quasigroups isomorphic? Are they ever? One can certainly create the same table by different definitions. For example, if one starts by placing 1 in the 0, 1 spot, one will get the table for Nim.
- How do the size and location of the seed affect the properties of the quasigroup?
- Does it matter if the seed is greater than or less than the corresponding Nim value? If it is less than the corresponding nim value, it affects its options, otherwise not.
- What happens if one defines more than one seed?
- Do generalized greedy quasigroups appear in analysis of combinatorial games? There is the characterization mentioned in a previous chapter, but are there other, more interesting applications.
- Can other conditions be imposed on the quasigroup? In particular, the all idempotent quasigroup mentioned earlier seems interesting.

Tri-quasigroups were discovered while searching for an algebraic characterization of Wythoff quasigroups. Can the diagonal structure be accounted for algebraically? The identities in tri-quasigroups that have been discovered so far are interesting and display remarkable symmetry.

Are there others? Is there a better characterization? Can similar algebras be formed to characterize pandiagonal latin squares?

Although many interesting results have been discovered so far, there is much work to be done. Greedy quasigroups seem to be important structures, perhaps the reader will be able to extend and apply these results.

APPENDIX A. Prover9 Generated Proofs

This appendix contains proofs generated automatically by Prover9. Prover9 is an automated theorem prover that is the successor of Otter written by W. McCune. Prover9 is distributed under the terms of the GNU General Public License (v2) and is free to download from McCune's website.

In order to use text based input, it was necessary to change the symbols for the binary operations. The following table gives the conversions: (No changes were needed for $/, \backslash$)

\cdot	$*$
\rightarrow	$+$
\leftarrow	$-$
\nearrow	\wedge
\downarrow	\otimes
\uparrow	$ $
\nwarrow	\sim

Proofs

Proofs not using loop structure

Proof that $(x \leftarrow y) \downarrow (x \backslash y) = y$.

===== prooftrans =====

Prover9 (32) version September-2006, September 2006. Process 2968 was started by Owner on YOUR-C018499B1B, Mon Apr 23 14:17:00 2007 The command was "bin/prover9 -f ls.in".

==== end of head =====

==== PROOF =====

% ----- Comments from original proof -----

% Proof 1 at 0.01 (+ 0.03) seconds.

% Length of proof is 8.

% Level of proof is 3.

% Maximum clause weight is 11.

% Given clauses 26.

1 (x -y) @ (x y) = y # label(goal). [goal].

2 x * (x y) = y. [assumption].

12 (x | y) @ y = x. [assumption].

15 (x * y) | y = x -(x * y).[assumption].

16 (c1 -c2) @ (c1 c2) != c2.[deny(1)].

25 x | (y x) = y -x.[para(2(a,1),15(a,1,1)),rewrite(2(4))].

33 (x -y) @ (x y) = y. [para(25(a,1),12(a,1,1))].

34 \$ F. [resolve(33,a,16,a)].

==== end of proof =====

Proof that $(x/y) \rightarrow (x \uparrow y) = x$.

==== prooftrans =====

Prover9 (32) version September-2006, September 2006.

Process 2348 was started by Owner on YOUR-C018499B1B,

Tue Apr 24 10:33:36 2007

The command was "bin/prover9 -f ls.in".

==== end of head =====

```

===== PROOF =====

% ----- Comments from original proof -----
% Proof 1 at 0.01 (+ 0.05) seconds.
% Length of proof is 8.
% Level of proof is 3.
% Maximum clause weight is 11.
% Given clauses 22.

1 (x / y) + (x | y) = x # label(goal). [goal].
3 (x / y) * y = x. [assumption].
6 x + (x -y) = y. [assumption].
14 (x * y) | y = x -(x * y). [assumption].
15 (c1 / c2) + (c1 | c2) != c1. [deny(1)].
23 (x / y) -x = x | y. [para(3(a,1),14(a,1,1)),rewrite(3(4)),flip(a)].
28 (x / y) + (x | y) = x. [para(23(a,1),6(a,1,2))].
29 $ F. [resolve(28,a,15,a)].

===== end of proof =====

Proof that  $(x/y) \rightarrow (x \uparrow y) = x$ .

===== prooftrans =====
Prover9 (32) version September-2006, September 2006.
Process 3124 was started by Owner on YOUR-C018499B1B,
Thu Apr 26 15:38:34 2007
The command was "bin/prover9 -f ls.in".
===== end of head =====

```

===== PROOF =====

```
% ----- Comments from original proof -----
% Proof 1 at 0.01 (+ 0.00) seconds.
% Length of proof is 8.
% Level of proof is 3.
% Maximum clause weight is 11.
% Given clauses 22.

1 (x / y) + (x | y) = x # label(goal). [goal].
3 (x / y) * y = x. [assumption].
6 x + (x -y) = y. [assumption].
14 (x * y) | y = x -(x * y). [assumption].
15 (c1 / c2) + (c1 | c2) != c1. [deny(1)].
23 (x / y) -x = x | y. [para(3(a,1),14(a,1,1)),rewrite(3(4)),flip(a)].
28 (x / y) + (x | y) = x. [para(23(a,1),6(a,1,2))].
29 $ F. [resolve(28,a,15,a)].
```

===== end of proof =====

Proofs using loop structure

Proof that

$$x \uparrow x = 0 \leftarrow x$$

$$(0 \leftarrow x) \downarrow x = x$$

$$(0 \leftarrow x) \nearrow x = x$$

===== prooftrans =====

Prover9 (32) version September-2006, September 2006.

Process 688 was started by Owner on YOUR-C018499B1B,

Fri Apr 27 11:14:50 2007

The command was "bin/prover9 -f ls.in".

==== end of head =====

==== PROOF =====

% ----- Comments from original proof -----

% Proof 1 at 0.03 (+ 0.03) seconds.

% Length of proof is 7.

% Level of proof is 3.

% Maximum clause weight is 11.

% Given clauses 14.

1 x | x = 0 -x # label(goal). [goal].

8 0 * x = x. [assumption].

17 (x * y) | y = x -(x * y). [assumption].

18 0 -c1 != c1 | c1. [deny(1)].

19 c1 | c1 != 0 -c1. [copy(18),flip(a)].

32 x | x = 0 -x. [para(8(a,1),17(a,1,1)),rewrite(8(4))].

33 \$ F. [resolve(32,a,19,a)].

==== end of proof =====

==== PROOF =====

% ----- Comments from original proof -----

% Proof 2 at 0.03 (+ 0.03) seconds.

% Length of proof is 8.

```

% Level of proof is 3.
% Maximum clause weight is 11.
% Given clauses 25.

3 (0 -x) @ x = x # label(goal). [goal].
8 0 * x = x. [assumption].
14 (x | y) @ y = x. [assumption].
17 (x * y) | y = x -(x * y). [assumption].
21 (0 -c3) @ c3 != c3. [deny(3)].
32 x | x = 0 -x. [para(8(a,1),17(a,1,1)),rewrite(8(4))].
37 (0 -x) @ x = x. [para(32(a,1),14(a,1,1))].
38 $ F. [resolve(37,a,21,a)].

===== end of proof =====

===== PROOF =====

% ----- Comments from original proof -----
% Proof 3 at 0.03 (+ 0.03) seconds.
% Length of proof is 10.
% Level of proof is 3.
% Maximum clause weight is 11.
% Given clauses 25.

2 (0 -x) ~ x = x # label(goal). [goal].
8 0 * x = x. [assumption].
14 (x | y) @ y = x. [assumption].
16 x ~ (x @ y) = y. [assumption].

```

```

17 (x * y) | y = x -(x * y). [assumption].
20 (0 -c2) ~ c2 != c2. [deny(2)].
29 (x | y) ~ x = y. [para(14(a,1),16(a,1,2))].
32 x | x = 0 -x. [para(8(a,1),17(a,1,1)),rewrite(8(4))].
39 (0 -x) ~ x = x. [para(32(a,1),29(a,1,1))].
40 $ F. [resolve(39,a,20,a)].

```

===== end of proof =====

Proof that $x \downarrow (0 \rightarrow x) = 0 \rightarrow x$.

===== prooftrans =====

Prover9 (32) version September-2006, September 2006. Process 2208 was started by
Owner on YOUR-C018499B1B, Mon Apr 23 10:28:25 2007 The command was "bin/prover9 -f
ls.in".

===== end of head =====

===== PROOF =====

% ----- Comments from original proof -----

% Proof 1 at 0.01 (+ 0.00) seconds.

% Length of proof is 10.

% Level of proof is 4.

% Maximum clause weight is 11.

% Given clauses 29.

```

1 x @ (0 + x) = 0 + x # label(goal). [goal].

```

```

6 0 * x = x. [assumption].

```

```

11 x -(x + y) = y. [assumption].

```

```

13 (x | y) @ y = x. [assumption].
16 (x * y) | y = x -(x * y). [assumption].
17 c1 @ (0 + c1) != 0 + c1.[deny(1)].
30 x | x = 0 -x. [para(6(a,1),16(a,1,1)),rewrite(6(4))].
34 (0 -x) @ x = x. [para(30(a,1),13(a,1,1))].
39 x @ (0 + x) = 0 + x. [para(11(a,1),34(a,1,1))].
40 $ F. [resolve(39,a,17,a)].

```

==== end of proof =====

Proof that $x \searrow (0 \rightarrow x) = (0 \rightarrow x)$.

==== prooftrans =====

Prover9 (32) version September-2006, September 2006. Process 3992 was started by Owner on YOUR-C018499B1B, Mon Apr 23 10:32:06 2007 The command was "bin/prover9 -f ls.in".

==== end of head =====

==== PROOF =====

% ----- Comments from original proof -----

% Proof 1 at 0.01 (+ 0.00) seconds.

% Length of proof is 12.

% Level of proof is 4.

% Maximum clause weight is 11.

% Given clauses 30.

```

1 x ~ (0 + x) = 0 + x # label(goal). [goal].

```

```

6 0 * x = x. [assumption]. 11 x -(x + y) = y. [assumption].

```

```

13 (x | y) @ y = x.[assumption].

```

```

15 x ~ (x @ y) = y. [assumption].
16 (x * y) | y = x -(x * y). [assumption].
17 c1 ~ (0 + c1) != 0 + c1. [deny(1)].
27 (x | y) ~ x = y. [para(13(a,1),15(a,1,2))].
30 x | x = 0 -x. [para(6(a,1),16(a,1,1)),rewrite(6(4))].
35 (0 -x) ~ x = x. [para(30(a,1),27(a,1,1))].
40 x ~ (0 + x) = 0 + x. [para(11(a,1),35(a,1,1))].
41 $ F. [resolve(40,a,17,a)].

```

==== end of proof =====

Proof that $x \searrow (0 \rightarrow x) = x \downarrow (0 \rightarrow x)$.

==== prooftrans =====

Prover9 (32) version September-2006, September 2006.

Process 3556 was started by Owner on YOUR-C018499B1B,

Mon Apr 23 10:32:42 2007

The command was "bin/prover9 -f ls.in".

==== end of head =====

==== PROOF =====

% ----- Comments from original proof -----

% Proof 1 at 0.01 (+ 0.00) seconds.

% Length of proof is 15.

% Level of proof is 5.

% Maximum clause weight is 11.

% Given clauses 30.

```

1 x ~ (0 + x) = x @ (0 + x) # label(goal). [goal].

```

```

6 0 * x = x. [assumption].

```

```

11 x -(x + y) =y. [assumption].
13 (x | y) @ y = x. [assumption].
15 x ~ (x @ y) = y. [assumption].
16 (x * y) | y = x -(x * y). [assumption].
17 c1 ~ (0 + c1) != c1 @ (0 + c1). [deny(1)].
27 (x | y) ~ x = y. [para(13(a,1),15(a,1,2))].
30 x | x = 0 -x. [para(6(a,1),16(a,1,1)),rewrite(6(4))].
34 (0 -x) @ x = x. [para(30(a,1),13(a,1,1))].
35 (0 -x) ~ x = x. [para(30(a,1),27(a,1,1))].
39 x @ (0 + x) = 0 + x. [para(11(a,1),34(a,1,1))].
40 c1 ~ (0 + c1) != 0 + c1. [back_rewrite(17),rewrite(39(10))].
41 x ~ (0 + x) = 0 + x. [para(11(a,1),35(a,1,1))].
42 $ F. [resolve(41,a,40,a)].

```

===== end of proof =====

Proof that

$$x \downarrow (0 \rightarrow x) = 0 \rightarrow x \quad (\text{A.1})$$

$$x \nearrow (0 \rightarrow x) = 0 \rightarrow x \quad (\text{A.2})$$

===== prooftrans =====

Prover9 (32) version September-2006, September 2006.

Process 2568 was started by Owner on YOUR-C018499B1B,

Fri Apr 27 14:41:35 2007

The command was "bin/prover9 -f ls.in".

===== end of head =====

op(500,infix,"|").

op(500,infix,"^").

```

op(500,infix,"+").
op(500,infix,"-").
op(500,infix,"~").
op(500,infix,"@").

```

```

===== end of input =====

```

```

===== PROOF =====

```

```

% ----- Comments from original proof -----

```

```

% Proof 1 at 0.01 (+ 0.00) seconds.

```

```

% Length of proof is 10.

```

```

% Level of proof is 4.

```

```

% Maximum clause weight is 11.

```

```

% Given clauses 27.

```

```

1 x @ (0 + x) = 0 + x # label(goal). [goal].

```

```

7 0 * x = x. [assumption].

```

```

11 x -(x + y) = y.[assumption].

```

```

13 (x | y) @ y = x. [assumption].

```

```

16 (x * y) | y = x -(x * y). [assumption].

```

```

17 c1 @ (0 + c1) != 0 + c1. [deny(1)].

```

```

29 x | x = 0 -x. [para(7(a,1),16(a,1,1)),rewrite(7(4))].

```

```

32 (0 -x) @ x = x.[para(29(a,1),13(a,1,1))].

```

```

36 x @ (0 + x) = 0 + x. [para(11(a,1),32(a,1,1))].

```

```

37 $ F. [resolve(36,a,17,a)].

```

```

===== end of proof =====

```

===== PROOF =====

```

% ----- Comments from original proof -----
% Proof 2 at 0.01 (+ 0.00) seconds.
% Length of proof is 17.
% Level of proof is 4.
% Maximum clause weight is 11.
% Given clauses 30.

2 x ~ (0 + x) = 0 + x # label(goal). [goal].
3 x * (x y) = y. [assumption].
7 0 * x = x. [assumption].
9 (x ^ y) + y = x. [assumption].
10 (x + y) ^ y = x. [assumption].
11 x -(x + y) = y. [assumption].
13 (x | y) @ y = x. [assumption].
15 x ~ (x @ y) = y. [assumption].
16 (x * y) | y = x -(x * y). [assumption].
18 c2 ~ (0 + c2) != 0 + c2. [deny(2)].
21 0 x = x. [para(7(a,1),3(a,1))].
24 (x ^ y) -x = y. [para(9(a,1),11(a,1,2))].
26 (x | y) ~ x = y. [para(13(a,1),15(a,1,2))].
27 x | (y x) = y -x. [para(3(a,1),16(a,1,1)),rewrite(3(4))].
35 (x -y) ~ y = x y. [para(27(a,1),26(a,1,1))].
44 x ~ y = (y ^ x) y. [para(24(a,1),35(a,1,1))].

```

45 \$ F. [back_rewrite(18),rewrite(44(5),10(5),21(5)),xx(a)].

===== end of proof =====

Proof that

$$x \leftarrow x = x \uparrow 0 \quad (\text{A.3})$$

$$x \rightarrow (x \uparrow 0) = x \quad (\text{A.4})$$

$$x \nearrow (x \uparrow 0) = x \quad (\text{A.5})$$

===== prooftrans =====

Prover9 (32) version September-2006, September 2006.

Process 3312 was started by Owner on YOUR-C018499B1B,

Fri Apr 27 14:58:37 2007 The command was "bin/prover9 -f ls.in".

===== end of head =====

===== PROOF =====

% ----- Comments from original proof -----

% Proof 1 at 0.06 (+ 0.01) seconds.

% Length of proof is 7.

% Level of proof is 3.

% Maximum clause weight is 11.

% Given clauses 14.

1 x -x = x | 0 # label(goal). [goal].

8 x * 0 = x. [assumption].

17 (x * y) | y = x -(x * y).[assumption].

18 c1 -c1 != c1 | 0. [deny(1)].

```

19 c1 | 0 != c1 -c1. [copy(18),flip(a)].
32 x | 0 = x -x. [para(8(a,1),17(a,1,1)),rewrite(8(4))].
33 $ F. [resolve(32,a,19,a)].

===== end of proof =====

===== PROOF =====

% ----- Comments from original proof -----
% Proof 2 at 0.06 (+ 0.01) seconds.
% Length of proof is 9.
% Level of proof is 2.
% Maximum clause weight is 11.
% Given clauses 14.

3 x ^ (x | 0) = x # label(goal). [goal].
8 x * 0 = x. [assumption].
9 x + (x -y) = y.[assumption].
11 (x + y) ^ y = x. [assumption].
17 (x * y) | y = x -(x * y). [assumption].
21 c3 ^ (c3 | 0) != c3. [deny(3)].
26 x ^ (y -x) = y. [para(9(a,1),11(a,1,1))].
32 x | 0 = x -x. [para(8(a,1),17(a,1,1)),rewrite(8(4))].
35 $ F. [back_rewrite(21),rewrite(32(4),26(5)),xx(a)].

===== end of proof =====

===== PROOF =====

```

```

% ----- Comments from original proof -----
% Proof 3 at 0.06 (+ 0.01) seconds.
% Length of proof is 7.
% Level of proof is 2.
% Maximum clause weight is 11.
% Given clauses 14.

2 x + (x | 0) = x # label(goal). [goal].
8 x * 0 = x. [assumption].
9 x + (x -y) = y. [assumption].
17 (x * y) | y = x -(x * y). [assumption].
20 c2 + (c2 | 0) != c2. [deny(2)].
32 x | 0 = x -x. [para(8(a,1),17(a,1,1)),rewrite(8(4))].
36 $ F. [back_rewrite(20),rewrite(32(4),9(5)),xx(a)].

===== end of proof =====

===== prooftrans =====

Prover9 (32) version September-2006, September 2006.
Process 1724 was started by Owner on YOUR-C018499B1B, Fri Apr 27 14:52:00 2007
The command was "bin/prover9 -f ls.in".

===== end of head =====

===== PROOF =====

% ----- Comments from original proof -----
% Proof 1 at 0.03 (+ 0.05) seconds.

```

```

% Length of proof is 11.
% Level of proof is 4.
% Maximum clause weight is 11.
% Given clauses 29.

1 (x @ 0) + x = x @ 0 # label(goal). [goal].
7 x * 0 = x. [assumption].
8 x + (x -y) = y. [assumption].
14 (x @ y) | y = x. [assumption].
16 (x * y) | y = x -(x * y). [assumption].
17 c1 @ 0 != (c1 @ 0) + c1. [deny(1)].
18 (c1 @ 0) + c1 != c1 @ 0. [copy(17),flip(a)].
30 x | 0 = x -x. [para(7(a,1),16(a,1,1)),rewrite(7(4))].
34 (x @ 0) -(x @ 0) = x. [para(30(a,1),14(a,1))].
40 (x @ 0) + x = x @ 0. [para(34(a,1),8(a,1,2))].
41 $ F. [resolve(40,a,18,a)].

===== end of proof =====

===== PROOF =====

% ----- Comments from original proof -----
% Proof 2 at 0.03 (+ 0.05) seconds.
% Length of proof is 12.
% Level of proof is 4.
% Maximum clause weight is 11.
% Given clauses 29.

```

2 $(x @ 0) \wedge x = x @ 0$ # label(goal). [goal].

7 $x * 0 = x$. [assumption].

8 $x + (x - y) = y$. [assumption].

10 $(x + y) \wedge y = x$. [assumption].

14 $(x @ y) \mid y = x$. [assumption].

16 $(x * y) \mid y = x -(x * y)$. [assumption].

19 $(c2 @ 0) \wedge c2 \neq c2 @ 0$. [deny(2)].

24 $x \wedge (y -x) = y$. [para(8(a,1),10(a,1,1))].

30 $x \mid 0 = x -x$. [para(7(a,1),16(a,1,1)),rewrite(7(4))].

34 $(x @ 0) -(x @ 0) = x$. [para(30(a,1),14(a,1))].

42 $(x @ 0) \wedge x = x @ 0$. [para(34(a,1),24(a,1,2))].

43 \$ F. [resolve(42,a,19,a)].

===== end of proof =====

BIBLIOGRAPHY

- [1] Adelson-Velskym, G. M. (1988) *Algorithms for Games* New York: Springer-Verlag Inc.
- [2] Afsarinejad, K. (1986). *Self Orthogonal Knut Vik Designs* Statistics and Probability Letters, 4, 289.
- [3] Afsarinejad, K. (1987). *On Mutually Orthogonal Knut Vik Designs* Statistics and Probability Letters, 5, 323-324.
- [4] Afsarinejad, K. (1987). *Semi Knut Vik Designs* Statistics and Probability Letters, 6, 243-245.
- [5] Albert, A. A. (1943. *Quasigroups I*. Transactions of the American Mathematical Society. 54, no 3, 507-519.
- [6] lavi, Y, Lick, D. R., Liu, J. (1994) *Strongly diagonal Latin squares and permutation cubes*. Congr. Numerantium 102, 65-70.
- [7] Atken, A. O. L., Hay, L.,Larson, R. G. (1983). *Enumeration and Construction of Pandiagonal Latin Squares of Prime Order*. Computers and Mathematics with Applications., 9 no 2, 267-292.
- [8] Bailey, R (2006) *Partially Balanced Designs* Encyclopedia of Statistitcal Sciences. John Wiley and Sons.
- [9] Berlekamp, E. R, Conway, J. H., Guy, R.K. (2001) *Winning Ways for your Mathematical Plays* vol 1. Natick, MA: A K Peters, Ltd.

- [10] Berlekamp, E. R, Conway, J. H., Guy, R.K. (2003) *Winning Ways for your Mathematical Plays* vol 2. Natick, MA: A K Peters, Ltd.
- [11] Bhattacharjee, M et. al. *Notes on Infinite Permutation Groups*. Lecture Notes In Mathematics, Springer-Verlag, New York 1998.
- [12] Blass, U. and Fraenkel A. S.(1990) *The Sprague-Grundy Function for Wythoff's Game* Theoretical Computer Science Vol 75 311-333.
- [13] Bruck, R. H. (1944) *Some results from the theory of quasigroups*. Transactions of the American Mathematical Society, 51 no 1. 19-52.
- [14] Conway, J. H. (2001). *On Numbers and Games*. Natick, MA: A K Peters, Ltd.
- [15] Coxeter, H. S. M. (1953) *The Golden Section, Phyllotaxis, and Wythoff's Game*. Scripta Mathematica. Vol 19 135-143.
- [16] Demain, Erik D. *Playing Games with Algorithms: Algorithmic Combinatorial Game Theory*. Lecture Notes in Computer Science 2136 2001. 18-33.
- [17] Denes, J and Keedwell A. D. (1974). *Latin Squares and their Applications*. New York: Academic Press.
- [18] Denes, J and Keedwell, A. D. (1988). *Latin Squares and 1-factorizations of complete graphs: (I) Connections bewteen the enumeration of latin squares and r-factorizations of labelled graphs*. Ars Combinatoria 25 (1988) pp 109-126.
- [19] Dixon, J and Mortimer, B. *Permutation Groups* Springer-Verlag, New York, 1996.
- [20] Dress, A. (1999) *Additive periodicity of the Sprague-Grundy function of certain nim games*. Advances in Applied Mathematics. vol 22, 1999. pp 249-270.
- [21] Evans, A. B. *On Strong complete mappings*. Congressus Numerantium. vol 70 pp 241-248.
- [22] Fraenkel, A. S. (1982) *How to beat your Wythoff Games' Opponents on Three Fronts*. American Math Monthly Vol 89 353-361.

- [23] Frankel, A. S.(1991) *Complexity of Games* in: *Combinatorial Games* in: Providence, RI: American Mathematical Society.
- [24] Fraenkel, A.S(1998) *How Far can Nim in disguise be stretched?* Journal of Combinatorial Theory Ser A vol 84 pp. 146-156.
- [25] Fraenkel, A. S.(2002). *Complexity, Appeal and Challenges of Combinatorial Games*. Theoretical Computer Science 313 393-415.
- [26] Gardner, M (1989). *Penrose Tiles and Trapdoor Ciphers*. New York: W.H. Freeman and Company
- [27] Gilbert, W. J and Nicholson, K. W. (2004). *Modern Algebra with Applications*. Hoboken, New Jersey: John Wiley and Sons, Inc.
- [28] Guy, R. K. (1991). *What is a Game?* in: *Combinatorial Games*. Providence, RI: American Mathematical Society.
- [29] Halmos, P, R. (1960). *Naive Set Theory*. Princeton, NJ: D. Van Nostrand Company, Inc.
- [30] Hedayat, A. and Federer, W. T. (1975). *On the Existence of Knut Vik Designs for All Even Orders*. The Annals of Statistics Vol 3, no. 2 445-447.
- [31] Hedayat, A. (1977). *A Complete Solution to the Existence and Nonexistence of Knut Vik Designs and Orthogonal Knut Vik Designs*. Journal of Combinatorial Theory (A) 22, 331-337.
- [32] Landman, H. A. (2002). *A Simple FSM-Based Proof of the Additive Periodicity of the Sprague-Grundy Function of Wythoff's Game*, in: *More Games of No Chance*, Proc MSRI Workshop on Combinatorial Games, July 2000, Berkeley, CA: MRSI Publ. Vol. 42 Cambridge: Cambridge University Press.
- [33] Plamback, T. E. (1992) *Daisies, Kayles, and the Silbert-Conway decomposition in misère octal games*. Theoretical Computer Science vol 96 361-388.

- [34] Schumer, P. D.(2004) *Mathematical Journeys*. Hoboken, New Jersey: John Wiley and Sons, Inc.
- [35] Silber, R.(1976) *A Fibonacci Property of Wythoff Pairs* Fibonacci Quarterly vol 17 380-384.
- [36] Silber, R.(1977) *Wythoff's Nim and Fibonacci Representations* Fibonacci Quarterly vol 15 85-88.
- [37] Smith, J. D. H. (2006) *An Introduction to Quasigroups and Their Representations*. Boca Raton, FL: Chapman and Hall CRC.
- [38] Smith, J. D. H. and Romanowska, A. B.(1999). *Post-Modern Algebra*. New York: John Wiley and Sons, Inc.
- [39] Stromquist, W. and Ullman, D. (1993) *Sequential compunds of combinatorial games* Theoretical Computer Science vol 119 311-321.
- [40] Suppes, P. (1960). *Axiomatic Set Theory*. Princeton, NJ:D. Van Nostrand Company, Inc.
- [41] Wolfe, David and Fraser, W. *Counting the number of games*. Theoretical Computer Science vol 313 527-532.

ACKNOWLEDGEMENTS

I would like to take this opportunity to thank those individuals who helped me with my thesis. First, I would like to thank my advisor, Dr. JDH Smith, who first expressed interest in my ideas and encouraged me greatly. His advise about what questions to ask was superb.

I am also indebted to my friend Jeremy Alm, who acted as a sounding board for my ideas and provided useful insights and asked good questions to aid my thinking. I also wish to thank my former office mate Ajith Gunaratne who helped my figure out how to use Matlab which was a valuable tool for this research.